Nonlinear excitations in a classical planar Heisenberg ferromagnet in an external field (CPHFF) are studied. Taking a classical counterpart of the spin-raising operator as a relevant field variable, we establish a close correspondence between the CPHFF and a complex scalar field (CSF) in which each atom in a complex lattice field, while coupled with its neighbours, sits on $\phi'$-like on-site potential with saturable nonlinearity; In their static form CPHFF equations and CSF equations are identical to each other. In the continuum limit the CSF takes a semi-classical form of a Bose liquid with nonlinearity, however, characteristic of classical spin system. Solutions to the field equations are studied by using the continuum approximation. In one-dimensional case moving domain-wall solutions associated with symmetry-breaking states are obtained for the CPHFF and the CSF. In two- and three-dimensional cases static solutions to the field equations are obtained in the form of vortex solutions in close analogy to the case of the Ginzburg-Pitaevskii equation in the theory of superfluidity.

§ 1. Introduction

The planar Heisenberg ferromagnet in an external field (PHFF) is defined by the Hamiltonian:

$$
\hat{H} = -\epsilon \sum_n \vec{S}_n^2 - \sum_{n,m} [J(n, m)(\vec{S}_n^x \vec{S}_m^x + \vec{S}_n^y \vec{S}_m^y) + J'(n, m)\vec{S}_n^z \vec{S}_m^z].
$$

Here $\vec{S}_n = (\vec{S}_n^x, \vec{S}_n^y, \vec{S}_n^z)$ is the spin angular momentum with magnitude $S$ on the lattice site $n$, $\vec{S}_n^a (a = x, y, z)$ being its Cartesian $a$-component, and $\epsilon > 0$ is the external field. The coupling constants $J(n, m)$ and $J'(n, m)$ are taken to be all positive and assumed to depend only on the coordinate difference between the lattice sites $n$ and $m$. This is an anisotropic Heisenberg ferromagnet with an easy plane of magnetization interacting with the external field perpendicular to the easy plane.**

Namely, $J$ and $J'$ satisfy the condition that the ground state

---

*Throughout this paper a $q$-number and its corresponding $c$-number are written by a symbol with and without "hat", respectively.

** We do not mean by planar that the axial components of the spins are forced to lie in a plane or that the lattice is two-dimensional. Here the spins are three-dimensional, while the lattice under consideration is either one-, two- or three-dimensional.
of the system without the external field has a ferromagnetic spin alignment, with 
\( \langle \hat{S}_n^+ \rangle \) in an arbitrary direction in the xy-plane,

\[
\langle \hat{S}_n^+ \rangle = \langle \hat{S}_n^x + i \hat{S}_n^y \rangle = S \langle \hat{q}_n \rangle \neq 0, \quad \langle \hat{S}_n^x \rangle = 0, \quad (1.2)
\]

where an angular bracket denotes an average. If \( \varepsilon \) is less than some critical field strength, the ground state still has a ferromagnetic alignment of spins, but it now also has \( \langle \hat{S}_n^z \rangle \neq 0 \). It has been used as a reasonable model of certain types of ferromagnetic insulators.\(^5\) In the specific case \( j(n, m) = 0 \),

\[
\hat{H} = -\varepsilon \sum_n \hat{S}_n^x - \sum_{n, m} (\hat{S}_n^x \hat{S}_m^x + \hat{S}_n^y \hat{S}_m^y)
\]

is the Hamiltonian of the so-called XY model in a transverse field (XYMTF). In one-dimensional (1d) case for \( S = 1/2 \) with nearest neighbour interactions the XYMTF is\(^2\) together with the Ising model in a transverse field (IMTF),\(^2\) among few many-body problems which can be solved exactly. The physical properties of the PHFF have received particular attention since it is similar to superfluid He\(^4\) in that the ground state of its ordered phase exhibits broken symmetry with respect to a continuous symmetry of the Hamiltonian. Matsubara and Matsuda were the first to use it as a quantum lattice fluid model of liquid He\(^4\).\(^5\) Halperin and Hohenberg have developed a hydrodynamic theory of spin waves to gain insight into the foundation of two-fluid hydrodynamics.\(^6\) Nagaoka discussed the problem of superfluidity by making a correspondence between a Bose system and the PHFF from the viewpoint of gauge symmetry and diagonal- and off-diagonal long-range orders.\(^7\) In these works, however, little attention was paid to nonlinear excitations in the PHFF.

On the other hand, there has recently been growing and developing interest in soliton-like excitations in one- and two-dimensional classical Heisenberg spin systems. One of remarkable results from mathematical viewpoint is proofs of complete integrability of the 1d continuous isotropic Heisenberg model\(^8\) and of the 1d spin system described by the Landau-Lifshitz equation.\(^9\) In a previous paper, which will hereafter be referred to as (1),\(^1\) we have studied soliton-like excitations in classical generalized Heisenberg spin systems which include the classical PHFF (CPHFF) as a specific case. It has been shown that nonlinear excitations in classical spin systems are very rich as in the case of plasma physics and hydrodynamics. For the CPHFF, however, only static domain-wall solutions were obtained.

The purpose of the present paper is three-fold: (1) By taking a classical counterpart \( S_n^+ \) of \( \hat{S}_n^+ \) as a relevant field variable, we make a close correspondence between the CPHFF and a complex scalar field (CSF) having the classical form of Boson field with nonlinearity, however, characteristic of
classical spin system. The CSF so introduced is therefore more similar to the superfluid He$^4$ than the CPHFF or the PHFF. (2) In 1d case we obtain moving domain wall solutions to nonlinear differential equations for the CPHFF and the CSF as a generalization of the results obtained in (1). (3) In two- and three-dimensional cases we obtain static solutions to the field equations which take the form of vortex solutions similar to those of the Ginzburg-Pitaevskii equation.\textsuperscript{13} This paper is organized as follows. In § 2 a general theory is formulated to get equations of motion for $\phi_n$ as well as for the conventional two angles of rotation for spin vector from the classical Hamiltonian $H$. Equations of motion for the CSF are then introduced in close similarity to the semi-classical form of Boson-field equations. In § 3 explicit expressions for the equations of motion are obtained in the form of nonlinear differential-difference equations for the CPHFF and the corresponding CSF. These two types of equations are then transformed into nonlinear differential equations by using a continuum approximation. In § 4 1d case is studied to obtain moving domain-wall solutions to the differential equation for the CPHFF and the CSF. A brief study is made in § 5 to elucidate the existence of vortex solutions as static solutions to the differential equations in two- and three-dimensional cases. The last section is devoted to brief remarks on results obtained in this paper.

§ 2. Classical spin system and complex scalar field

Let us begin the discussion with a study of a classical spin system more general than that considered in § 1. Let us assume that the Hamiltonian $H$ of the system is written by a set $\{S_n\}$ of spin vectors, namely $H = H(\{S_n\})$. Conventionally, the components $S_n^a (a = x, y, z)$ of $S_n$ are parametrized by two angles of rotation

$$S_n^x = S \sin \theta_n \cos \varphi_n, \quad S_n^y = S \sin \theta_n \sin \varphi_n, \quad S_n^z = S \cos \theta_n. \quad (2\cdot1)$$

Equations of motion obeyed by $\theta_n$ and $\varphi_n$ are given by\textsuperscript{12,14}

$$\dot{\theta}_n = (1/S \sin \theta_n) \partial H/\partial \varphi_n, \quad \dot{\varphi}_n = -(1/S \sin \theta_n) \partial H/\partial \theta_n. \quad (2\cdot2)$$

Equations (2·2) imply that $S_n^x = m_n = S \cos \theta_n$ and $\varphi_n$ constitute canonically conjugate variables, namely

$$\dot{\varphi}_n = \partial H/\partial m_n, \quad \dot{m}_n = - \partial H/\partial \varphi_n. \quad (2\cdot3)$$

It is often convenient to introduce another variable to characterize the classical

\textsuperscript{1} We use units $\hbar = 1$ throughout this paper.
spin system under consideration. In (I) a stereographic variable\(^{(11)}\)

\[ \mu_n = \tan(\theta_n/2) \exp(i\varphi_n) \]  

(2.4)

was introduced to express the equations of motion in the form

\[ i\mu_n = [(1+|\mu_n|^2)/2S]\partial H/\partial\mu_n^* \quad \text{and c.c.} \]  

(2.5)

It has been shown that it is useful for the case of isotropic Heisenberg ferromagnets to study pseudo-particle or instanton solutions in two-dimensional case\(^{(15)}\) and to see a formal similarity of a static form of Eq. (2.5) to the Ernst equation in axisymmetric gravitational field in three-dimensional case.\(^{(16)}\) In this paper we take

\[ \psi_n = S_n^+ / S = \sin \theta_n \exp(i\varphi_n) = (2/S)^{1/2} \rho_n^{1/2} \exp(i\varphi_n) \]  

(2.6)

as a relevant field variable, in addition to the angle variables \(\theta_n\) and \(\varphi_n\), to elucidate a formal similarity of a complex scalar field yet to be introduced to a classical form of the Boson field. This is done by first rewriting the equations of motion in terms of \(\psi_n\) as

\[ i\dot\psi_n (1-|\psi_n|^2)^{-1/2} = (2/S) \partial H/\partial\psi_n^* \quad \text{and c.c.} \]  

(2.7a)

by the use of Eqs. (2.2) and (2.6) and then by taking the continuum approximation to get

\[ i\dot\psi (1-|\psi|^2)^{-1/2} = (2/S) \partial H/\partial\psi^* \quad \text{and c.c.} \]  

(2.7b)

The factor \((1-|\psi_n|^2)^{-1/2}\) or \((1-|\psi|^2)^{-1/2}\) on the left-hand sides is characteristic of the classical spin system. We now introduce in this connection a complex scalar field (CSF) governed by the equation

\[ i\dot\phi_n = (2/S) \partial H/\partial\phi_n^* \quad \text{and c.c.} \quad \text{for a discrete field,} \]  

(2.8a)

\[ i\dot\phi = (2/S) \partial H/\partial\phi^* \quad \text{and c.c.} \quad \text{for a continuous field.} \]  

(2.8b)

In terms of \(\theta_n\) and \(\varphi_n\) Eq. (2.8a) is rewritten as

\[ \dot\theta_n = (1/S \sin \theta_n \cos \theta_n) \partial H/\partial\theta_n, \quad \dot\varphi_n = -(1/S \sin \theta_n \cos \theta_n) \partial H/\partial\varphi_n. \]  

(2.9)

Equations (2.9) are different from the corresponding equations (2.2) only by the factor \(1/\cos \theta_n\) on the right-hand side. Here \(\rho_n = (S/2)|\phi_n|^2 = (S/2)\sin^2 \theta_n\) and \(\varphi_n\) constitute canonically conjugate variables, namely
\[
\rho_n = \frac{\partial H}{\partial \varphi_n} \quad \text{and} \quad \dot{\varphi}_n = -\frac{\partial H}{\partial \rho_n} \quad \text{for a discrete field,} \tag{2·10a}
\]
\[
\dot{\rho} = \frac{\partial H}{\partial \varphi} \quad \text{and} \quad \dot{\varphi} = -\frac{\partial H}{\partial \rho} \quad \text{for a continuous field,} \tag{2·10b}
\]
in contrast with the case of the classical spin system. Several remarks are in order on the CSF introduced here: (1) The equations of motion (2·8) have formally the same form as those for Bose systems in the semi-classical approximation.\(^{17}\) In Eqs. (2·8), however, nonlinearity inherent in the classical spin system is preserved. In particular, Eqs. (2·10b) are identical to the superfluid equations of motion discussed by Anderson,\(^{18}\) provided the quantity \(\rho\) defined here is identified with the condensate number density. (2) Static solutions to the equations for the original spin system and the CSF are obtained by one and the same equations.

\[
\frac{\partial H}{\partial \theta_n} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \varphi_n} = 0 \tag{2·11a}
\]

or

\[
\frac{\partial H}{\partial \varphi_n'} = 0 \quad \text{and c. c.} \tag{2·11b}
\]

These equations give the energy minimum states of the classical spin system, which also turn out to be approximate energy minimum states of the original quantal spin system obtainable by the use of a variational method.\(^{12}\)

\section*{§ 3. The CPHFF, its corresponding CSF and equations of motion}

In this section we illustrate the formal theory in § 2 to the case of the CPHFF and its corresponding CSF by giving explicit expressions for the equations of motion. We first write the Hamiltonian of the CPHFF in terms of \(\varphi_n\) as

\[
\frac{H}{\mathcal{S}} = \frac{1}{4} \sum_{n,m} [J_{s}(n, m)|\psi_m - \psi_n|^2 + J_{s'}(n, m)(1-|\psi_m|^2)^{1/2} - (1-|\psi_n|^2)^{1/2})^2 + \sum_n \nu(|\psi_n|^2), \tag{3·1}
\]

where

\[
J_{s}(n, m) = 2SJ(n, m), \quad J_{s'}(n, m) = 2SJ'(n, m) \tag{3·2}
\]

and

\[
\nu(|\psi_n|^2) = -(1/2) [J_s(0) - J_s'(0)] |\psi_n|^2 - \epsilon (1-|\psi_n|^2)^{1/2} \tag{3·3}
\]

\[
- \epsilon - (1/2) [J_s(0) - J_s'(0) - \epsilon]|\psi_n|^2 + (\epsilon/8)|\psi_n|^4 \equiv -\epsilon + \nu_0(\psi_n^2)
\]

\[
\text{for } |\psi_n|^2 < 1. \tag{3·3'}
\]

Here the quantities \(J_{s}(0)\) and \(J_{s'}(0)\) are the values of
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\[ J_s(0) = \sum_n J_s(n, m) \exp[i \mathbf{q} \cdot (m - n)] \]

and

\[ J'_s(0) = \sum_n J'_s(n, m) \exp[i \mathbf{q} \cdot (m - n)] \] (3.4)

at \( q = 0 \), respectively. It is understood that for the CPHFF or the PHFF the inequality

\[ J_s(0) > J'_s(0) \] (3.5)

holds. Equation (3.1) may also be considered as the Hamiltonian for a complex lattice field in which each atom, while sitting on the on-site potential \( \nu(|\varphi_n|^2) \), couples with its neighbouring atoms with harmonic force constants \((1/2)J_s(n, m)\) and anharmonic force constants \((1/2)J'_s(n, m)\), respectively. The on-site potential so introduced has a nonlinearity characterized by the factor \((1 - |\varphi_n|^2)^{1/2}\), which is similar to, but stronger than, that of the conventional \( \varphi^4 \) potential. Both of \( \nu(|\varphi_n|^2) \) and the \( \varphi^4 \)-type potential \( \nu_0(|\varphi_n|^2) \) defined by Eq. (3.3') are of double-well type provided the condition

\[ J_s(0) - J'_s(0) > 0 \] (3.6)

is satisfied. In the mean-field picture this is nothing but the condition for the existence of a nonvanishing value of \( \langle S_{n+} \rangle = S \langle \varphi_n \rangle \), i.e. for the appearance of symmetry-breaking state in the PHFF.\(^{9,12}\)

We give the remark in passing that the CSF for the CPHFF is similar to a real scalar field (RSF) introduced in another previous paper\(^9\) for the classical IMTF (CIMTF), the Hamiltonian of which is given by

\[ H = -\varepsilon \sum_n S_n^z - \sum_{n,m} J(n, m) S_n^z S_m^z. \] (3.7)

In this case

\[ S_n^z / S \equiv \rho_n \] (3.8)

is taken as a relevant field variable, in terms of which Eq. (3.7) is rewritten as

\[ H/S = -\varepsilon \sum_n [1 - \rho_n^2 - (\rho_n^2 / \varepsilon^2)]^{1/2} \]

\[ - [J_s(0)/2] \sum_n \rho_n^2 + (1/4) \sum_{n,m} J_s(n, m)(\rho_m - \rho_n)^2. \] (3.9)

The corresponding RSF is then defined by the Hamiltonian

\[ H/S = \sum_n (1/2 \varepsilon^2) \dot{\rho}_n^2 + \ddot{\rho}_n + (1/4) \sum_{n,m} J_s(n, m)(\rho_m - \rho_n)^2, \] (3.10)
where
\[ \hat{\nu}(\rho_n) = -\left[ J_s(0)/2 \right] \rho_n^2 - \epsilon (1 - \rho_n^2)^{1/2} \] (3.11)
is the on-site potential which is of the same form as \( \nu(|\phi|^2) \) defined by Eq. (3.3). As shown in the previous paper, \(^{19}\) Eq. (3.10) is the Hamiltonian of lattice dynamics in which an \( n \) atom in a lattice with atomic mass \( 1/e^2 \) and displacement \( \rho_n \) from its equilibrium position, while sitting on its on-site potential \( \hat{\nu}(\rho_n) \), couples with its neighbouring atoms with harmonic force constants \( (1/2) J_s(n, m) \). Here nontrivial solutions to Eqs. (2.11a) are given by
\[ \varphi = 0 \text{ or } \pi \quad \text{for all } n \] (3.12a)
and by the solution of the equation
\[ \sum_n J_s(n, m)(\rho_n - \rho_n) + [\partial \hat{\nu}(\rho_n)/\partial \rho_n] = 0. \] (3.12b)
It is also seen that CXYMTF has a closer similarity to the CIMTF.
By the use of a continuum approximation Eq. (3.1) reduces to
\[ H/S = \int \left[ \frac{dr}{a^d} \right] \left( \frac{\sigma^2}{2} J_s(0) \nabla \phi^2 + \eta \left\{ \nabla (1 - |\phi|^2)^{1/2} \right\}^2 + \nu(|\phi|^2) \right] \]
\[ = \int \left[ \frac{dr}{a^d} \right] H(r). \] (3.13)
Here the system under consideration has been assumed to constitute a cubic lattice with lattice constant \( a \) and dimensionality \( d \). Here we have also assumed that \( J_s(\mathbf{q}) \) and \( J_s'(\mathbf{q}) \) in the long wavelength limit take the form
\[ J_s(\mathbf{q}) = J_s(0) \left[ 1 - \left( a_1 q_x^2 + a_2 q_y^2 + a_3 q_z^2 \right) a^2 \right], \]
\[ J_s'(\mathbf{q}) = J_s'(0) \left[ 1 - \left( a_1 q_x^2 + a_2 q_y^2 + a_3 q_z^2 \right) a^2 \right] \] (3.14)
with
\[ \eta = J_s'(0)/J_s(0) \quad (\times 1) \] (3.15)
and rescaled the coordinate variables as
\[ x' = x/a^{1/2} \to x, \quad y' = y/a^{1/2} \to y, \quad z' = z/a^{1/2} \to z. \] (3.16)
In Eq. (3.14) \( q_x, q_y \) and \( q_z \) are the Cartesian components of the wave vector \( \mathbf{q} \) and the \( a' \)s are dimensionless constants. It is seen that for the CXYMTF (\( \eta = 0 \)) the Hamiltonian functional \( H(r) \) defined in Eq. (3.13) is identical in form to the Ginzburg-Pitaevskii free energy functional in the theory of superfluidity, \(^{13}\) provided the potential \( \nu(|\phi|^2) \) is replaced by the conventional \( \varphi^4 \) potential \( \nu_0(|\phi|^2) \).
and the condition (3·6) is satisfied at the same time. The similarity of the CPHFF and the CXYMTF in particular to the superfluid He⁴ is already apparent at this stage.

We are now in a position to obtain explicit expressions for the field equations for the CPHFF and its corresponding CSF. From Eqs. (1·1), (2·1) and (2·6) it is a straightforward matter to show that explicit expressions for Eqs. (2·2), (2·9), (2·7a) and (2·7a) are given by (see also Eqs. (3·8a) and (3·8b) in (1))

\begin{align}
\gamma_1 \theta_n &= \sum_m J_s(n, m) \sin \theta_m \sin(\varphi_n - \varphi_m), \\
\gamma_1 \sin \theta_n \varphi_n &= -\epsilon \sin \theta_n \\
&+ \sum_m [J_s(n, m) \cos \theta_n \sin \theta_m \cos(\varphi_n - \varphi_m) \\
&- J_s'(n, m) \sin \theta_n \cos \theta_m], \\
\gamma_2 i\varphi_n &= \epsilon \varphi_n (1 - |\psi_n|^2)^{-1/2} \\
&- \sum_m [J_s(n, m) \varphi_m - \varphi_n (1 - |\varphi_n|^2)^{-1/2} J_s'(n, m) (1 - |\varphi_m|^2)^{1/2}], \\
\end{align}

where the factors \( \gamma_1 \) and \( \gamma_2 \) are defined by

\begin{align}
\gamma_1 &= \begin{cases} 1 & \text{for the CPHFF} \\ \cos \theta_n & \text{for the CSF.} \end{cases} \\
\gamma_2 &= \begin{cases} 1 & \text{for the CPHFF} \\ (1 - |\psi_n|^2)^{-1/2} & \text{for the CSF.} \end{cases}
\end{align}

We first observe that here a nontrivial solution to Eqs. (2·11) corresponding to a spatially uniform spin alignment is given by

\begin{align}
\varphi_n \text{ is independent of } n \quad \text{or} \quad \varphi_n &= \varphi_0, \quad (3·20a) \\
\theta_n &= \cos^{-1}[\epsilon/\{J_s(0) - J_s'(0)]\to \theta_0 \quad (3·20b)
\end{align}

or

\begin{equation}
\varphi_n = \{1 - \epsilon^2/[J_s(0) - J_s'(0)]^2\}^{1/2} \exp(i\varphi_0) \equiv \psi_0. \quad (3·21)
\end{equation}

The symmetry-breaking state (3·20) or (3·21), which is realizable under the condition (3·6), is doubly degenerate with respect to the \( \theta \)'s except the degeneracy with respect to the \( \varphi \)'s. Equation (3·21) is an explicit expression for the first of Eqs. (1·2), in which the average is taken with respect to the coherent state,²⁰ namely

\begin{equation}
\varphi_n = \langle \theta_n \psi_n | \psi_n | \theta_n \varphi_n \rangle \quad (3·22)
\end{equation}

with

\begin{equation}
|\theta_n \varphi_n \rangle = \exp(\xi_n S_n^- - \xi_n^* S_n^+)|S_n \rangle, \quad \xi_n = (\theta_n/2) \exp(i\varphi_n), \quad (3·23)
\end{equation}

where \(|S_n \rangle\) is the normalized "spin up" state. The energy eigenvalue of the
symmetry-breaking state giving the ground state of the CPHFF or the PHFF obtained by using the variational calculation is lower than that of the state \( \varphi_n = \text{const} \) and \( \theta_n = 0 \) or \( \psi_n = 0 \) corresponding to trivial solutions of Eqs (2.11), if the condition (3.6) is satisfied.

Equations (3.17) and (3.18) in the continuum approximation take the form

\[
\begin{align*}
\gamma_1 \frac{\partial \theta}{\partial t} & = -J_s(0) a^2 (2 \cos \theta \nabla \theta \cdot \nabla \varphi + \sin \theta \Delta \varphi), \\
\gamma_1 \sin \theta \frac{\partial \varphi}{\partial t} & = -\epsilon \sin \theta + J_s(0)(1-\eta) \sin \theta \cos \theta \\
& + J_s(0) a^2[(\cos^2 \theta + \eta \sin^2 \theta) \Delta \theta \\
& - \sin \theta \cos \theta \{ (\nabla \varphi)^2 + (1-\eta)(\nabla \theta)^2 \}], \\
\gamma_2 i \dot{\psi} & = \epsilon \psi (1-|\psi|^2)^{1/2} - J_s(0)(1-\eta) \phi \\
& - J_s(0) a^2 [\Delta \psi - \eta \phi (1-|\phi|^2)^{1/2}](1-|\psi|^2)^{1/2}. 
\end{align*}
\]

Equation (3.24a) is rewritten in the form of conservation law as

\[
\begin{align*}
(\partial / \partial t) \cos \theta & = J_s(0) a^2 \nabla \cdot (\sin^2 \theta \nabla \varphi) & \text{for the CPHFF,} \\
(1/2)(\partial / \partial t) \sin^2 \theta & = -J_s(0) a^2 \nabla \cdot (\sin^2 \theta \nabla \varphi) & \text{for the CSF.}
\end{align*}
\]

For the CXYMTF Eq. (3.25) reduces to

\[
\gamma_2 i \dot{\psi} = \psi (1-|\psi|^2)^{-1/2} - J_s(0) \psi - J_s(0) a^2 \Delta \psi .
\]

The corresponding CSF equation which is obtained by putting \( \gamma_2 = 1 \) is a spin-version of the Gross-Pitaevskii equation or the nonlinear Schrödinger equation:\textsuperscript{13}

\[
i \dot{\psi} + [J_s(0) - \epsilon] \psi + J_s(0) a^2 \Delta \psi - (\epsilon/2)|\psi|^2 \psi = 0 .
\]

Equation (3.28) itself is obtained by replacing \( \psi(\epsilon |\psi|^2) \) by \( \psi_0(\epsilon |\psi|^2) \).

A little more insight into the properties of Eqs. (3.24) ~ (3.26) can be gained by introducing a vector function

\[
v = 2J_s(0) a^2 \nabla \varphi ,
\]

analogous to the superfluid velocity \( v_s \) in the theory of liquid \( \text{He}^4 \). We are then interested in rewriting Eqs. (3.24) or Eq. (3.25) in forms somewhat similar to equations in hydrodynamic theory. This is done immediately for Eqs. (3.26) as follows:

\[
\begin{align*}
\partial m / \partial t & = \nabla \cdot (\rho v) & \text{for the CPHFF,} \\
\partial \rho / \partial t + \nabla \cdot (\rho v) & = 0 & \text{for the CSF.}
\end{align*}
\]
It is to be noted that Eq. (3·26) or Eq. (3·30) holds for the isotropic Heisenberg ferromagnets as well as for the CPHFF and the CXYMTF. An equation obeyed by the velocity function, however, generally takes different form for different model of classical spin system. Here we present an explicit form for the equation only for the CXYMTF. Combining Eqs. (3·24b) and (3·29), we then obtain

$$\frac{\partial v}{\partial t} = 2J_s(0)\sigma^2 \nabla (\partial H/\partial m) \quad \text{for the CPHFF},$$  
$$\left( \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) \nu = \nabla \Pi \quad \text{for the CSF},$$  

where

$$\Pi = 2J_s(0)^2 \sigma^2 \left( \frac{\Delta (\rho^{1/2})/\rho^{1/2}}{2\nu} \right) - 2\varepsilon J_s(0)\sigma^2 (1 - \rho)^{1/2}. \quad (3·32)$$

Equations (3·30a) and (3·31a) are coincident with the result of Halperin and Hohenberg, where $\rho$ is identified as the superfluid density. On the other hand, Eqs. (3·30b) and (3·31b) have just the form of hydrodynamic equations, the former and the latter corresponding to the equation of continuity and the Bernoulli equation, respectively. It is seen that the CSF has a more direct similarity to superfluid He$^4$ than the original CPHFF itself.

§ 4. One-soliton solutions in one-dimensional case

Here we are concerned with one-dimensional case in which spatial variation of $\theta$ and $\phi$ or $\psi$ is taken to be in the direction of the $x$-axis. We seek particular solutions of the form

$$\theta = \theta(x - vt) = \theta(\xi), \quad \phi = \Omega t + \phi(x - vt) = \Omega t + \phi(\xi), \quad (4·1)$$

where $v$ and $\Omega$ are constants representing linear and angular velocities, respectively. By the use of Eqs. (3·26), Eqs. (3·24) are then integrated once to give

(i) for the CPHFF

$$a \phi_t \sin^2 \theta + v ' \cos \theta = C_1', \quad (4·2a)$$
$$a^2 [1 - (1 - \eta)\sin^2 \theta] \phi_t^2 + \sin^2 \theta + \left[ \left( C_1' - v' \cos \theta \right) \phi_t^2 / \sin^2 \theta \right] + \left[ \phi_t + \left( \Omega + \frac{1}{\lambda} \right) \cos \theta - 2C_2' \right] = 0. \quad (4·2b)$$

(ii) for the CSF

$$a \phi_t \sin^2 \theta - (v' / 2) \sin^2 \theta = C_1', \quad (4·3a)$$
$$a^2 [1 - (1 - \eta)\sin^2 \theta] \phi_t^2 + (1 - \eta - \Omega ' ) \sin^2 \theta \left[ C_1' + (v' / 2) \sin^2 \theta \phi_t^2 + \left( \phi_t + \left( \Omega' \phi_t^2 / \sin^2 \theta \right) + (2/\lambda) \cos \theta - 2C_2' \right] = 0. \quad (4·3b)$$
In the above equations we have put 

\[ \varphi = d\varphi/dz, \quad \theta = d\theta/dz, \quad \Omega = \Omega/J_s(0), \]
\[ v' = v/J_s(0)a, \quad C'_1 = C_1/J_s(0)a, \quad C'_2 = C_2/J_s(0), \quad \lambda = J_s(0)/\epsilon \]

(4·4)

to express the result of calculations in dimensionless form. Equations (4·2) are identical to Eqs. (5·4a) and (5·5) in (1). There only static domain-wall solutions were obtained by taking \( C'_1 = v' = 0 \). In this paper it is shown that moving domain-wall solutions to Eqs. (4·2) and (4·3) can be obtained as a generalization of the previous result, under the following boundary condition:

\[ \theta = \pm \theta_0 \text{ for } x = \pm \infty \text{ or } \theta = \mp \theta_0 \text{ for } x = \pm \infty \]

(4·5)
in the static limit \( v \to 0 \) and \( \Omega \to 0 \). The solutions connect the regions ordered with spins in the \( \pm \theta_0 \) wells to the regions ordered with spins in the \( \mp \theta_0 \) wells. This can be done by first re-expressing Eqs. (4·2b) and (4·3b) in the form

\[ \frac{\partial^2 \Theta}{\partial x^2} + V(\Theta) = 0 \]

(4·6)

with

\[ V(\Theta) = -\left(\cos \Theta - \cos \Theta_0\right)^2/\left(\eta' + \cos^2 \Theta\right) \]

(4·7)

by taking the integral constants \( C'_1 \) and \( C'_2 \) as given below. In the above equations the quantity \( \Theta \) is defined by

\[ \cos \theta = A \cos \Theta. \]

(4·8)

Writing in the form (4·6), we can get an insight into the global behavior of the solutions for \( \Theta \) or \( \theta \) by identifying \( \Theta \) and \( \xi \) as a particle position and time in Newtonian mechanics. Explicit expressions for the quantities \( \tilde{a}, A, \Theta_0, \eta' \) and the integral constants are given by

(i) for the CPHFF

\[ \tilde{a} = a, \quad A = \left[1 - \left(v^2/(1 - \eta)\right)\right]^{1/2}, \]

(4·9a)

\[ A \cos \Theta_0 = \left[(1/\lambda) + \Omega'/\left(1 - \eta\right)\right], \]

(4·10a)

\[ \eta' = \eta/\left(1 - \eta - v^2\right), \]

(4·11a)

\[ C'_1 = v'[\left(1/\lambda\right) + \Omega']/\left(1 - \eta\right), \]

(4·12a)

\[ C'_2 = \left[(1 - \eta)/2\right]\left[1 + \left(1/\lambda\right) + \Omega'\right]/\left(1 - \eta \eta'\right). \]

(4·13a)

(ii) for the CSF

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\* In the 1d case we will use a simplified notation to abbreviate \( \partial A/\partial x \) and \( \partial^2 A/\partial x^2 \) as \( A_1 \) and \( A_{11} \), respectively.
A schematic feature of $V(\theta)$ is shown in Fig. 1. The existence of $2\Theta_0$-kink solutions is easily seen, though the integration in the formal solutions of Eq. (4·6),

$$\int [-V(\theta)]^{-1/2} d\theta = \pm (x - vt)/\tilde{a},$$

cannot be done analytically. Here the plus and minus signs on the right-hand side correspond to kink- and anti-kink solutions, respectively. By inserting Eqs. (4·7) and (4·14) back into Eqs. (4·2a) and (4·3a), solutions for $\varphi$ can be obtained as follows:

$$\varphi = \Theta t + \left\{ \begin{array}{ll}
\frac{v' A}{a} \int \frac{\cos \Theta_0 - \cos \theta(\xi)}{1 - A^2 \sin^2 \theta(\xi)} d\xi + \varphi_0 & \text{for the CPHFF} \\
\frac{v'}{2a}(x - vt) + \varphi_0 & \text{for the CSF},
\end{array} \right.$$
\[ \int [-V(\theta)]^{-1/2} d\theta = \pm (x/a) \quad \text{and} \quad \varphi = \varphi_0 \quad (4\cdot16) \]

with
\[ \bar{a} = a, \quad A = 1, \quad \cos \Theta_0 = 1/\lambda(1-\eta) = \cos \theta_0, \]
\[ \eta' = \eta/(1-\eta), \quad C_1' = 0, \quad C_2' = [(1-\eta)/2][1+1/\lambda^2(1-\eta)^2]. \quad (4\cdot17) \]

Equations (4·16) and (4·17) are one static domain-wall solutions obtained in (1). For the CXYMTF the potential function \( V(\Theta) \) takes the form
\[ V(\Theta) = -(\cos \Theta - \cos \Theta_0)^2/\cos^2 \Theta. \quad (4\cdot18) \]

Equation (4·14) can then be integrated to give
\[ \Theta + 2 \cot \Theta_0 \tanh^{-1}[\cot(\Theta_0/2)\tan(\Theta/2)] = \pm (x-vt)/a, \quad (4\cdot19) \]

which, together with Eqs. (4·15) with \( \eta = 0 \), constitutes one-soliton solutions to Eqs. (3·24). Here Eqs. (4·9)~(4·13) reduce to:

(i) for the CXYMTF
\[ \bar{a} = a, \quad A = (1-v'^2)^{1/2}, \quad A \cos \Theta_0 = (1/\lambda) + O', \quad \eta' = 0, \quad (4\cdot20a) \]
\[ C_1' = v'[(1/\lambda) + O'], \quad C_2' = (1/2)[1+(1/\lambda) + O]^\gamma]. \quad (4\cdot21a) \]

(ii) for the CSF
\[ \bar{a} = a/[1+(v'/2)^2-O'^2]^{1/2}, \quad A = 1, \]
\[ \cos \Theta_0 = (1/\lambda)[1-O'-(v'/2)^2], \quad \eta' = 0, \quad (4\cdot20b) \]
\[ C_1' = 0, \quad C_2' = (1/2)[1-O'-(v'/2)^2+1/\lambda^2(1-O'-(v'/2)^2)]. \quad (4\cdot21b) \]

In the case of the CXYMTF the solution for \( \varphi \) is obtained by inserting the solution of Eq. (4·19) into Eq. (4·15a). Here the integral still cannot be done analytically, since the solution for \( \theta \) has not been given explicitly as a function of \( x-vt \). It is seen that the solution for \( \theta \) is of the same form as that in the case of the CIMTF.21

The solution for \( \psi \) can be obtained either by inserting the above result into Eq. (2·6) or by directly solving Eq. (3·25). Before closing this section we give the remark that Eqs. (3·27) and (3·28) for the CSF in the 1d case are given by
\[ i\psi_t + \psi + \psi_{xx} - (1/\lambda)\psi(1-|\psi|^2)^{-1/2} = 0, \quad (4\cdot22) \]
\[ i\psi_t + (1-(1/\lambda))\psi + \psi_{xx} - (1/2\lambda)|\psi|^2\psi = 0, \quad (4\cdot23) \]

where we have used reduced variables \( t' = f_s(0)t-i, \quad x' = x/a-x \). One-soliton (dark soliton) solution to Eq. (4·23) was first obtained by Tsuzuki to study the solution of the Gross-Pitaevskii equation22 and later by Hasegawa and Tappert in...
their study of nonlinear optical pulses in dispersive dielectric fibers. Hirota obtained exact $N$-envelope soliton solutions by the use of his own method (the Hirota method).

§ 5. Vortex solutions in higher-dimensional cases

In studying nonlinear excitations and soliton-like solutions in higher-dimensional cases, we limit our discussion to static solutions to Eqs. (3.24) and (3.25). We observe that a particular solution of Eq. (3.24a) with $\theta = 0$ is the solution of the equation

$$\nabla \theta \cdot \nabla \varphi = 0 \quad \text{or} \quad \Delta \varphi = 0.$$  \hfill (5.1)

Let us introduce the cylindrical coordinate $r, \phi, z$ and the polar coordinate $r, \psi$ for three- and two-dimensional cases, respectively. Then, the most typical example of solutions which satisfy Eq. (5.1) is a single vortex-like solution for which $\theta$ and $\varphi$ or $\psi$ take the form

$$\theta = \theta(r) \quad \text{and} \quad \varphi = q\phi \quad \text{with} \quad q = \pm 1, \pm 2, \ldots,$$

$$\psi = \sin \theta(r) \exp(iq\phi) = F(r) \exp(iq\phi).$$  \hfill (5.2'')

The velocity function $v$ is then given by

$$v = (2J_s(0)a^2q/r)\hat{\phi} \quad \text{or} \quad \nabla \varphi = (q/r)\hat{\phi},$$  \hfill (5.3)

where $\hat{\phi}$ is a unit vector in the direction of $\phi$. It is seen that the static solutions are neatly separated into two parts, the solution of the Laplace equation and the solution for $\theta(r)$ or $F(r)$. Here we illustrate this situation for Eq. (3.25), the static form of which is given by

$$a^2[\Delta \psi - \eta \psi(1-|\psi|^2)^{-1/2} \Delta (1-|\psi|^2)^{1/2} + (1-\eta)\psi$$

$$- (1/\lambda)\psi(1-|\psi|^2)^{-1/2} = 0.$$  \hfill (5.4)

It is of interest to note in passing that the equation

$$a^2\Delta \psi + \psi - (1/\lambda)\psi(1-|\psi|^2)^{-1/2} = 0,$$  \hfill (5.5)

which holds for the CXYMTF, is entirely identical to the static equation which exists for the CIMTF, namely to the differential equation obtainable from Eq. (3.12b) by using the continuum approximation, provided $\psi$ is taken to be real. Putting Eq. (5.2'') into Eq. (5.4), we obtain
For arbitrary value of $\eta$, Eq. (5.6) cannot be solved analytically. Here we obtain an asymptotic form of the solution satisfying the boundary condition:

$$F(r) = \begin{cases} 
0 & \text{for } r = 0 \\
[1 - (1/\sqrt{\eta(1-\eta)})]^{1/2} & \text{for } r = \infty.
\end{cases} \quad (5.8a)$$

Equation (5.8b) corresponds to the symmetry-breaking solution (3.21). A straightforward calculation leads to the result:

$$F(r) = \left[ 1 - \frac{1}{\sqrt{\eta(1-\eta)}} \right]^{1/2} - \frac{q^2}{\sqrt{\eta(1-\eta)^2 - 1}} \frac{1}{r^{1/2}} \quad (5.9a)$$

where $C$ is a constant. The solution (5.9) is similar to the result obtained by Ginzburg and Pitaevskii.\textsuperscript{13} Equations (5.2) and (5.9) represent a single-vortex solution. The existence of multi-vortex solutions is yet to be studied.

In the case of the classical isotropic Heisenberg ferromagnet ($\eta = 1$) the solution can be obtained analytically. Here it is convenient to treat Eq. (3.24b). Using Eqs. (5.2) and (5.3), we then obtain

$$\frac{d}{dr} \left( \tilde{r} \frac{d\theta}{d\tilde{r}} \right) - (q^2/2 \tilde{r}) \sin 2\theta = 0. \quad (5.10)$$

A particular solution of this equation satisfying the boundary condition

$$\theta = 0 \quad \text{for } r = 0 \quad \text{and} \quad \theta = \pi/2 \quad \text{for } r = \infty \quad (5.11)$$

is given by

$$\theta = 2 \tan^{-1} \tilde{r}^{\eta/2} \quad \text{with} \quad \phi = \varphi \phi. \quad (5.12)$$

In two-dimensional case the solution so obtained is coincident with the one pseudo-particle solution obtained by Belavin and Polyakov.\textsuperscript{15} Vortex solutions of similar nature for the two-dimensional classical planar Heisenberg ferromagnet without an external field were also obtained by Hikami and Tsuneto under the boundary condition (4.11).\textsuperscript{25}
§ 6. Concluding remarks

We have shown that nonlinear excitations in the CPHFF are rich in the sense that the field equations in the continuum approximation admit domain-wall-type soliton solutions in 1d case and vortex-type static soliton solutions in two- and three-dimensional cases.* In studying the problem we have made a correspondence between the CPHFF and the CSF to elucidate the physical picture of the CPHFF itself by taking \((S_n + iS_n^*) / S\) as a relevant field variable. The CSF so introduced is shown to be similar to the complex \(\varphi^4\) field or the superfluid He\(^4\), the similarity being closer for the case of the CXYMTF. The situation here is somewhat analogous to the case of the CMTF, in which the corresponding RSF is shown to be similar to the “real” \(\varphi^4\) field appearing in, say, the problem of structural phase transition in solid state physics.\(^{19}\) The CSF or the CPHFF as well as the RSF here are, however, different in an important respect from the conventional \(\varphi^4\) field in that the potential \(\psi(|\phi|^2)\) defined by Eq. (3·3) has, though of double-well type under the condition \(\lambda > 1\), saturable nonlinearity.

Much remains to be done for the CPHFF or the PHFF both from mathematical and physical sides. One thing that readily comes to mind from mathematical viewpoint is an examination of the existence of two- or multi-soliton solutions to Eq. (4·22), which is closest to the conventional 1d nonlinear Schrödinger equation\(^{27}\) among the differential equations discussed in this paper. We note in this connection that the nonlinear differential equation

\[
i\psi_t + \phi_{xx} - \phi(1 - |\phi|^2)^{-1/2}(1 - |\phi|^2)^{1/2} = 0
\]

(6·1)

having the same nonlinearity as that of Eqs. (4·22) is equivalent to the nonlinear Schrödinger equation, since it is the nonlinear differential equation in terms of \(\phi\) for the CSF for the 1d isotropic continuous Heisenberg ferromagnet, namely it is derived from the 1d form of Eq. (3·25) with \(\gamma_2 = 1\), \(\epsilon = 0\) and \(\eta = 1\). Recently, a complete integrability of several nonlinear differential equations having different form of saturable nonlinearity has been studied by Wadati and others.\(^{28}\) Another mathematical problem here is the existence of static multi-vortex solutions or time-dependent solutions to Eqs. (3·24) or (3·25) in two- and three-dimensional cases.

From physical side implications of the results obtained here to the superfluid He\(^4\) is the problem yet to be studied. Another important problem here is to examine interrelationship between exact solutions which have been known to exist in several 1d quantal spin systems by using the Bethe ansatz or the Jordan-Wigner

* Here we use the terminology of “soliton” in a less restrictive sense in accordance with advocation by Lee (see Ref. 26)).
transformation and soliton solutions in the corresponding classical spin system. This kind of problem has already been studied by several workers. Thacker et al. and others\textsuperscript{29--31} have made an attempt to elucidate the relation between the exact solution to the quantal nonlinear Schrödinger equation and the corresponding classical one by using the inverse scattering method. Faddeev and his collaborators have studied the quantal inverse scattering method for the nonlinear Schrödinger equation, the sine-Gordon equation and the XYZ spin model.\textsuperscript{32}

References


See also, C. P. Enz, Rev. Mod. Phys. 46 (1974), 705.


11) A. A. Belavin and A. M. Polyakov, JETP Letters 22 (1975), 245.

12) F. J. Ernst, Phys. Rev. 167 (1968), 1175.


