Renormalization Group Approach to the Hard Mode Instability

Jun-Ichiro TADA and Satoshi TAKADA
Institute of Physics, University of Tsukuba, Ibaraki 305
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Critical properties of the hard mode instability are investigated on the basis of a simple model using the response function analogously to that in dynamical critical phenomena. It is shown that Wilson’s renormalization group method can be applied although some features are more complex in this case. Consequently, critical exponents are shown to be the same as those of the two-component $\phi^4$-TDGL model.

Recently considerable progress has been made in the study of critical properties near instabilities in non-linear systems far from equilibrium. As a consequence, it has been clarified that the critical behaviour is analogous to that in the equilibrium phase transition. These works, however, have been confined to the study of the uniform soft mode instability so far. Especially, Dewel et al.\textsuperscript{1} and Fukuyama and Mori\textsuperscript{2} have investigated Schlögl’s model\textsuperscript{3} using the method of Wilson’s renormalization group\textsuperscript{4,5} and have shown that the critical behaviour near the uniform soft-mode instability is very similar to that of the one-component $\phi^4$-TDGL model in the theory of equilibrium critical phenomena.

In this article, we shall show that the analogy holds for the hard mode instability which appears in chemical reaction system such as Brusselator,\textsuperscript{6} and that Wilson’s renormalization group method can be applied also to this case.

In order not to be encumbered by incidentals, let us introduce a simple model which exhibits only the hard mode instability,

\begin{equation}
(\partial_t - D_1 V^2) x_1(r, t) = D_1 x_1(r, t) + \omega_c x_1(r, t) - 2\omega_1 \{ x_1(r, t) \}^3 + \eta_1(r, t),
\end{equation}

where $x_1(r, t)$'s are fluctuations around a steady state, $D_1$'s are diffusion constants, $\omega_1$'s are parameters, $\omega_c$ is the hard mode (critical) frequency, and $\eta_1$'s are coupling constants of the third order non-linear processes. Here $\eta_i(r, t)$'s denote random forces\textsuperscript{7} which can be derived from reduction of fast processes.\textsuperscript{2,8} We assume for simplicity that $\eta_i(r, t)$ and $\eta_2(r, t)$ are statistically independent and their distribution are Gaussian white. An additional condition, $|\eta_i| \ll \omega_c \ (i=1, 2)$ should be satisfied so that only the hard mode instability occurs.

The linear stability analysis\textsuperscript{6} leads that the hard mode instability occurs at $\omega_c = 0$, where $\Delta = (\omega_1 + \omega_2)/2$. Existence of a limit cycle of the system can be shown for $\Delta > 0$ by reductive perturbation method.\textsuperscript{9,10}

Making Fourier transformation of variables in Eqs. (1) with respect to time and spatial coordinates, we have

\begin{equation}
G_{\omega}(\bm{k}, \omega; \Delta) x(\bm{k}, \omega) = \Theta(\bm{k}, \omega) + N(\bm{x}(\bm{k}, \omega); U, \omega),
\end{equation}

where

\begin{align*}
x(\bm{k}, \omega) &= \left( \begin{array}{c} x_1(\bm{k}, \omega) \\ x_2(\bm{k}, \omega) \end{array} \right), \\
\Theta(\bm{k}, \omega) &= \left( \begin{array}{c} \eta_1(\bm{k}, \omega) \\ \eta_2(\bm{k}, \omega) \end{array} \right),
\end{align*}

and

\begin{equation}
N(\bm{x}(\bm{k}, \omega); U, \omega) = \left( \begin{array}{c} N_1(\bm{x}(\bm{k}, \omega); U, \omega) \\ N_2(\bm{x}(\bm{k}, \omega); U, \omega) \end{array} \right).
\end{equation}
\[ N_{\alpha}(x; \mathbf{k}, \omega; U) = (2\pi)^{d-1} U_{\alpha\alpha} \]
\[ \times \int d^d \mathbf{k}' \; d\omega' \; d^d \mathbf{k}'' \; d\omega'' \; s_{\alpha}(\mathbf{k}'', \omega'') \]
\[ \times x_{\alpha}(\mathbf{k}', \omega') \; s_{\alpha}(\mathbf{k} - \mathbf{k}' - \mathbf{k}'', \omega - \omega' - \omega''). \]

Here Greek suffices take values 1 or 2, the exponent \( d \) represents the spatial dimensionality of the system, and non-linear vertex \( U \) has the following elements:

\[ U_{\alpha\beta} = \begin{cases} -2U, & \text{for } \alpha = \beta = \gamma = \delta = 1, \\ -2U, & \text{for } \alpha = \beta = \gamma = \delta = 2, \\ 0, & \text{otherwise.} \end{cases} \]

The free propagator \( G_0(\mathbf{k}, \omega; \mathcal{D}) \) in Eq. (2) is given by

\[ G_0(\mathbf{k}, \omega; \mathcal{D}) = \left( i\omega + D_k k^2 - \Delta_k \right)^{-1}, \]

where \( \lambda_\alpha(\mathbf{k}; \mathcal{D}) \) are given up to the order of \( k^2 \) as

\[ \lambda_\alpha(\mathbf{k}; \mathcal{D}) \sim \Delta - c k^2 + \left\{ \frac{\omega_\perp^2 - (D_1 + D_2)^2}{8\omega_\perp} + \left( D_1 - D_2 \right) \Delta \right\}. \]

Here suffix \( D \) denotes a value in the new representation and \( c = (D_1 + D_2)/2 \).

To investigate the effect of non-linear terms on the critical behaviour, we adopt the procedure of renormalization group. Thus true response function \( \delta G(\mathbf{k}, \omega; \mathcal{D}) \) satisfies the following Dyson’s equation:

\[ \delta G^{-1}(\mathbf{k}, \omega; \mathcal{D}) = \delta G_0^{-1}(\mathbf{k}, \omega; \mathcal{D}) - \Sigma(\mathbf{k}, \omega; \mathcal{D}, U; \Lambda), \]

where \( \Lambda \) denotes cutoff wave number of the order of the inverse diffusion length \( \lambda \) and \( \mu = \left\{ \mu_1 + \mu_2 \right\} \cdot \text{Tr} \mathbf{R}/4 \). Here \( \mathbf{R} \) is the correlation matrix of random force \( \theta(\mathbf{k}, \omega) \).

As the system has non-vanishing hard mode frequency at the instability point, the first and the second diagonal elements of the free propagator \( \delta G_0(\mathbf{k}, \omega; \mathcal{D}) \) cannot diverge at the same time; the former diverges at \( \omega = \omega_\perp \) and \( k = \mathcal{D} = 0 \), while the latter does at \( \omega = -\omega_\perp \) and \( k = \mathcal{D} = 0 \). Similarly the first and second diagonal elements of true response function in the diagonal form have a singularity at \( k = 0, \mathcal{D} = \mathcal{D}' \) for \( \omega = \omega_\perp \) and \( \omega = -\omega_\perp \) respectively, where \( \mathcal{D}' \) is the renormalized critical value of \( \mathcal{D} \) to be determined. As is easily seen, the parameter \( \mathcal{D} \) in the present model plays the same role as the temperature in the equilibrium phase transition. Thus critical exponents are defined as follows:

\[ \xi \propto (\mathcal{D} - \mathcal{D}')^{1-\nu}, \]

\[ \delta G_{11}(0, \omega; \mathcal{D}) = \delta G_{22}(0, -\omega; \mathcal{D}) \propto (\mathcal{D} - \mathcal{D}')^{-\nu}. \]

\[ \delta G_{11}(\mathbf{k}, \omega; \mathcal{D}'') = \delta G_{22}(\mathbf{k}, -\omega; \mathcal{D}'') \propto \left( \frac{\omega - \omega_\perp}{\omega_\perp} \right)^{1-\nu}. \]

where \( \xi \) is the coherence length of the system. In the classical theory, these exponents have values \( \nu = 1/2, \gamma = 1 \) and \( \eta = 2 \).

In the present analysis, the Kadanoff transformation \( \mathbf{R}_b \) and the scale transformation \( \mathbf{R}_s \) are applied to the response function directly, i.e.,

\[ \mathbf{R}_s \delta G^{-1}(\mathbf{k}, \omega; \mathcal{D}) = \delta G^{-1}(\mathbf{k}, \omega; \mathcal{D}') - \Sigma(\mathbf{k}, \omega; \mathcal{D}', \mathbf{R}_s), \]

\[ \mathbf{R}_s \mathbf{k} = b \mathbf{k}, \quad \mathbf{R}_s \omega = b^2 \omega, \quad \mathbf{R}_s \mathcal{D} = b \mathcal{D}. \]
Fig. 1. (a) An example of critical vertices. Numbers fixed on arrows show elements of free propagators. (b) An example of non-critical vertices. (c) Examples of effective four point vertices consisting of pairs of three point vertices connected with non-critical propagators. Here wavy lines represent non-critical propagators. These two terms renormalize the critical vertex shown in Fig. 1(a).

\[ R_{\gamma} u = b^\gamma u, \quad R_{\gamma} G = b^{-\gamma} G, \]

where tilders denote renormalized values, \( \Sigma \) denotes the long range part of the self-energy in which the wave number of the all internal propagators run \( 0 < k < A/b \) in integrations, and \( b \) is an arbitrary constant greater than unity. Note that we assume throughout this paper that \( \Lambda \xi > 1. \)

It should be noted that there appear most-divergent diagrams and less-divergent diagrams in the same order of interaction \( u. \)

The most-divergent diagrams are what include only critical vertices: Only two kinds of elements in the non-linear vertex \( U_{11,11,10} \) and \( U_{10,1,10} \) which have the same value \( - u \) at \( k = A = 0 \) are critical vertices, where brackets in the suffix represent permutations of suffixes. Examples of critical and non-critical vertices are shown in Figs. 1(a) and 1(b) respectively. In the most-divergent diagrams, examples of which are shown in Fig. (3), \( \omega_c \)-dependence in those contributions can be removed by appropriate shift of external frequencies \( \omega \) to \( \omega \pm \omega_c \). It is convenient to introduce the critical propagators defined by \( G_c(\mathbf{k}, \omega; \Delta) = \Sigma G(\mathbf{k}, \omega - \omega_c, \Delta) = G(\mathbf{k}, \omega + \omega_c; \Delta) \) which does not include \( \omega_c \) as a parameter and diverges at \( k = \omega = D = 0. \)

Thus by shifting external frequencies \( \omega \) to \( \omega \pm \omega_c \), e.g., \( \Sigma_1(\mathbf{k}, \omega \pm \omega_c, \Delta, U, L) \), we have the most divergent diagrams which consist of only critical vertices and critical propagators and do not have \( \omega_c \) as a parameter. In less-divergent diagrams we cannot remove \( \omega_c \) dependence by shifting the external frequencies and there remains at least one non-critical propagator such as \( G_c(\mathbf{k}, \omega \pm \omega_c, \Delta) \) which does not diverge at \( k = \omega = D = 0. \)

Thus far as critical properties are concerned, only the most-divergent terms are dominant in the vicinity of the instability point, and hence only those terms relate to the renormalization group procedure.

Requiring scale invariance for coefficients of \( k^2 \) and \( \omega \) in the critical propagator without performing Kadanoff transformation, we obtain \( z = 6 = g = 2. \) The scale transformation gives \( \nu = 4 - d \), which shows that the critical dimensionality of the system is 4. This is the same result as that in the equilibrium phase transition in the \( \phi^4 \)-TDGL model.

It should be emphasized that non-critical propagators need not be scaled, since they remain finite at the instability point where the coherence length of the system diverges. This leads that the critical dimensionality of the hard mode instability in those models containing second order non-linear processes such as Brusselator and Oregonator also becomes 4, because at least one of the propagator connected to a three point vertex
Fig. 2. (a) Diagrams which renormalize \( u \). A broken line denotes \((-2c\xi^d)^{-1}\) where \( \Lambda/b < q < \Lambda \) is the wave number of the line. The self-energy parts Eqs. (3) having two internal lines with short wave numbers and one line with a long wave number reduce to the long range self energy part \( \Sigma_{2} \) of the type of Fig. 3(b) with the resormalized vertex. (b) A diagram which renormalizes \( \Lambda \). must be non-critical one due to conservation law. In other words, a pair of three-point vertex connected with a non-critical propagator acts as an effective four-point critical vertex as shown in Fig. 1(c). In contrast with the hard mode instability, those systems containing second order non-linear processes in equilibrium critical phenomena such as the \( \phi^4 \)-TDGL model has the critical dimensionality equal to 6.12

First we carry out the renormalization group analysis of non-linear coupling constant \( u \). We extract terms proportional to \( \beta u \equiv \ln b \) from two kinds of diagrams shown in Fig. 2(a) representing vertex corrections. Similarly the renormalization group analysis of the parameter \( \Lambda \) is carried out on the diagram shown in Fig. 2(b). The results are summarized into following equations:

\[
\frac{\beta u}{\beta \Lambda} = u \left( \epsilon - \frac{5K_{4}}{2c^{2}u} \right),
\]

where \( \epsilon = 4 - d \) and \( K_{4} = (8\pi^{2})^{-1} \). It is seen from Eqs. (6) that \( u \) and \( \Lambda \) have non-zero value \( u^{*} = 2c^{2}/5K_{4} \) and \( \Lambda^{*} = \epsilon c A^{d}/5 \) at the fixed point which is stable for \( d < 4 \). Linearizing Eqs. (6) around the non-zero fixed point, we obtain the critical exponent \( \nu \) up to the order of \( \epsilon \) as \( \nu = 1/2 + \epsilon/10 \).

There are three kinds of diagrams shown in Fig. 3(a) which contribute to the exponent \( \eta \), and each types of these three diagrams contribute twice according to the way of contractions of inner lines. Straightforward calculation of \( \epsilon \)-expansion analysis gives \( \eta = \epsilon^{2}/50 \). The exponent \( \eta' \) is obtained as \( \eta' = \epsilon^{2}/100 \) using the same set of diagrams. Similarly, with the diagram shown in Fig. 3(b), the critical exponent \( \gamma \) is obtained as \( \gamma = 1 + \epsilon/5 \). These values of critical exponents are the same as those of the

Fig. 3. (a) Three kinds of diagram which contribute to \( \eta \) and \( \eta' \). (b) The diagram which contributes to \( \gamma \).
two-component $\phi^4$-TDGL model in the theory of equilibrium critical phenomena. So far as critical properties are concerned, the two systems show the same behaviour and have the same critical indices.

In this paper we have investigated critical properties of the hard mode instability on a simple model. There are three consequences obtained throughout this work. First, it is shown that the critical behaviour of the hard mode system can be investigated by Wilson's renormalization group method, although the system has non-vanishing critical frequency at the critical point. Second, existence of non-critical propagators which do not contribute to the singular behaviour of the hard mode system at the instability point is shown. The appearance of these non-critical propagators is the most characteristic property in the hard mode system. In case of equilibrium phase transition, a system containing second order non-linear processes such as the $\phi^3$-TDGL model has the critical dimensionality 6, while the critical dimensionality of a hard mode system containing second order non-linear processes such as Brusselator and Oregonator is lowered to 4 due to the presence of non-critical propagators. Third, as the result of Wilson's renormalization group and $\varepsilon$-expansion methods, it is shown that the critical exponents in the hard mode instability are exactly the same as those of the two-component $\phi^4$-TDGL model in equilibrium phase transition within the lowest order in $\varepsilon$. This may not be surprising, since there exist two conjugate critical modes in case of the hard mode instability.