Signature Scheme of the Shell Model and Nuclear Collective Excitations

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(Received May 10, 1980)

The symmetries between the proton and neutron systems in a nucleus are described in terms of the signatures. Three kinds of signatures (i.e., the \(pn\), \(pn\) and \(p\alpha\) (or \(n\alpha\)) signatures) are introduced and they are related to the appearance of various collective motions such as rotations or vibrations. It is shown that the \(pn\)-signature scheme provides a band-like structure of deformed nuclei, while the features associated with the quadrupole vibration are obtained in the \(pn\)- and \(p\alpha\)-signature scheme. From the detailed investigation of the \(p\alpha\)-symmetric wave functions, it is concluded that the regularities provided by the present shell model are more analogous to those of the \(\gamma\)-unstable surface vibration model (Wilets-Jean model) rather than to those of harmonic phonon model.

§ 1. Introduction

One of the well established nuclear models is the collective model proposed by Bohr and Mottelson.1 Macroscopically the excitations such as rotations or vibrations have been well studied with the model. In its simplest forms, the model is based on the two concepts, namely the band structure for rotational nuclei and the multi-phonon structure for vibrational ones. In the band structures a series of well deformed states is excited or de-excited via strong \(E2\) transitions and are relatively independent of other series of states (the side bands). The quantitative aspects of energy level and \(E2\) transitions are simply described in terms of the moment of inertia and the intrinsic deformation. In the quadrupole vibration model, the states are classified by the phonon number \(N\). In the harmonic phonon model the states of the same \(N\) are degenerate and form the phonon multiplets. Strong \(E2\) transitions are expected only between the states with \(\Delta N = 1\).

Another established model is the nuclear shell model, which is based on the assumption of the independent particle motions. Many aspects of nuclei near closed shell are well explained by the model. In spite of the restriction on the number of active nucleons and on the available shell orbits, usually the calculated results of the shell model appear in full complexity. Resultant wave functions do not suggest the existence of simple regularity on energies or electromagnetic transitions. This tendency is quite natural if one considers the importance of the
Pauli principle and the complexity of the effective one- and two-body Hamiltonians. In this sense it might be hard to expect that the application of the shell model to more complex system yields simple regularities such as the band structures or the phonon multiplets.

It is, however, acceptable that the shell model in principle can describe any nuclear collective motions.\(^2\) Therefore it is strongly required to investigate the underlying concepts which elucidate the regularities hidden in the complicated shell-model wave functions.

For this purpose we emphasize the importance of the "signature" concept which describes simple symmetry between the proton- and neutron-systems in nuclei. This concept has been originally introduced to the shell model for the \(1\sbar m\) nuclei.\(^3\) In this paper we propose the applicability of this concept to understand the collective properties which microscopic wave functions possess.

In § 2 we briefly review the basic idea of the three kinds of signatures, i.e., \(pn-, p\bar{n}-, \) and \(pp/(or \, n\bar{n}-)\) signatures. Regularities of the \(E2\) transition matrices derived in the signature scheme are presented in § 3. In § 4 several applications of the signature scheme and resultant regularities are shown to be related to the collective excitation modes. More detailed discussions including the discrepancy between the shell-model regularities and the harmonic phonon scheme are presented in § 5.

§ 2. Definition of the \(pn-, p\bar{n}-, \) and \(pp/(or \, n\bar{n}-)\) signatures

In the case where active protons and neutrons occupy the same orbit \(j,\) eigenstates of the Hamiltonian show some primitive symmetries in the proton-neutron coupling representation, i.e., in the \(|\sbar m\rangle\langle m|\rangle|\sbar m\rangle\langle m|\rangle_{n,J}\rangle\) basis, where \(a\) represents quantum numbers other than the spin \(J.\)

If the Hamiltonian

\[ H = H_0 + V_{pp} + V_{nn} + V_{pn} \]  (1)

has the charge symmetric effective interactions, i.e., the equality of \(V_{pp}\) and \(V_{nn},\) it is invariant under the transformation \(X(p \rightarrow n)\) which changes all protons to neutrons and all neutrons to protons (the \(pn\)-transformation).

By the transformation an eigenstate \(\Psi_{ji}(m, m'),\) the \(i\)-th state of spin \(J\) in the \((j^n)_{p}(j^n)_{n}\) system, is transformed to the corresponding eigenstate in the mirror nuclei \(\Psi_{ji}(m', m).\) As the special case of the \(pn\)-transformation, the eigenstates of the self-conjugate nuclei \((j^n)_{n}\) system are transformed to themselves.\(^4\)

\[ X(p \rightarrow n) \Psi_{ji}(m, m) = S_{nn}(J) \Psi_{ji}(m, m). \]  (2)
where the eigenvalue $S_{pn}$ is $+1$ or $-1$. Thus each state in the self-conjugate nucleus possesses a definite signature $S_{pn}$. Hereafter we call it the $pn$-signature so as to avoid a confusion with other signatures introduced later.

The wave functions of the positive (negative) $pn$-signature states are described as a linear combination of the positive (negative) basic vectors. The positive and negative basic vectors $\phi_{pn}^{\pm}$ are defined as

$$
\phi_{pn}^{\pm}(a_1J_1, a_2J_2; J) = \{ j^m(a_1J_1)b_{\pmb{J}_{1}n}^m(a_2J_2)nJ \} \pm (-)^{J_{1}+J_{2}-J} \times \{ j^m(a_2J_2)b_{\pmb{J}_{2}n}^m(a_1J_1)nJ \} / \sqrt{2(1+\delta(a_1J_1, a_2J_2))}.
$$

This equation indicates that the basis $\phi_{pn}(a_1J_1, a_2J_2; J)$ belongs either to the positive or to the negative basis in accordance with the phase factor $(-)^{J_{1}+J_{2}-J}$.

The $pn$-signature $S_{pn}$ introduced here is related to the isobaric spin $T$ in the $N=Z$ nucleus. The states in the self-conjugate nucleus are divided into two groups; one is the states of $T=even$ and the other of $T=odd$. Each group corresponds either to positive- or to negative-signature under the relation of $S_{pn} = (-)^{T-Z}$. Thus in this case the $pn$-signature is nothing but a simplification of the isospin concept. We, however, insist on the importance of this concept in its applicability.

In the $j^n$ configuration we can find another symmetry under the $pn$-transformation associated with particle-hole transformation, i.e., all protons in the $j$-orbit are changed into neutron-holes in the $j$-orbit and vice versa. This transformation will be referred to as the $p\bar{n}$-transformation and represented by the operator $X(p\rightarrow \bar{n})$. Corresponding eigenfunctions in two nuclei with the configuration of $(j^{m})_{p}(j^{-m})_{n}$ and $(j^{m})_{p}(j^{m})_{n}$ become conjugate under the $p\bar{n}$-transformation when the Hamiltonian is charge symmetric. This conjugation is well known in the nuclear shell model and a pair of nuclei are called the cross-conjugate nuclei. The operation of $X(p\rightarrow \bar{n})$ yields another kind of signature for the eigenstates of the self-cross-conjugate nucleus which has the $(j^{m})_{p}(j^{m})_{n}$ configuration,

$$
X(p\rightarrow \bar{n}) \Psi_{j^{m}}(m, -m) = S_{p\bar{n}}(J_{1}) \Psi_{j^{m}}(m, -m),
$$

where $S_{p\bar{n}}$ is $+1$ or $-1$. In this paper the signature $S_{p\bar{n}}$ will be called the $p\bar{n}$-signature.

The positive and negative bases $\phi_{p\bar{n}}^{\pm}$ are given in a similar way to those for the $pn$-symmetric case (Eq. (3)),

$$
\phi_{p\bar{n}}^{\pm}(a_1J_1, a_2J_2; J) = \{ j^m(a_1J_1)b_{\pmb{J}_{1}n}^m(a_2J_2)nJ \} \pm (-)^{J_{1}+J_{2}-J} \times \{ j^m(a_2J_2)b_{\pmb{J}_{2}n}^m(a_1J_1)nJ \} / \sqrt{2(1+\delta(a_1J_1, a_2J_2))}.
$$

This symmetry was first introduced as the even-odd symmetry of the $f_{7/2}^{m}$ wave functions and applied to the explanation of the vanishing $\beta$-decay matrix.
element in $A=48$ nuclei.\(^7\)

The third symmetry in the single $j$ shell model is the symmetry which appears in the identical particle system under the particle-hole transformation $X(p\rightarrow\tilde{p})$ or $X(n\rightarrow\tilde{n})$. For half-filled configuration $j^{j+1/2}$ of protons (or neutrons), a signature $S_{p\tilde{p}}$ which is called the $p\tilde{p}$-signature is defined,

$$X(p\rightarrow\tilde{p})\Psi_j^{(j)}(j + \frac{1}{2}, 0) = S_{p\tilde{p}}(J)\Psi_j^{(j)}(j + \frac{1}{2}, 0).$$

Originally the concept of this signature was introduced by Bell.\(^9\)

According to the group theory the $p\tilde{p}$- or $n\tilde{n}$-signature closely relates to the seniority $v$ of the identical system,\(^7\) i.e.,

$$|j^{j+1/2}\beta eJ\rangle = (-)^{(j+1/2)}|j^{j+1/2}\beta eJ\rangle$$

if $j + \frac{1}{2} =$ even,

$$|j^{j+1/2}\beta eJ\rangle = (-)^{(j+1/2)}|j^{j+1/2}\beta eJ\rangle$$

if $j + \frac{1}{2} =$ odd,

where $\beta$ is an additional quantum number other than the seniority $v$ and the spin $J$.

§ 3. Selection rules of the \(E^2\) transitions

One of the advantages of adopting the signature scheme is seen in the derivation of simple selection rules of the electromagnetic transitions. In the following we briefly review the selection rules of \(E^2\) transitions which closely relate to nuclear quadrupole collectivities.

The electric quadrupole operator is written as a sum of the isoscalar and isovector parts,

$$Q_n = \sum r_i^2 Y_{2\mu}(\theta_i, \phi_i)$$

$$= (e_p + e_n) \sum r_i^2 Y_{2\mu}(\theta_i, \phi_i) / 2 - (e_p - e_n) \sum \tau_i r_i^2 Y_{2\mu}(\theta_i, \phi_i) / 2.$$\(^8\)

For the \(pn\)-symmetric configuration in which we can define the \(pn\)-signatures as a quantum number, the reduced matrix element of the \(E^2\) transition $\langle \Psi_\mu^p | Q | \Psi_\nu^\tilde{p} \rangle$ is also expressed as $\langle X | \Psi_\mu^p | Q | X | \Psi_\nu^\tilde{p} \rangle$, where $Q = X Q X^{-1}$ and $X$ stands for $X(p\rightarrow\tilde{n})$. The $pn$-transformed \(E^2\) operator $\tilde{Q}$ is obtained by interchanging $e_p$ and $e_n$ in Eq. (8),

$$\tilde{Q} = (e_p + e_n) \sum r_i^2 Y_{2\mu}(\theta_i, \phi_i) / 2 + (e_p - e_n) \sum \tau_i r_i^2 Y_{2\mu}(\theta_i, \phi_i) / 2.$$\(^9\)

Using the $pn$-signatures, $S$ and $S'$, the reduced matrix elements are calculated by the following two equations:
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\[ \langle \Psi_j^{(i)} | Q | \Psi_j^{(i')} \rangle = (e_p + e_n)A - (e_p - e_n)B \]
\[ = \{(e_p + e_n)A + (e_p - e_n)B \}|SS'\],

where

\[ A = \langle \Psi_j^{(i)} | \sum r_i^2 Y_2(\theta_i, \phi_i) | \Psi_j^{(i')} \rangle / 2 \]

and

\[ B = \langle \Psi_j^{(i)} | \sum \tau_i r_i^2 Y_2(\theta_i, \phi_i) | \Psi_j^{(i')} \rangle / 2. \]

Since the two different equations (10) and (11) are derived for the same physical quantity, it follows that the scalar part \( A \) must vanish for the transitions between two states of different \( pn \)-signatures (\( SS' = -1 \)) and the isovector part \( B \) must vanish for the transitions between two states of the same \( pn \)-signature (\( SS' = 1 \)). Therefore, we can summarize the selection rule of the \( E2 \) transitions in the \( pn \)-symmetric configurations as

\[ \langle \Psi_j^{(i)}(m, m) | Q | \Psi_j^{(i)}(m, m) \rangle \]
\[ = \begin{cases} (e_p + e_n)A & \text{for } SS' = 1, \\ -(e_p - e_n)B & \text{for } SS' = -1. \end{cases} \]

Since the enhancement of the \( E2 \) transitions comes from the isoscalar part \( A \), it is expected that any strong \( E2 \) transitions would occur without changing the \( pn \)-signature. On the other hand, transitions with \( pn \)-signature change are usually hindered.

As for the \( E2 \) selection rule in the \( pn \)-symmetric configuration, the derivation can be proceeded in an analogous way to Eqs. (8)~(11) except that the \( pn \)-transformed \( E2 \) operator \( \tilde{Q} \) is written as

\[ \tilde{Q} = X(p \rightarrow \tilde{n})QX(p \rightarrow \tilde{n})^{-1} \]
\[ = -(e_p + e_n)\sum r_i^2 Y_2(\theta_i, \phi_i) / 2 - (e_p - e_n)\sum \tau_i r_i^2 Y_2(\theta_i, \phi_i) / 2. \]

Finally, the following rules are derived for the \( E2 \) transitions in the \((j^m)_{p}(j^-m)_{n}\) configuration,

\[ \langle \Psi_j^{(i)}(m, -m) | Q | \Psi_j^{(i)}(m, -m) \rangle \]
\[ = \begin{cases} (e_p + e_n)A & \text{for } SS' = -1, \\ -(e_p - e_n)B & \text{for } SS' = 1, \end{cases} \]

where \( A \) and \( B \) are given by Eq. (12). This rule suggests that the strong \( E2 \) transitions in the \( \tilde{p} \tilde{n} \)-symmetric configurations are expected only between the states of different \( \tilde{p} \tilde{n} \)-signatures, and the transitions between states of the same \( \tilde{p} \tilde{n} \)-signature are usually hindered. For the \( E2 \) operator with the same effective
charges for protons and neutrons, the latter transitions are forbidden. Furthermore for this quadrupole operator all eigenstates do not possess the moment in the \( p\bar{n} \)-symmetric configuration, because the moments are given by the matrix elements between the same \( p\bar{n} \)-signature states. Vanishing of the quadrupole moments suggests that the \( p\bar{n} \)-symmetric states can be related to motions of the spherical nuclei.

For the \( p\bar{f} \) (or \( n\bar{n} \)) symmetric configuration, the \( E2 \) selection rule is easily derived and is expresses as

\[
\langle \Psi_{j'p}(j+\frac{1}{2},0)|Q|\Psi_{j'p}(j+\frac{1}{2},0)\rangle = \left\{ \begin{array}{ll}
e_p & \text{for } SS' = -1, \\
0 & \text{for } SS' = 1, \end{array} \right.
\]

(16)

where \( S \) and \( S' \) are the \( p\bar{f} \)-signatures.

The selection rule of Eq. (16) shows a tendency similar to the rule of Eq. (15). Thus it is expected that the \( p\bar{f} \) (or \( n\bar{n} \)) symmetric states also relate to the spherical nuclei. It must be remarked on the selection rule (16) that the equivalent rule is derived from the number dependence of reduced matrix elements of the even-rank tensors in the seniority scheme.

§ 4. Signature scheme and nuclear collective regularities

The concept of \( p\bar{n} \)-signature has already been applied to the studies of the \( 1_{17/2} \) and \( 19_{23/2} \)-nuclei. The \( E2 \) and \( M1 \) transitions or \( \beta \)-decay\(^{30}\) observed for the self-cross-conjugate nuclei are compared with the selection rules in the signature scheme to test the \( j^n \) configurational assumptions. Many interesting results have been derived from such phenomenological investigations. In this paper, however, we emphasize the importance of qualitative aspects of the signature scheme in connection with the nuclear collective excitations such as rotations and vibrations. Energies and wave functions are calculated with conventional effective two-body interactions in order to know the ordinary distributions of the signatures. Several configurations of \( (j_1)^p(j_2)^n \) are adopted for the investigations.

4.1. \( p\bar{n} \)-symmetry and vibrational collectivity

The calculated spectrum of the \( (9/2)^p \times (9/2)^n \) configuration is shown in Fig. 1. The delta interactions are adopted for \( V_{pp} \) and \( V_{nn} \) and the pure quadrupole forces for \( V_{pn} \). Above the \( J=0^+ \) ground state of the positive \( p\bar{n} \)-signature \((0_{\text{g.s}}(+))\), the \( J=2^+ \) state of the negative \( p\bar{n} \)-signature \((2_i^+(-))\) is obtained as the first excited state. Since the quadrupole matrix elements for the
proton-particle and for the neutron-hole have the opposite signs in the $p\bar{n}$-symmetric system, i.e., $\langle j^m(a_J)Q^n||j^m(a_J)\rangle = -\langle j^m(a_J)Q^n||j^m(a_J)\rangle$, the negative-signature component ($\left| J_p=2_1^{+}\times J_n=0^+_1; 2^+ \right> \sim \left| J_p=0^+_1 \times J_n=2^+_1; 2^+ \right>$) / $\sqrt{2}$ gains the proton-neutron interaction energy which is the attractive quadrupole force and becomes the major component in the $J=2_1^+(-)$. It must be noticed that, due to the conservation of the $p\bar{n}$-signature, the $\left| J_p=2_1^+\times J_n=2^+_1; 2^+ \right>$ component does not admix in the $J=2_1^+(-)$ state. The $J=2_1^+(-)$ state mostly exhausts the sum rule of the quadrupole excitation from the $0^{+}_{\text{odd}}(+) \text{ state}$ and is approximated by

$$\left| 2_1^+(-) \right> \sim \frac{1}{\sqrt{N(2_1^+)}} Q 0^{+}_{\text{odd}}(+)$$

(17)
where \( N \) is the normalization constant and \( Q \) is the quadrupole operator (8) with \( \epsilon_p = \epsilon_n \). From Eq. (15), it can be seen that the \( 2t^- (-) \) state as well as all other states in the \( pn^- \)-symmetric configuration does not possess the quadrupole moment.

Nearly degenerate two states of \( J=4t^+(+) \) and \( 2t^+(+) \) are obtained above the \( 2t^- (-) \) state. The main components in both states are the \( |J_p=2t^- \times J_n=2t^-; J> \) and \( (|J_p=0t^- \times J_n=J; J> + |J_p=J \times J_n=0t^-; J>)/\sqrt{2} \) ones. The quadrupole transitions from these states to \( 2t^- (-) \) are classified as a favoured one according to the signature selection rule and large reduced matrix elements are obtained in the calculation (see Table I). Therefore it is suggested that these states are approximately reproduced by the equations

\[
|4t^+(+)> \sim \frac{1}{\sqrt{N(4t^-)}} (Q|2t^- (-)>)^{(J=4)}
\]

and

\[
|2t^+(+)> \sim \frac{1}{\sqrt{N(2t^-)}} (Q|2t^- (-)>)^{(J=2)}.
\]

It must be noted here that the approximation (19) holds because of the vanishing of the quadrupole moment of the \( 2t^- \) state in the \( pn^- \)-symmetric configurations. If the \( J=3t^+ \) state had non-vanishing quadrupole moment, i.e., \( <2t^+|Q|2t^-> \neq 0 \), the vector \( (Q|2t^->)^{1/2} \) becomes non-orthogonal to the \( |2t^-> \) vector. Therefore it is only

| Table I. Relative \( <J|Q|J'> \) values in the \( (9/2)p^4(9/2)n^-4 \) configuration. Reduced matrix elements \( <J(S)Q|J'(S')> \) which are larger than half of \( <2t^-(-)|Q|2t^-(-)> \) are shown for low-lying states. |
|-----------------|-----------------|-----------------|
| \( J(S_{pn}) \) | \( J'(S_{pn}) \) | \( <J(S_{pn})|Q|J'(S_{pn})> \) |
| \( 2t^- (-) \)   | \( 0t^- (+) \)   | 100 (norm)  |
| \( 2t^+(+), (-) \) | \( 2t^+(+), (-) \) | 154          |
| \( 2t^- (-) \)   | \( 2t^- (-) \)   | 154          |
| \( 3t^- (-) \)   | \( 3t^- (-) \)   | 154          |
| \( 4t^- (-) \)   | \( 4t^- (-) \)   | 154          |
| \( 5t^- (-) \)   | \( 5t^- (-) \)   | 154          |
| \( 6t^- (-) \)   | \( 6t^- (-) \)   | 154          |
| \( 7t^- (-) \)   | \( 7t^- (-) \)   | 154          |
| \( 8t^- (-) \)   | \( 8t^- (-) \)   | 154          |
expected in the $p\bar{n}$-symmetric configurations, and probably in the $p\bar{p}$- or $n\bar{n}$-symmetric configurations, that the $(Q \cdot Q)^{J=2}\langle 0_{\text{ind}}|\rangle$ approximation is valid for the $J=2s^+$ state.

For each excited state we can define the multiplicity of strong quadrupole decays down to the ground state. As known from the $(Q \cdot Q)^{J=2}\langle 0_{\text{ind}}|\rangle$ approximations, the $J=2s^+(+)\text{ and } 4s^+(+)\text{ states decay to the } 0_{\text{ind}}\text{ state through strong quadrupole transitions twice. For these states the multiplicity of strong quadrupole transitions } M\text{ is defined two. As for the } M=3\text{ states we can classify the } J=6s^-( -), 4s^-( -), 3s^-( -) \text{ and } 0s^-( -)\text{ states. The } p\bar{n}\text{-signature scheme provides the unique classification of the states according to the multiplicity } M, \text{ even when there exist plural branches of the strong quadrupole decays. The result of the classification is shown in Table II. Here the transitions of which the reduced matrix element } |\langle J' \mid Q \mid J \rangle|\text{ are larger than half of } |\langle 0_{\text{ind}} \mid Q \mid 2s^+ \rangle|\text{ are regarded as the strong ones (Table I).}

Now the correspondence between the $M$-multiplicity state in the present model and the $M$-phonon state in the usual vibrational model becomes clear from the relation

$$(Q^M)^{J=0}\langle 0_{\text{ind}}|\rangle \sim |(a^+)^M|\rangle\langle 0|,$$  \hspace{1cm} (20)

where $a^+$ is a creation operator of phonons.

In the multiplicity representation, quadrupole transitions of $\Delta M=1$ are enhanced but the $\Delta M=2$ transitions are forbidden due to the signature selection rule. The $\Delta M=0$ transitions as well as the quadrupole moments are also forbidden. Now we notice that this selection rules are equivalent to those of the harmonic-phonon model if one regards $\Delta M$ above as the change of phonon numbers. Thus it is strongly suggested that the shell-model wave functions of the $p\bar{n}$-symmetry possess substantial similarity to those of the vibrational model.

Table II, however, suggests some differences between the harmonic-phonon model and the present shell model. States corresponding to several phonon-states are missing in the shell model. This discrepancy is usually inevitable when we adopt the conventional effective interactions. This will be discussed in detail later.

<table>
<thead>
<tr>
<th>Multiplicity</th>
<th>Shell model state $J^+(S_{p\bar{n}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M=0$</td>
<td>$0_{\text{ind}}(+)\text{ and } 2s^-(+)$</td>
</tr>
<tr>
<td>$M=1$</td>
<td>$2s^-(+)$ and $2s^+(+)$</td>
</tr>
<tr>
<td>$M=2$</td>
<td>$6s^-(+), 4s^-(+), 3s^-(+)\text{ and } 0s^+(+)$</td>
</tr>
<tr>
<td>$M=3$</td>
<td>$8s^+(+), 6s^+(+), 5s^+(+), 4s^+(+)$ and $2s^+(+)$</td>
</tr>
</tbody>
</table>
4.2. \textit{pn}-symmetry and band structures

The appearance of rotation-like spectra in the simple proton-neutron configuration such as $j_p^2 \times j_n^2$ have been discussed in Refs. 11) and 12). Empirically it was suggested that the large \(j_p\) and \(j_n\) orbits are favoured to produce the rotational spectrum. In this section we show a simplified calculation and investigate relation between the \textit{pn}-symmetric structures and the band-like structures.

The calculated spectrum for the \((i_{13/2})^2(i_{13/2})n^2\) configuration is shown in Fig. 2. Similar to Ref. 12) the delta plus strong quadrupole forces have been
assumed for effective proton-neutron interaction $V_{pn}$ and also for $V_{pp}$ and $V_{nn}$. The $pn$-signatures are indicated in the brackets for each level. Positive $pn$-signatures are obtained for yrast states, i.e., $0_{\text{ind}}^{(+)}$, $2_{1}^{(+)}$, $4_{1}^{(+)}$, and $6_{1}^{(+)}$. The resultant wave functions yield large matrix elements of the quadrupole transitions between these states and also large quadrupole moment for each state (Table III). The quadrupole transitions and moments of which reduced matrix elements are larger than half of $\langle 0_{\text{ind}}^{(+)} | Q | 2_{1}^{(+)} \rangle$ are indicated in Fig. 2. From the discussion in § 3 (Eq. (13)), the coexistence of the enhanced quadrupole transitions and moments are allowed in the $pn$-symmetric configurations.

The yrast states can be approximately created by successive operations of the quadrupole operator $Q$ to the $J=0_{\text{ind}}^{(+)}$ wave function. In the $pn$-symmetric configuration this procedure does not yield another branch of the states other than yrast ones, e.g., $J=2_{1}^{+}$ or $4_{1}^{+}$. This is in strong contrast with the situation in the $pn$-symmetric configuration (see Fig. 1). The reason why only the yrast states are produced by the multiple $Q$ operations to the $J=0_{\text{ind}}$ state in the $pn$-symmetric configuration can be understood in the following. A $J=2^{+}$ state which would presumably be excited from the $J=2_{1}^{(+)}$ state through the strong quadrupole transition should be approximately expressed by a vector $\{ Q | 2_{1}^{(+)} \}$.
large quadrupole moment of the $J=2^+_1(+)\) state $\langle 2^+_1(+)\|Q\|2^+_1(+)\rangle$, however, suggests a large overlap between $\{ Q\|2^+_1(+)\rangle \}^{\pm 2}$ and $|2^+_1(+)\rangle$. Therefore, if one takes account of the orthogonalization, the $\{ Q\|2^+_1(+)\rangle \}^{\pm 2}$ vector loses the physical significance as an eigenstate. Generally the $\{ Q\|J\rangle \}^{\pm}$ vector cannot be new eigenstate in the $pn$-symmetric system.

Now a series of levels $J=0^+_{2n}(+)$, $2^+_1(+)\), $4^+_1(+)\) and $6^+_1(+)\) can be considered to correspond to the ground band in the deformed nuclei. In Fig. 2 we notice another series of the states which starts from the $J=1^+_1(+)\) states. This series of negative $pn$-signature states also makes a band-like structure. It is easily expected from the selection rule (13) that the quadrupole transitions between these two bands are hindered. Thus many calculated properties which are closely related to the band structures are well understood in terms of the $pn$-signature concept. It must be also noticed that the theoretical investigation to obtain the vibrational aspect seems to be meaningless for the $pn$-symmetric system with any manipulation of the effective Hamiltonian.

4.3. Generalization of the signature concept to the $j_p+j_n$ space

In the medium-weight or heavy nuclei, active protons and neutrons occupy the different orbits. Therefore the signature concepts originally defined in the $j_p=j_n$ configuration must be generalized to the $j_p\neq j_n$ space.

It has been shown in Ref. 12) that the band-like structures can be obtained for the large $j_p$ and $j_n$ configuration like $(h_1^1l_2^1)+p(i_{13}^1l_2^1)n^2$ with the enhanced quadrupole forces for $V_{pp}$, $V_{nn}$ and $V_{pn}$. In this section we reproduce the shell model calculation for a configuration of $(1h_{11}^1l_2^1)+(i_{13}^1l_2^1)n^2$ and try to understand the results qualitatively in terms of the $pn$-signature.

The energy spectra of the $h_{11}^1l_2^1$ proton system and the $i_{13}^1l_2^1$ neutron system are quite similar. Furthermore the resemblance between the quadrupole matrix elements $\langle J_1\|Q_p\|J_2\rangle$ and $\langle J_1\|Q_n\|J_2\rangle$ suggests a nearly symmetric situation under the strong quadrupole interactions. Therefore it is expected that resultant wave functions might approximately hold the $pn$-symmetry (the quasi $pn$-symmetry) especially for the low spin states.

A generalization of the $pn$-transformation for the $(h_{11}^1l_2^1)+p(i_{13}^1l_2^1)n^2$ system is possible with the following definition:

$$X(p\rightarrow n)|h_{11}^1l_2^1(J_1)p^1i_{13}^1l_2^1(J_2)n^2; J\rangle = (-1)^{l_1+l_2}|h_{11}^1l_2^1(J_2)p^1i_{13}^1l_2^1(J_1)n; J\rangle.$$ \hspace{1cm} (21)

Using this operator $X(p\rightarrow n)$ the generalized $pn$-signature $S_{pn}$ is defined for each eigenstate $\Psi_f^{(n)}(h_{11}^1l_2^1, i_{13}^1l_2^1)$ as an expectation value

$$S_{pn} = \langle \Psi_f^{(n)}(h_{11}^1l_2^1, i_{13}^1l_2^1)|X(p\rightarrow n)\Psi_f^{(n)}(h_{11}^1l_2^1, i_{13}^1l_2^1)\rangle.$$ \hspace{1cm} (22)
It must be noticed that the generalized signature \( S_{pn} \) above is no longer +1 or -1, because of the asymmetry between \( V_{pp} \) and \( V_{nn} \) matrix elements and of some vanishing components in \( X(p\rightarrow n) \Psi_{f}^{(i)}(\tilde{h}_{1/2}, \tilde{i}_{3/2}) \) which originate from

\[
X(p\rightarrow n)|\tilde{h}_{1/2}(J)\tilde{i}_{3/2}(12^{+}); J> = 0 .
\]

The calculated spectrum shown in Fig. 3 exhibits the rotation-like structure. The generalized \( pn \)-signature values \( S_{pn} \) are given in the brackets. It is remarkable that \( S_{pn} \) are very close to +1.0 for the yrast states and also -1.0 for the other excited states. Although the selection rule of the quadrupole

![Fig. 3. Energy spectrum of the \((h_{1/2})^{2}(i_{13/2})_{n}^{2} \) configuration and the generalized \( pn \)-signatures. The delta plus quadrupole interaction (\( K=0.02 \)) is assumed to \( V_{pp} \), \( V_{nn} \) and \( V_{pn} \) (see Ref. 12). All calculated levels (\( J \leq 5 \)) below 10 MeV are shown with the generalized \( pn \)-signatures \( S_{pn} \) defined by Eq. (22).](https://academic.oup.com/ptp/article-abstract/65/2/533/1929367)
transitions (13) is not exactly valid for the present system due to the non-equality of $\langle j_1 \| Q_0 \| j_1 \rangle$ and $\langle j_2 \| Q_0 \| j_2 \rangle$, the tendency of the calculated transitions is quite selective, which has been discussed in Ref. 12) quantitatively. The transitions within the yrast states and also within the other excited states started from the $J = 1^+$ state are enhanced, while the transitions between the different groups are very much hindered.

Similarly in the previous section, we can say that the band-like structure is obtained with the background of the quasi $\rho\nu$-symmetry. It should also be pointed out that the $\rho\nu$-signature concept is more significant than the isospin concept in the $j_\rho \neq j_\nu$ configuration. Generally the isospin $T$ has less meaning in the neutron-excess nucleus, because the low-lying states possess the isospin of $T = (N - Z)/2$ uniquely.

The generalization of the $\rho\nu$-signature to the $j_\rho \neq j_\nu$ orbital space can be extended for the $(h_{11/2})^2(i_{33/2})n^{-2}$ configuration in a similar way to that of the $\rho\nu$-signatures (Eqs. (21) and (22)). In Fig. 4 the energy levels calculated for the $(h_{11/2})^2(i_{33/2})n^{-2}$ configuration are shown together with the generalized $\rho\nu$-signatures,

$$S_{\rho\nu} = \langle \Psi_j^{(i)}(h_{11/2}, i_{33/2}) X (\rho \rightarrow \nu) \Psi_j^{(i)}(h_{11/2}, i_{33/2}) \rangle.$$  

(23)

Fig. 4. Energy spectrum of the $(h_{11/2})^2(i_{33/2})n^{-2}$ configuration and the generalized $\rho\nu$-signatures. The same interaction as in Fig. 3 is assumed. All calculated levels ($J \leq 6$) below 3.5 MeV are shown with the generalized $\rho\nu$-signatures $S_{\rho\nu}$ defined by Eq. (23).
The spectrum which has been obtained by using the same effective interactions for the \((h_{11/2})^2(i_{13/2})^n\) configuration (Fig. 3) exhibits a similar tendency to that in Fig. 1. In fact the calculated \(S_{\pi\bar{\pi}}\) are very close to either +1 or -1 and the quadrupole transitions are quite selective. Again the states can be classified according to the multiplicity \(M\) of the strong quadrupole transitions, i.e., \(0|\text{out}(S_{\pi\bar{\pi}} = 1.0) for M=0, 2^+(S_{\pi\bar{\pi}} = -0.999) for M=1, 4^+(S_{\pi\bar{\pi}} = 0.997) for M=2, 6^+(S_{\pi\bar{\pi}} = -0.989), 4^+(S_{\pi\bar{\pi}} = -0.953) and 3^+(S_{\pi\bar{\pi}} = -0.999) for M=3 and so on. Though there exists some discrepancy from Table I for the \((9/2)^2(9/2)\) case due to the simpler assumption of the configuration, essentially the situation is same as in § 4.1. Thus the quasi \(\pi\bar{\pi}\)-symmetry structure is found to be presented for the \((h_{11/2})^2(i_{13/2})^n\) system which reveals a vibration-like spectrum.

§ 5. Discussion

In § 4.1 we emphasized the qualitative similarities of the quadrupole transitions and moments between the harmonic vibrational model and the \(j^n\) shell-model wave functions in the \(\pi\bar{\pi}\)-symmetric configurations. However Table II suggests there exists a discrepancy between these two models. Some phonon states have no corresponding state in the \(\pi\bar{\pi}\)-signature scheme. Those are the 2-phonon 0+ state, 3-phonon 2+ state and 4-phonon 0+, 2+ and 4+ states. In the result of the shell model calculations in the \(\pi\bar{\pi}\)-symmetry, the \(J=0^+(+)\) state with the multiplicity \(M=2, J=2^+(-)\) with \(M=3, J=0^+(+), 2^+(+)\) and \(4^+(+)\) with \(M=4\) are missing. In this sense the present shell-model result resembles the surface oscillation model proposed by Wilets and Jean\(^{13}\) rather than the harmonic phonon model. In the Wilets-Jean model, the \(\gamma\)-instability is taken into account to the macroscopic potential for the surface oscillations. The classification of the vibrational states differs from the usual phonon model. They introduced the quantum number \(n_\gamma\), which can be interpreted as a phonon seniority number, and showed the states with \(n_\gamma=0\) become low in energy when one adopts the \(\gamma\)-independent potential. The states with \(n_\gamma>0\) are 2-phonon 0\(^+\) (\(n_\gamma=1\)), 3-phonon 2\(^+\) (\(n_\gamma=1\)) and 4-phonon 2\(^+\) (\(n_\gamma=1\), \(4^+(n_\gamma=1)\) and \(0^+(n_\gamma=2)\), which correspond to the missing states in the present shell model. Thus it is concluded that the regularity obtained in the present \(\pi\bar{\pi}\)-symmetric system has a good correspondence to the \(\gamma\)-unstable macroscopic collective model\(^{13}\) rather than to the harmonic vibrator's.

In the following we discuss why the states corresponding to the 2-phonon 0\(^+\) or 3-phonon 2\(^+\) are not obtained in the present shell model. A \(J=0^+\) state which is presumably corresponding to the 2-phonon 0\(^+\) state should be a positive \(\pi\bar{\pi}\)-signature state and be approximated by

\[
|0^+(+)> \sim \frac{1}{\sqrt{N(0)}} \{ Q|2^+(-)> \}^{0_1}
\]
The second-step approximation is valid due to the fact that the $2_1^+(-)$ state is well represented by Eq. (17). Now we consider the orthogonalization of the $0^+(+)$ state given by Eq. (24) to the other $J=0^+$ state of the positive $p\bar{n}$-signature, e.g., $J=0_{\text{g.s.}}^+(+)$. It must be noticed here that if the Hamiltonian is composed of the quadrupole interaction, i.e., $H=k(Q\cdot Q)$, the state given by Eq. (24) becomes identical to the $0_{\text{g.s.}}^+(+)$ state since the $0_{\text{g.s.}}^+(+)$ state is an eigenstate of the quadrupole Hamiltonian. Though the Hamiltonian which was used in § 4 is not purely quadrupole one, it contains strong quadrupole component in the effective proton-neutron interactions. The $0_{\text{g.s.}}^+(+)$ wave function is expected to have a large overlap with the ground state of the pure quadrupole Hamiltonian. Actually this overlap becomes 0.99. Thus it is clear now that the $0^+(+)$ state defined by Eq. (24) loses the 2-phonon character through the orthogonalization process. And the reason why the state $\{Q\cdot Q\}^{(0)}[2,(-)]$ of the 3-phonon character loses its physical significance in the shell model can be explained by a similar way. Therefore we conclude that at least within the single $j$ configurations dominance of the quadrupole component in the usual Hamiltonian naturally favours the regularities of the $\gamma$-unstable vibration model rather than those of the harmonic phonon model.

It must be added that the present classification of the levels in the $p\bar{n}$-symmetry has strong coincidence to the $O(6)$ classification proposed recently in the interacting boson model.\(^1\)\(^4\) Actually the levels listed in Table II is equivalent to the $(8,0)$ representation of the $O(6)$ group. Furthermore the fact that $O(6)$-like spectra are obtained also in the proton-particle and neutron-hole system by the interacting proton- and neutron-boson calculation\(^1\)\(^5\) may suggest the important role of the $p\bar{n}$-symmetry for the appearance of $O(6)$ spectrum.

For the collective rotational excitations, we emphasized the importance of the $pn$-symmetric structures. We, however, know more refined shell-model description of the rotation, namely the $SU(3)$ model by Elliott.\(^1\)\(^6\) The idea of $SU(3)$ can be applied to the microscopic description of the rotational motions of heavy nuclei.\(^1\)\(^7\)\(^,\)\(^8\) It is shown for some light nuclei with $N=Z$ like $^{20}\text{Ne}$ that the $pn$-symmetry exactly holds. It will be an interesting problem whether the quasi $pn$-symmetry is preserved when the pseudo $SU(3)$ coupling scheme is valid. For this purpose the generalization of the signature concepts to the more general configuration presumed for heavy nuclei is strongly recommended.

In summary we have shown the importance of the signature concepts for understanding of the collective feature which the shell-model wave functions possess. The regularity obtained in the $pn$-symmetric system is shown to have a common aspect to the band structure in the deformed nuclei. Whereas the
regularity obtained in the $pn$-symmetric system where the $pn$-signatures are defined has a strong resemblance to the vibrational mode proposed by Wilets and Jean or the $O(6)$ regularity which has been found in many observed spectra of transitional nuclei.

The author wishes to thank Professor T. Marumori for his interest and encouragement during the course of this work. He is indebted to Professor A. Arima for valuable discussion. His thanks are also due to Dr. S. Hayakawa for careful reading of manuscript and valuable comments. He also thanks the Nishina Memorial Foundation for financial support. Numerical calculations were carried out with the use of FACOM M-180IIAD at Institute for Nuclear Study of University of Tokyo.

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