Decays of Pseudoscalar Mesons

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Decays of pseudoscalar mesons are studied by paying attention to the property that $|\psi_v(0)|^2$ is proportional to $M_v^2$, where $M_v$ is the mass of vector meson and $\psi_v(0)$ is the origin value of wave function for the vector meson. It is shown that (i) the data for the $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$ processes can be explained well and (ii) for the heavy pseudoscalar mesons, $\Gamma(\eta \rightarrow \gamma\gamma)/e_\phi^2$ is roughly the same, of the order of 30 keV, for the $\eta$, $\eta'$ and $\eta''$. By making use of the above result, the hadronic decay width of the $\eta$ is estimated at about 5 MeV. The $\eta$ mass is discussed on the basis of the decay width for $\Upsilon \rightarrow \gamma\eta$, and some predictions for decays of pseudoscalar and vector mesons in the $t\bar{t}$ system are made.

§ 1. Introduction

On the basis of the quark model, Royen and Weisskopf\(^1\) studied the hadron decay processes and gave both a formula

$$\Gamma(V(Q\bar{Q}) \rightarrow e^+e^-) = \frac{16\pi e^2}{M_V^3}|\psi_v(0)|^2 e_\phi^2$$

for the leptonic decay width of vector meson and an expression for $2\gamma$-decay of pseudoscalar meson by assuming the decay process shown in Fig. 1, for example,

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \frac{e^2}{16\pi} \left[ \frac{\sqrt{3}}{3} M_\pi^{-3/2} \psi_\pi(0) + \frac{\sqrt{3}}{3} M_\eta^{-3/2} \psi_\eta(0) \right]^2 \mu_p^2 M_\pi^3,$$

where $V(Q\bar{Q})$ is the lowest vector meson in the $Q\bar{Q}$ system, $e_\phi$ is the quark charge, $M_V$ is the mass of the $V(Q\bar{Q})$, $\psi_V(0)$ is the value of wave function at the origin for the vector meson and $\mu_p$ is the total proton magnetic moment: $\mu_p = 2.79 (e/2M_p)$.

In their estimation of the decay widths of mesons, they assumed

Fig. 1. 2\(\gamma\)-decay process of pseudoscalar meson.
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\[ |\psi_v(0)|^2 = (M_v/M_\pi)|\psi_\pi(0)|^2 \]

with

\[ |\psi_\pi(0)|^2 = (1/2)M_\pi^3 \]  \hspace{1cm} (3)

This assumption seems to be natural. In the case of the \( V(Q\bar{Q}) \) mesons, however, the proportionality of \(|\psi_v(0)|^2\) with the meson mass does not hold as is mentioned below. According to experimental data, the value of \( \Gamma(V(Q\bar{Q})\rightarrow e^+e^-)/e_\phi^2 \) is roughly the same for the \( \rho, \phi, \phi/J \) and \( \tau \), that is, \[ \Gamma(V(Q\bar{Q})\rightarrow e^+e^-)/e_\phi^2 \approx 11 \text{ keV} \].  \hspace{1cm} (4)

From Eqs. (1) and (4) it follows that

\[ |\psi_v(0)|^2 \propto M_v^2. \]  \hspace{1cm} (5)

Thus \(|\psi_v(0)|^2\) is not proportional to \( M_v \) but to \( M_v^2 \), so far as the \( V(Q\bar{Q}) \) mesons are concerned. This is consistent with the result obtained by Hirata et al. \[ (2) \]

In view of the above fact we think it is necessary to re-examine the decay widths of pseudoscalar mesons not only in the light quark region but also in the heavy quark region, because the heavier the mass of vector meson is, the larger the effect of the property (5) becomes. It is shown that the experimental results for the decay widths of the \( \pi^\pm \rightarrow 2\gamma, \eta \rightarrow 2\gamma \) and \( \eta' \rightarrow 2\gamma \) processes can be explained well by using the relation (5) and introducing the reasonable mixing angle for the \( \eta-\eta' \).

For the heavy mesons we wish to emphasize the following points:

(i) If the \( V(Q\bar{Q}) \) is heavier than a pseudoscalar meson \( \eta_0 \) in the \( Q-\bar{Q} \) system, needless to say, \( \eta_0 \rightarrow 2\gamma \) is one of the most important decay processes and \( \Gamma(\eta_0 \rightarrow 2\gamma)/e_\phi^4 \) does not depend strongly on flavor, that is,

\[ \Gamma(\eta_0 \rightarrow 2\gamma)/e_\phi^4 \approx 30 \text{ keV}. \]  \hspace{1cm} (6)

(ii) If the ratio \( \Gamma(\eta_0 \rightarrow \text{hadrons})/\Gamma(\eta_0 \rightarrow 2\gamma) \) is given by \( (2/9) (a_s(m_\phi)/a e_\phi^2) \), the above result (i) leads to the conclusion that \( \Gamma(\eta_0 \rightarrow \text{hadrons}) \) is proportional to \([\log(m_\phi/m_0)]^{-2}\), because \( a_s(m_\phi) \propto 1/\log(m_\phi/m_0) \) as is seen from the concept of asymptotic freedom, where \( m_0 \) is a parameter.

(iii) As in the case of the process \( \psi/J \rightarrow \gamma\eta_0 \), the decay width \( \Gamma \) or branching ratio \( B \) for the decay \( \Gamma \rightarrow \gamma\eta_0 \) depends strongly on the mass of the \( \eta_0 \). On the basis of flavor independence of the hadron mass splittings, the \( \eta_0 \) mass is estimated at about 9.34 GeV and a relation

\[ M_v^2 \Gamma(V(Q\bar{Q})\rightarrow \gamma\eta_0)/e_\phi^2 \approx \text{const} \]

is pointed out.

Moreover the decay widths for \( \eta \rightarrow 2\gamma \) and \( \phi(f^0) \rightarrow \gamma\eta \) are predicted in the
last section.

§ 2. Two-gamma decays of low-lying pseudoscalar mesons

In this section, the decay widths for $2\gamma$-decays of the low-lying pseudoscalar mesons are calculated along the same line with the study in Ref. 1), except for the $M_\nu$-dependence of $|\varphi_\nu(0)|^2$. We adopt here the following interactions:

\[ H_{\nu\gamma} = e f_{\nu\gamma} \gamma_\mu A_\mu \]  

(7)

for vector-photon coupling, and

\[ H_{\nu\pi} = 2 \mu_{\nu\pi} e a_\gamma d_\mu A_\mu(x) \partial_\tau V(x) P(x) \]  

(8)

for vector-pseudoscalar-photon coupling, where we use the same notation as in Ref. 1). The vector and pseudoscalar states ($|V\rangle$ and $|P\rangle$) in the quark-antiquark system can be represented by the following $SU(6)$ wave functions:

\[ |V(s_z=0)\rangle = (1/\sqrt{2}) \sum_i a_i \left[ |\bar{Q}_i(\uparrow)Q_i(\downarrow)\rangle + |\bar{Q}_i(\downarrow)Q_i(\uparrow)\rangle \right] \]

\[ |P\rangle = (1/\sqrt{2}) \sum_i a_i \left[ |\bar{Q}_i(\uparrow)Q_i(\downarrow)\rangle - |\bar{Q}_i(\downarrow)Q_i(\uparrow)\rangle \right] \]

The expressions for the $f_{\nu\gamma}$ and $\mu_{\nu\pi}$ are given by

\[ f_{\nu\gamma} = 2 \sum_i a_i^\gamma e_\gamma^i M_\nu^{-3/2} \varphi_\nu(0) \]  

(9)

and

\[ \mu_{\nu\pi} = e \langle P| \sum_i (e_\gamma^i/2m_\gamma^i)\sigma_z |V(s_z=0)\rangle \]

\[ = -2e \sum_i a_i^{\gamma'} a_i^{\gamma'} \frac{e_{\gamma'}^{i}}{2m_{\gamma'}^{i}}, \]  

(10)

where $e_\gamma^i$ and $m_\gamma^i$ are, respectively, the charge and effective mass of the quark $i$, the sum over $i$ goes over the quarks in the $V(P)$ meson and $a_i^{\gamma'}(a_i^{\gamma'})$ is the Clebsch-Gordan coefficient of the quark $i$ in the $V(P)$ meson.

The matrix element for the $P\rightarrow 2\gamma$ process shown in Fig. 1 can easily be calculated by using the expressions (7) and (8). Thus we get

\[ \Gamma(P\rightarrow 2\gamma) = 3(e^2/16\pi) \left| \sum_{\nu\gamma} \mu_{\nu\gamma} f_{\nu\gamma} \right|^2 M_{\rho\pi}^3, \]  

(11)

where $M_{\rho\pi}$ is the mass of the pseudoscalar meson ($P$) and the factor 3 on the right-hand side of Eq. (11) comes from the color freedom.\footnote{For the reason why the $\mu_\nu^2$ is contained in Eq. (2), see Eq. (21).} In the case $\pi^0\rightarrow 2\gamma$,
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for example, Eq. (11) is reduced to the form of Eq. (2) apart from the color factor. It is conventional to describe the \( \eta \) and \( \eta' \) in terms of an octet-singlet angle \( \theta \), that is,*

\[
\eta = \eta_s \cos \theta + \eta_t \sin \theta,
\]
\[
\eta' = -\eta_s \sin \theta + \eta_t \cos \theta,
\]
where
\[
\eta_t = (1/\sqrt{3})(u\bar{u} + d\bar{d} + s\bar{s}),
\]
\[
\eta_s = (1/\sqrt{6})(u\bar{u} + d\bar{d} - 2s\bar{s}).
\]

Then, the decay widths for the \( \eta \to 2\gamma \) and \( \eta' \to 2\gamma \) processes are expressed in the following forms:

\[
\Gamma(\eta \to 2\gamma) = 3(e^2/16\pi) \left| \sqrt{\frac{2}{3}}(\cos \theta + \sqrt{2}\sin \theta) \times \left( M_{\eta}^{-3/2}\psi_\omega(0) + \frac{1}{9} M_{\omega}^{-3/2}\psi_\omega(0) \right) \right|^2 \mu_\eta^2 M_\eta^3
\]

and

\[
\Gamma(\eta' \to 2\gamma) = 3(e^2/16\pi) \left| \sqrt{\frac{2}{3}}(\sin \theta - \sqrt{2}\cos \theta) \times \left( M_{\eta'}^{-3/2}\psi_\omega(0) + \frac{1}{9} M_{\omega}^{-3/2}\psi_\omega(0) \right) \right|^2 \mu_{\eta'}^2 M_{\eta'}^3.
\]

Hereafter we refer to as Cases (I) and (II) when the \( |\psi_\nu(0)|^2 \) is proportional to \( M_\nu \) and \( M_\nu^2 \), respectively. Strictly speaking,

\[
|\psi_\nu(0)|^2 = \begin{cases} 
  k_1(M_\nu/M_\pi)M_\pi^3 & \text{for Case (I)}, \\
  k_2(M_\nu/M_\pi)^2M_\pi^3 & \text{for Case (II)}.
\end{cases}
\]

As was mentioned before, Royen and Weissskopf\textsuperscript{11} considered Case (I) with \( k_1 = 1/2 \). If the color factor 3 is taken into account, however, the \( k_1 \) must be changed from 1/2 to 1/6. In our Case (II), the parameter \( k_2 \) is determined so that the experimental result (4) may be reproduced. As the result we obtain

* In this section, the decay widths for \( \eta \to 2\gamma \), \( \eta \to 2\gamma \) and \( \eta' \to 2\gamma \) are estimated with the assumption that \( m_\eta = m_\eta \equiv m_\eta \). In the next section, needless to say, we take into account the difference between \( m_\eta(M_\pi) \) and \( m_\eta \) (see Eq. (25)).
Table I. Values of the coefficients $A$, $B$, $C$ and $D$ in the expression of $\Gamma(P \rightarrow 2\gamma) = A + B \cos 2(\alpha - \beta) + C \cos 2(\beta - \gamma) + D \cos 2(\beta - \gamma - \alpha)$.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$\pi^0 \rightarrow 2\gamma$ (eV)</th>
<th>$\eta \rightarrow 2\gamma$ (keV)</th>
<th>$\eta' \rightarrow 2\gamma$ (keV)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case (I)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A$</td>
<td>7.34</td>
<td>1.12</td>
<td>5.52</td>
</tr>
<tr>
<td>$B$</td>
<td>7.34</td>
<td>0.23</td>
<td>1.12</td>
</tr>
<tr>
<td>$C$</td>
<td>0</td>
<td>-0.05</td>
<td>0.28</td>
</tr>
<tr>
<td>$D$</td>
<td>0</td>
<td>-0.48</td>
<td>2.55</td>
</tr>
<tr>
<td><strong>Case (II)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A$</td>
<td>7.33</td>
<td>1.13</td>
<td>5.57</td>
</tr>
<tr>
<td>$B$</td>
<td>7.33</td>
<td>0.23</td>
<td>1.12</td>
</tr>
<tr>
<td>$C$</td>
<td>0</td>
<td>-0.06</td>
<td>0.32</td>
</tr>
<tr>
<td>$D$</td>
<td>0</td>
<td>-0.55</td>
<td>2.90</td>
</tr>
</tbody>
</table>

$k_2 = 0.03$.

When the wave functions of vector mesons are expressed by

$$\phi_\mu(0) = |\phi_\mu(0)| \exp(2i\alpha), \quad \phi_\nu(0) = |\phi_\nu(0)| \exp(2i\beta)$$

the expression of $\Gamma(P \rightarrow 2\gamma)$ is reduced to the following form:

$$\Gamma(P \rightarrow 2\gamma) = A + B \cos 2(\alpha - \beta) + C \cos 2(\beta - \gamma) + D \cos 2(\beta - \gamma - \alpha).$$

In Table I are shown the values of $A$, $B$, $C$ and $D$, when

$$k_1 = 1/6, \quad k_2 = 0.03 \quad \text{and} \quad \theta = 11^\circ.$$  

The predictions for $\Gamma(\pi^0 \rightarrow 2\gamma)$, $\Gamma(\eta \rightarrow 2\gamma)$ and $\Gamma(\eta' \rightarrow 2\gamma)$ in Case (II) are similar to those in Case (I). This is because the $k_1$ and $k_2$ have been determined so that good fit to the data for leptonic decay width of the low-lying vector meson may be obtained. In the heavy quark region, however, there is a large difference between the predictions for $\Gamma(\eta_0 \rightarrow 2\gamma)$ in Cases (I) and (II), because $\Gamma(\eta_0 \rightarrow 2\gamma)/\Gamma(\eta \rightarrow 2\gamma)$ in Case (I) is almost proportional to $M_{\eta_0}^{-1}$, while in Case (II), $\Gamma(\eta_0 \rightarrow 2\gamma)/\Gamma(\eta \rightarrow 2\gamma) \approx \text{const}$ as is shown in § 3.

We suppose that Royen and Weisskopf considered a case in which there is a situation $\cos 2(\alpha - \beta) = \cos 2(\beta - \gamma) = \cos 2(\gamma - \alpha) = 0$ among the relative phases of $\phi_\mu(0)$, $\phi_\nu(0)$ and $\phi_\phi(0)$. In this case, the theoretical value of $\Gamma(\eta \rightarrow 2\gamma)$ is considerably larger than the experimental one. In order to amend the defect we try

*Exactly speaking, the theoretical $\Gamma(\eta \rightarrow 2\gamma)$ is 3.5 times as large as the observed one, although the experimental results for $\Gamma(\pi^0 \rightarrow 2\gamma)$ and $\Gamma(\eta \rightarrow 2\gamma)$ can be explained fairly well. Paying attention to the fact that $D$ for the case $\eta \rightarrow 2\gamma$ has a pretty large magnitude with negative sign (cf. Table I), we consider the case in which $\alpha = \gamma$ and $\cos 2(\alpha - \beta) = 0$ so that the predicted values of $\Gamma(\pi^0 \rightarrow 2\gamma)$ and $\Gamma(\eta \rightarrow 2\gamma)$ may not be so far from the observed ones, respectively.
Table II. Partial widths for the $\pi^0 \rightarrow 2\gamma$, $\eta \rightarrow 2\gamma$ and $\eta' \rightarrow 2\gamma$ decay processes.

<table>
<thead>
<tr>
<th></th>
<th>$\Gamma(\pi^0 \rightarrow 2\gamma)$ (eV)</th>
<th>$\Gamma(\eta \rightarrow 2\gamma)$ (eV)</th>
<th>$\Gamma(\eta' \rightarrow 2\gamma)$ (keV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical values in Case (II)</td>
<td>7.3</td>
<td>580</td>
<td>8.5</td>
</tr>
<tr>
<td>Experimental values</td>
<td>9.2±1.2</td>
<td>323±50</td>
<td>5.4±2.1</td>
</tr>
</tbody>
</table>

To adjust the $\alpha$, $\beta$ and $\gamma$ so that the predicted widths for $\pi^0 \rightarrow 2\gamma$, $\eta \rightarrow 2\gamma$ and $\eta' \rightarrow 2\gamma$ may be in good agreement with the observed ones. In Table II are shown the results for the case in which

$$\alpha = \gamma \text{ and } \cos 2(\alpha - \beta) = 0.$$  

Thus a better fit to the data for $\Gamma(P \rightarrow 2\gamma)$ can be obtained.

Incidentally we wish to add the following remark in connection with the expression (10) of $\mu_{VP}$. As is well known, the magnetic moment of the baryon is obtained by summing the quark moments in the $SU(6)$ wave function which is totally symmetric in spin and flavor. When $m_u = m_d$, for example,

$$\mu_b = e \langle p| \sum_{u,d} (e_d/2m_d')|\sigma_z|p\rangle = e/2m_u$$  

and

$$\mu_A = e \langle A| \sum_{u,d} (e_d/2m_d')|\sigma_z|A\rangle = -(1/3)e/2m_s.$$  

Using the observed values $\mu_b = 2.79(e/2M_b)$ and $\mu_A = -0.614(e/2M_b)$, we get $m_u = 336$ MeV and $m_s = 510$ MeV. It should be noted that these values of $m_u$ and $m_s$ are nearly equal to $M_b/2$ and $M_s/2$, respectively.

§ 3. Decays of heavy pseudoscalar mesons

We are interested not only in the $c\cdot\bar{c}$ and $b\cdot\bar{b}$ systems but also in the $t\cdot\bar{t}$ system. Since we have no information about the $t$ quark from the present experiments, however, it is necessary to introduce more assumptions in the study of the latter system compared with the former systems. By considering this situation, the pseudoscalar state in the $t\cdot\bar{t}$ system is discussed briefly in the last part of this section.

3.1. Decay widths for the $\eta_8 \rightarrow 2\gamma$ processes

The decay widths for $\eta_c \rightarrow 2\gamma$ and $\eta_b \rightarrow 2\gamma$ can be estimated by Eq. (11), where

$$\Gamma(\eta \rightarrow 2\gamma) = \frac{e^2}{16\pi} |f_{\eta \rightarrow 2\gamma}|^2$$

and

$$|f_{\eta \rightarrow 2\gamma}|^2 = \frac{1}{4} \frac{1}{m_\eta^2} \frac{1}{m_\gamma^2} \frac{1}{m_\gamma'P}.$$
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\[ \mu_{\nu \rho} = e \langle \eta_0 \rangle \sum (e_{\nu} / 2 m_{\nu} \rangle \sigma_{\nu} \mid V(Q\bar{Q}) \rangle \]

\[ = -2e \langle e / 2 m_{\rho} \rangle = \begin{cases} -\frac{4}{3}(m_u/m_c)\mu_{\nu} & \text{for the } c\bar{c} \text{ system,} \\ \frac{2}{3}(m_u/m_s)\mu_{\nu} & \text{for the } b\bar{b} \text{ system} \end{cases} \]

and

\[ f_{\nu \rho} = \begin{cases} \frac{4}{3}\phi_{\nu}(0)M_{\rho}^{-3/2} & \text{for the } \phi/J, \\ -\frac{2}{3}\phi_{\rho}(0)M_{\rho}^{-3/2} & \text{for the } \Gamma. \end{cases} \] (24)

Note that in 2\(\gamma\)-decays of heavy pseudoscalar mesons, there is no need to consider the effects due to the relative phases of wave functions as in the case of (14) or (15), because it is only the lowest vector meson \(V(Q\bar{Q})\) that contributes as the mesons in the intermediate state of the decay process for the \(\eta_0\)(cf. Fig. 1). In what follows we adopt the value of \(k_{2}\) in (17) which is consistent with (4). Moreover we assume the following values as the effective masses of quarks:

\[ m_u = m_d = 336 \text{ MeV,}\]
\[ m_s = 510 \text{ MeV,}\]
\[ m_c = 1550 \text{ MeV.} \]

and

\[ m_b = 4730 \text{ MeV.} \] (25)

It is said that in the crystal-ball experiments, the data have indicated the existence of a state with \(M = 2976 \pm 20\) MeV which may be regarded as the \(\eta_c\). Since we have no information about the \(\eta_b\), we tentatively consider the two cases in which the \(\eta_b\) mass is equal to 9.17 and 9.34 GeV, on the basis of the following considerations: Previously the \(\eta_b\) mass of 9.17 GeV was predicted by using the radial mixing model and it is consistent with the result derived from the Regge-pole theory. On the other hand, the observed equality for the \(\phi' - \phi\) and \(\Gamma' - \Gamma\) splittings motivates us to suggest that in the heavy quark region, the hadron mass splittings do not depend strongly on the mass of the quarks of which the hadrons are composed. If this is the case,

\[ M_{\Gamma} - M_{\phi'} \approx M_b - M_{\eta_b}, \] (26)

This leads to the prediction of the \(\eta_b\) with \(M \approx 9.34\) GeV which is consistent with that given by Bradley and Gault. For \(\eta_0 \rightarrow 2\gamma\), the following results can be derived from Eq. (11):

\[ \Gamma(\eta_c - 2\gamma) \approx 5.90 \text{ keV}(1.48 \text{ keV}) \] (27)

\[ \text{and} \]

\[ \text{This property has been introduced formally in the logarithmic potential suggested by Quigg and Rosner.} \]
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\[ \Gamma(\eta_b \to 2\gamma) \approx \begin{cases} 0.379 \text{ keV} (0.031 \text{ keV}) & \text{for } M_{\eta_b} = 9.17 \text{ GeV}, \\ 0.400 \text{ keV} (0.033 \text{ keV}) & \text{for } M_{\eta_b} = 9.34 \text{ GeV} \end{cases}, \quad (28) \]

where the values in the parenthesis show the results for Case (I). We can see the large difference between the predictions for \( \Gamma(\eta_b \to 2\gamma) \) in Cases (I) and (II).

Note that Eqs. (27) and (28) in Case (II) indicate the existence of a remarkable property

\[ \Gamma(\eta_c \to 2\gamma) / e_{\eta_c}^4 \approx \Gamma(\eta_b \to 2\gamma) / e_{\eta_b}^4 \approx 30 \text{ keV}. \quad (29) \]

This can be understood by the following considerations: From Eqs. (23) and (24), \( \mu_{\nu \ell} = -e(e_{\ell} / m_{\ell}) \) and \( f_{\nu \ell} = 2 e_{\nu} \psi(0) M_{\nu}^{-3/2} \). Therefore,

\[ |\mu_{\nu \ell} f_{\nu \ell}|^2 = 4 e^2 (e_{\nu}^4 / m_{\ell}^2) |\psi(0)|^2 M_{\nu}^{-3}. \quad (30) \]

It is said that \( M_{\nu} \approx 2 m_{\ell} \) (cf. the last part of § 2). In fact \( M_{\eta} / M_{\pi} \approx m_{\ell} / m_{e} \approx 0.327 \). The ratio \( M_{\eta} / M_{\pi} \) is equal to 0.325 and 0.319 corresponding to the cases in which the \( \eta_b \) has the mass of 9.17 and 9.34 GeV, respectively. Then, the following relation does hold approximately: In the heavy quark region,*

\[ M_{\eta_b}(Q' \bar{Q}') / M_{\eta_b}(Q \bar{Q}) = M_{\eta_b}(Q' \bar{Q}') / M_{\eta_b}(Q \bar{Q}) = m_{\ell} / m_{\eta}. \quad (31) \]

From Eqs. (5), (11), (30) and (31), the important relation**

\[ \Gamma(\eta_b \to 2\gamma) / e_{\eta_b}^4 \approx \text{const} \quad (32) \]

can be derived.

3.2. Hadronic decay widths of pseudoscalar mesons

It can be expected that the \( \eta_c \) has a larger width than the \( V(\bar{Q}Q) \) for which \( \Gamma \sim a^4 \). The decay processes shown in Fig. 2 may suggest a relation

\[ \Gamma(\eta_c \to gg - \text{hadrons}) / \Gamma(\eta_b \to 2\gamma) = (2/9)[a_s(m_{\ell}) / a e_{\nu}^2]^2 \quad (33) \]

which was pointed out by some authors.\textsuperscript{10} Then, Eqs. (32) and (33) predict

\* When flavor independence of the hadron mass splittings does hold.

\** This can be rewritten as follows:

\[ (k-1)M_{\eta}(Q \bar{Q}) = (k-1)M_{\eta}(Q \bar{Q}). \]

where \( k = M_{\eta}(Q' \bar{Q}') / M_{\eta}(Q \bar{Q}) \) and \( k' = M_{\eta_b}(Q' \bar{Q}') / M_{\eta_b}(Q \bar{Q}) \). Therefore, \( (k-1)(k-1) = M_{\eta}(Q \bar{Q}) / M_{\eta}(Q \bar{Q}) \). For charmonium, \( M_{\eta} / M_{\pi} \approx 1.04 \). Thus, \( k' \) is not equal to \( k \), but slightly larger than \( k \). In this meaning, Eq. (31) should not be taken so strictly.

\textsuperscript{10} In Case (I), on the other hand, \( |\psi(0)|^2 \approx M_{\nu} \). Then, \( \Gamma(\eta_b \to 2\gamma) / e_{\nu}^4 \) is proportional to \( M_{\nu}^{-1} \).
But $\Gamma(\eta_0 \rightarrow \text{hadrons})$ would be a decreasing function of $m_0$ (cf. Eq. (33)) because of the property of coupling constant $a_s(m_0) \propto 1/\log(m_0/m_c)$ which is derived from the concept of asymptotic freedom. Recently Wolf$^{31}$ has estimated $\Gamma(\eta_c \rightarrow \text{hadrons})$ on the assumption that the $^3S_1$ and $^1S_0$ states in the $c \bar{c}$ system have the same radial wave functions. According to his result,

$$\Gamma(\eta_c \rightarrow \text{hadrons}) \approx 5 \text{ MeV} \quad \text{for } a_s(m_c) = 0.2.$$ (34)

for $a_s = 0.2$. This is consistent with the value in (34).

We now think it worth while to examine the relation (33). For the $\eta'$ in which the nonstrange component is included, the component can decay into hadrons through strong interaction irrelevant to the OZI-rule.$^{12}$ Therefore, $\Gamma(\eta' \rightarrow \text{hadrons})/\Gamma(\eta' \rightarrow 2\gamma)$ ought to be larger than the value ($\approx 850$) given by Eq. (33). Nevertheless, the data$^{31}$ indicate that the branching ratio $B$ for the decay $\eta' \rightarrow 2\gamma$ is about 2%. How should we interpret this fact? The $V(Qar{Q})$ can be included in the intermediate $1^-$ state in Fig. 2(a). In other words, the decay process shown in Fig. 1 can be regarded as a special case of Fig. 2(a). In Fig. 2(b), on the other hand, the $V(Qar{Q})$ cannot be included in the intermediate 1$^-$ state, because the gluon and $V(Qar{Q})$ are the particles with and without color, respectively. Therefore it may be said that the large difference between the observed value of $\Gamma(\eta' \rightarrow \text{hadrons})/\Gamma(\eta' \rightarrow 2\gamma)$ and the prediction based on Eq. (33) mainly comes from the effects due to the decay process shown in Fig. 1.

As was mentioned in § 2, the decay widths for the $\pi^0 \rightarrow 2\gamma$, $\eta \rightarrow 2\gamma$ and $\eta' \rightarrow 2\gamma$ processes can be explained well by the model in which the decays of pseudoscalar mesons take place through the process of Fig. 1. Although $\Gamma(\eta_0 \rightarrow 2\gamma)$ for the $\eta_0$ with large mass has been estimated by using the model, does the same process also play the most important role in decays of heavy pseudoscalar mesons? If this is not the case, $\Gamma(\eta_0 \rightarrow 2\gamma)$ may be much smaller than the value given in Eq. (27) or (28) for Case (II), and the relation (33) holds approximately in the heavy quark
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Moreover it is important to add some remark on the hadronic decay width of the $\eta_0$. The existence of glueballs has been suggested by several authors. If the colorless glueballs really exist, the photons in Fig. 1 can be replaced by the glueballs with $J^{PC}=1^-$. Then, it may be expected that there is a relation similar to (33) between the partial decay widths $\Gamma(\eta_0\rightarrow\text{hadrons})$ and $\Gamma(\eta_0\rightarrow 2\gamma)$.

3.3. Decay process $V(\bar{Q}Q)\rightarrow \Upsilon \eta_0$ and the $\eta_0$ mass

The decay width for the $V(\bar{Q}Q)\rightarrow \Upsilon \eta_0$ process can be calculated by using the interaction (8) for the vector-pseudoscalar-photon coupling. The result is expressed as follows:

$$\Gamma(V(\bar{Q}Q)\rightarrow \Upsilon \eta_0) = \frac{\mu_{VP}^2}{24\pi} \left( \frac{M_v^2 - M_{PS}^2}{M_v} \right)^3.$$  (36)

In charmonium,

$$\Gamma(\psi\rightarrow \eta \eta_0) \approx 2.7 \text{ keV} \quad \text{for} \quad M_{\psi}=2.98 \text{ GeV}.$$  (37)

Note that $\Gamma(V(\bar{Q}Q)\rightarrow \Upsilon \eta_0)$ depends strongly on the $\eta_0$ mass. As was emphasized by Wolf, the predicted decay width $\Gamma(\psi\rightarrow \eta \eta_0)$ becomes 45 and 1.6 keV, when the $\eta_0$ mass is equal to 2.79 and 2.99 GeV, respectively. Fritzsch and Jackson have also concluded that either the $X(2.82)$ is not the $\eta_0$, or the naive approach to the $c\bar{c}$ states based on a simple non-relativistic dynamics is misleading, because the predicted width turns out to be almost an order of magnitude larger than the experimental upper limit.

Similarly the $\eta_b$ mass can be discussed on the basis of the width for the decay $\Upsilon \rightarrow \gamma \eta_b$. Equation (36) enables us to make the following predictions:

$$\Gamma(\Upsilon\rightarrow \eta_0 \eta_b) \approx \begin{cases} 1.12 \text{ keV} & \text{for } M_{\eta_0}=9.17 \text{ GeV}, \\ 0.082 \text{ keV} & \text{for } M_{\eta_0}=9.34 \text{ GeV}. \end{cases}$$

As is supposed, $\Gamma(\Upsilon\rightarrow \eta_0 \eta_b)$ depends strongly on the $\eta_b$ mass. But it is difficult at present to draw any conclusion about the $M_{\eta_0}$, because we have no data for the radiative decay of the $\Upsilon$.

Then let us tentatively try to estimate $\Gamma(V(\bar{Q}Q)\rightarrow \Upsilon \eta_0)/e\zeta^2$, since at first sight it may be supposed that there would be no large difference between the values of $\Gamma(V(\bar{Q}Q)\rightarrow \Upsilon \eta_0)/e\zeta^2$ for the $c\bar{c}$ and $b\bar{b}$ systems. The results are as follows:

$$\Gamma(\psi\rightarrow \eta \eta_0)/e\zeta^2 \approx 6.1 \text{ keV}$$

and

$$\Gamma(\Upsilon\rightarrow \eta_b \eta_b)/e\zeta^2 \approx \begin{cases} 10.1 \text{ keV} & \text{for } M_{\eta_b}=9.17 \text{ GeV}, \\ 0.74 \text{ keV} & \text{for } M_{\eta_b}=9.34 \text{ GeV}. \end{cases}$$
These values of $\Gamma(V(Q\bar{Q})\rightarrow\gamma\gamma_0)$ seem to imply that the $\eta_\rho$ mass of 9.17 GeV rather than 9.34 GeV is favorable.

In the case of flavor independence of the hadron mass splittings, however, it is not the $\Gamma(V(Q\bar{Q})\rightarrow\gamma\gamma_0)/e_0^2$ but the $M_v^2\Gamma(V(Q\bar{Q})\rightarrow\gamma\gamma_0)/e_0^2$ that can be regarded as a constant as is shown below. Because of flavor independence of $(M_v-M_{PS})$ and $(M_v+M_{PS})/M_v\approx 2$, $\Gamma(V(Q\bar{Q})\rightarrow\gamma\gamma_0)$ in Eq. (36) is proportional to $\mu^2_{V}(=e_0^2(e/m_Q)^2)$ or $a e_0^2/M_v^2$. In other words,

$$M_v^2\Gamma(V(Q\bar{Q})\rightarrow\gamma\gamma_0)/e_0^2\approx \text{const.} \quad (38)$$

For the $\eta_\rho$ mass of 9.34 GeV which has been predicted on the basis of flavor independence of the hadron mass splittings, the relation (38) can approximately be satisfied. Namely,

$$[\Gamma(\Upsilon\rightarrow\gamma\eta_\rho)/e_0^2]/[\Gamma(\psi\rightarrow\gamma\eta_\rho)/e_0^2][=(M_\phi/M_\Upsilon)^2\approx 0.11].$$

This can be regarded as a self-consistent result and it is possible to say that the predicted $\eta_\rho$ mass of 9.34 GeV is very promising.

3.4. Pseudoscalar state in the $t\bar{t}$ system

For the $t$ quark mass, various kinds of predictions have been made by many authors. Some of them have said that $m_t\approx 26$ GeV and others $m_t\approx 13$ GeV or 20 GeV. According to the recent data for $e^+e^-$ collisions up to $\sim 31$ GeV, however, there seems to be no evidence for the $V_t$. This probably means that the mass of the $t$ quark is larger than 15 GeV. In this paper we consider the following two cases and study the properties of the $\eta_t$: $m_t=20$ GeV and $m_t=26$ GeV.

Using the relation $M_v(Q\bar{Q})\approx 2 m_Q$ and the flavor independence of $M_v(Q\bar{Q})-M_{PS}(Q\bar{Q})$, we can predict the masses of the $V(t\bar{t})$ and $\eta_t$. That is,

$$M_v(t\bar{t})\approx \left\{ \begin{array}{ll} 40 \text{ GeV} & \text{for } m_t=20 \text{ GeV,} \\ 52 \text{ GeV} & \text{for } m_t=26 \text{ GeV.} \end{array} \right.$$  

Then, the decay widths for the processes $\eta_t\rightarrow 2\gamma$ and $V(t\bar{t})\rightarrow \eta_t$, can be estimated by the same way as in §§ 3.1 and 3.3. The results are as follows:***

*** Since the ratio $\xi=M_{\mu}(Q\bar{Q})/M_v(Q\bar{Q})$ is an increasing function of $m_Q$ under the assumption of flavor independence of the hadron mass splittings (cf. Eq. (26)), $\xi$ lies in a region from 0.96 to 1 as is seen from $M_{\mu}/M_v\approx 0.96$ for charmonium.

*** Some discussions about flavor independence of the hadron mass splittings will be made in another place.

*** Only the results for Case (II) are shown below (cf. Eq. (16)).
and
\[
\Gamma(\eta_i \rightarrow 2\gamma) = \begin{cases} 
6.60 \text{ keV} & \text{for } m_t = 20 \text{ GeV}, \\
6.62 \text{ keV} & \text{for } m_t = 26 \text{ GeV}.
\end{cases}
\]

Needless to say, the relations (32) and (38) emphasized in §§ 3.1 and 3.3 can approximately be satisfied by the values of \(\Gamma(\eta_i \rightarrow 2\gamma)\) and \(\Gamma(V(t\bar{t}) \rightarrow \gamma\eta_i)\), respectively.

References

2) For example, see, G. Weber, DESY 78/74 (November 1978).
7) For example, see, H. J. Lipkin, FERMILAB-Conf-78/73-THY, September 1978.
9) A. Bradley and F. D. Gault, preprint.
10) For example, see, M. Krammer and H. Krasemann, DESY 79/20 (April 1979) and Ref. 11.
11) G. Wolf, DESY 80/13 (February 1980).
13) For example, see, Particle Data Group, Phys. Letters 75B (1978), 1
19) For example, see, G. Wolf, DESY 80/13, February 1980.