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A Method of Calculating the Hydraulic Properties of Leaky Aquifer Systems

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This paper describes a method for calculating the hydraulic properties of esker aquifers where a leakage to the aquifer is induced by pumping. The method is an extension of the channel method described in an earlier paper. As an example of the applicability of the method a short description of a performed pumptest is given.

Notation

B	(m)	Channel width
BP'/m'	(m/s)	Channel leakage coefficient
E	(m)	Leakage factor
$D(w)$		Drain function
$D(w,x/E)$		Drain function for a leaky channel
Q	(m^3/s)	Pumpage from a well
r	(m)	Distance from pumping well
s	(m)	Draw down
S	($m^3/m.m.^2$)	Storage coefficient
P'	(m/s)	Permeability of semiconfining bed
SB	(m)	Channel storage
t	(min)	Pumping time
T	(m^2/s)	Transmissivity
TB	(m^3/s)	Channel conductivity
$W(u)$		Well function
x	(m)	Parallell distance from pumping well

Introduction

The standard methods of evaluation of the hydraulic properties of an aquifer based on data from a non-steady state pumping test are mainly applicable to two dimensional aquifers. Methods to evaluate hydraulic boundaries and other inhomogenities are exsisting but are very difficult to apply if the aquifer geometry leads to infinite series of boundaries generated by the image well theory.

Fig. 1 shows a somewhat simplified profile across an esker, which can be defined as a ridge-shaped deposition of glaciofluvial sand and gravel. In its most typical form it is deposited below the highest shore level and surrounded by fine grained (clay-silt) sea or lake sediments. The coarse sediments of the esker normally have very high permeability.

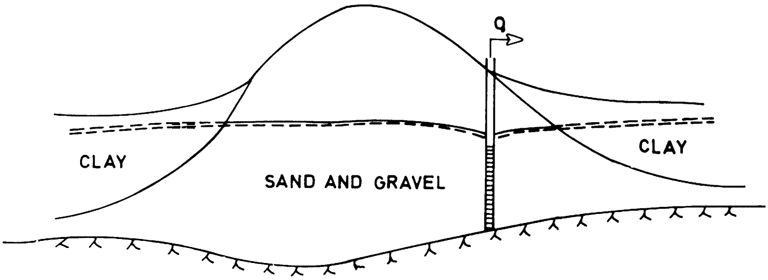


Fig. 1. Typical profile of an esker.

The Hydraulic Properties of an Esker

As can be seen from Fig. 1, the geological features of an esker, some difficulties will arise in the determination of the hydraulic properties. The reasons for this can be summarized as follows:

1. *Unconfined aquifers.* The central part of the esker is often unconfined which will introduce the problem of delayed yield during pumpage.

2. *High transmissivity.* The high transmissivity of an esker will give a very fast spreading of the cone of depression and consequently a fast reaction from the boundaries.

3. *Inhomogenities.* The homogenity across the esker is normally poor and difficult to judge. Along the esker the homogenity frequently is good.

4. *Hydraulic boundaries.* The esker is always bounded by two more or less parallell hydraulic boundaries which will introduce an infinite series of image wells to the analysis of the hydraulic properties.

The difficulties caused by these conditions are primarily that the influence from the boundaries will appear during the delayed yield period which will give problems to separate the different mirror wells. Secondly also the mirror wells will be influenced by delayed yield.

Groundwater Flow in an Esker by Pumpage from a Well

The width of an esker is normally about 300-600 m but the length of its individual groundwater basins can frequently extend to several kilometres. The flow in the aquifer can therefore be regarded as one-dimensional except in the vicinity of the pumpage well.

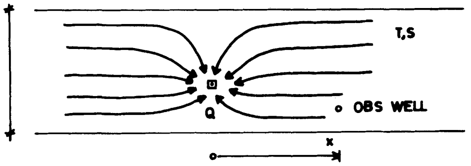


Fig. 2. Flow in an esker by pumpage from a well.

As it is shown by Gustafson (1974) the well can be replaced by a drain across the esker for observation wells that are sufficiently distant.

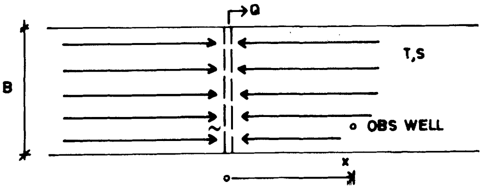


Fig. 3. Auxiliary hydraulic system for calculating the hydraulic properties of an esker.

The development of a flow equation for the auxiliary system can be made under the following assumptions:

The aquifer is homogeneous, isotropic and of infinite extension, the discharging drain penetrates the aquifer completely, the aquifer is bounded by impermeable strata above and below, the flow is laminar and unidimensional, the release of water from storage is instantaneous and in proportion to the decline in head, and the drain discharges water at a constant rate. Under these condition the flow in the auxiliary system leads to the following differential equation by nonsteady state conditions:

$$\frac{\partial^2 h}{\partial x^2} = \frac{S}{T} \frac{\partial h}{\partial t} \qquad (1)$$

$$h(x,0) = h(\infty,t) = h_0 \qquad (2)$$

$$\frac{\partial h}{\partial x}(0,t) = -\frac{Q}{2TB} \qquad (3)$$

This equation has been solved by Edelman (Huisman 1972) and Ferris (1949). The solution will be of the following kind:

$$s = \frac{1}{2\sqrt{\pi}} \cdot \frac{Qx}{TB} \cdot D(w) \qquad (4)$$

$$w = \frac{x^2 S}{4Tt} \qquad (5)$$

$$D(w) = \frac{e^{-w}}{\sqrt{w}} - \sqrt{\pi} + 2\int_0^{\sqrt{w}} e^{-x^2}\,dx \qquad (6)$$

Leaky Eskers

The described theory of channel formed aquifers is derived under the assumption that the parallell boundaries of the aquifer are impervious. This is however oftenly not the case. The draw down caused by the pumpage from the esker will normally induce a leakage from the finegrained sediments that form the aquifer boundaries, se Fig. 4.

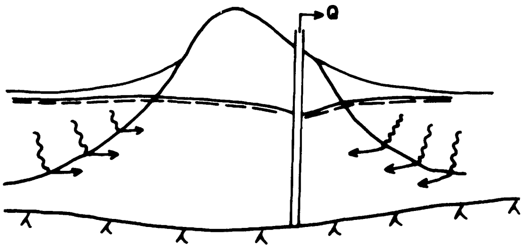


Fig. 4. Induced leakage by pumpage in an esker.

Another frequent situation is that the esker forms the bank of a river or a lake and that the pumpage induces a leakage from the surface water, se Fig. 5.

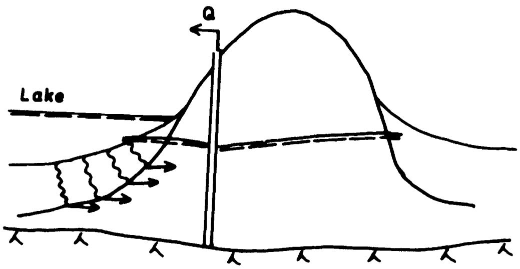


Fig. 5. Induced leakage from lake or stream.

The development of a flow equation for a leaky esker system can be made under the following assumptions:

- The aquifer is homogeneous, isotropic and of infinite extension.
- The discharging drain penetrates the aquifer completely.
- The aquifer is bounded by strata with a constant permeability considerably lower than the permeability of the aquifer thus giving a flow perpendicular to the groundwater flow of the aquifer.
- The flow is laminar and unidimensional.
- The release of water from storage is instantaneous and in proportion to the decline in head.
- The drain discharges water at a consant rate.
- The head of the reservoir from which leakage takes place is constant.

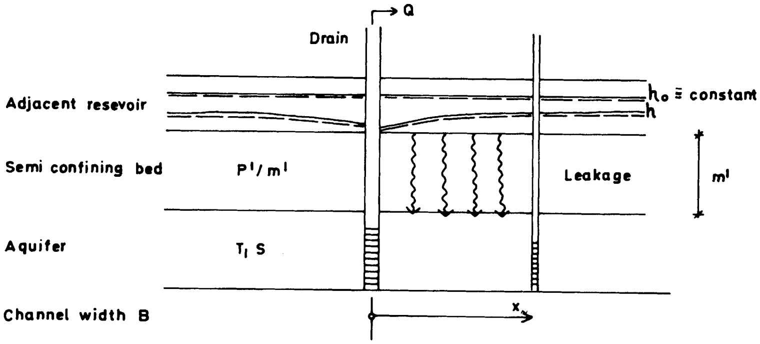


Fig. 6. Derivation of the drain function for leaky channel aquifers.

By introducing the leakage factor, $E = \sqrt{Tm^1/P^1}$, the flow in the leaky channel leads to the following differential equation by non steady state conditions:

$$\frac{\partial^2 s}{\partial x^2} - \frac{s}{E^2} \equiv \frac{S}{T} \cdot \frac{\partial s}{\partial t} \tag{7}$$

$$s(x,0) = s(\infty,t) = 0 \tag{8}$$

$$\frac{\partial h}{\partial x}(0,t) = -\frac{Q}{2TB} \tag{9}$$

The solution of this equation will be of the following kind:

$$s \equiv \frac{1}{2\sqrt{\pi}} \cdot \frac{Qx}{TB} \cdot D\left(w,\frac{x}{E}\right) \tag{10}$$

$$w = \frac{x^2 S}{4Tt} \tag{11}$$

$$D\left(w,\frac{x}{E}\right) \equiv e^{-(x/E)^2/4w}\left[D(w) + \int_0^{t^1} e^{\tau} \cdot D(x^1, t^1 - \tau)\, d\tau\right] \tag{12}$$

$$x^1 = \frac{x}{E} \tag{13}$$

$$t^1 \equiv \frac{(x/E)^2}{4w} \tag{14}$$

$$D(w) \equiv \frac{e^{-w}}{\sqrt{w}} - \sqrt{\pi} + 2\int_0^{\sqrt{w}} e^{-x^2}\, dx \tag{15}$$

The function $D(w, x/E)$ is hereafter called the drain function for leaky aquifers. It has been tabulated by the author and is given in appendix 1.

Steady State Conditions

After some time of pumping a steady state will occur because of the leakage. Under this condition Eq. (7) reduces to:

$$\frac{\partial^2 s}{\partial x^2} - \frac{s}{E^2} = 0 \tag{16}$$

The solution of the steady state equation, se ao Huisman (1972), will be of the following kind:

$$s = \frac{Q \cdot E}{2TB} \cdot e^{-x/E} \tag{17}$$

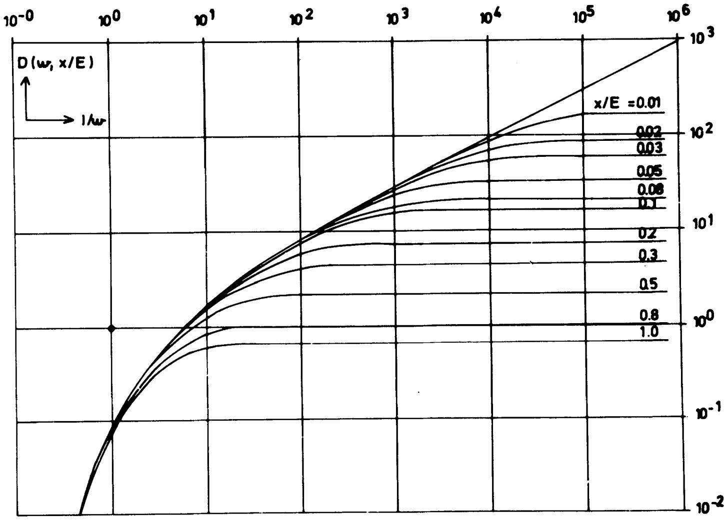


Fig. 7. Non-steady state type curves for leaky channel flow.

$$E \equiv \sqrt{\frac{T}{P^1/m^1}} \tag{18}$$

This equation will form a straight line in a semi-logarithmic diagram.

Evaluation of The Hydraulic Properties of a Leaky Esker

For practical use the following redefinition of the leakage factor can be made:

$$E = \sqrt{\frac{TB}{BP^1/m^1}} \tag{19}$$

The draw down can then be calculated as:

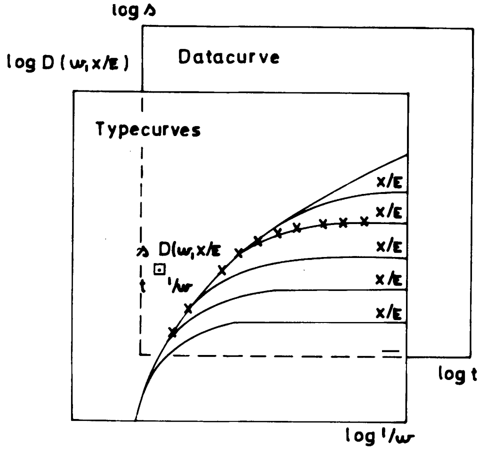
$$s = \frac{1}{2\sqrt{\pi}} \cdot \frac{Qx}{TB} D\left(w, \frac{x}{E}\right) \tag{20}$$

$$w = \frac{x^2 SB}{4TB \cdot t} \tag{21}$$

$$\frac{x}{E} = \sqrt{\frac{x^2 BP^1/m^1}{TB}} \tag{22}$$

Time-Draw Down Analysis

In the same manner as for non leaky systems a logarithmic plot of draw-down versus pumping time with the same log scale as the type curve is made. The determination of the hydraulic properties can then be made by curve fitting in the normal way, see Fig. 8.



$$TB \equiv 0.282\,\frac{Q\,x}{s}\cdot D(w,\,x/E)$$

$$SB = \frac{240\,TB\cdot t}{x^2}\cdot w$$

$$BP'/m' = \frac{(x/E)^2\cdot TB}{x^2}$$

Fig. 8. Time-draw down analysis for a leaky channel aquifer.

If pumping time is written in minutes and pumpage in m^3/s the hydraulic properties of the aquifer can be calculated by the formulas given in Fig. 8.

Steady-State Analysis

The draw down at steady state for a channel aquifer can be calculated with the Eqs. (20) and (21). In order to make the evaluation of the hydraulic properties a semi-logarithmic plot of draw down data for different observation wells versus distance to the pumping well is made, see Fig. 9.

The draw down in the aquifer at steady state can be calculated as:

$$s = \frac{Q\cdot E}{2TB}\cdot e^{-x/E} \qquad (23)$$

The distance at which the draw down is 10 times greater can be calculated as:

$$10\,s \equiv \frac{Q\cdot E}{2TB}\cdot e^{-x_{10}/E} \qquad (24)$$

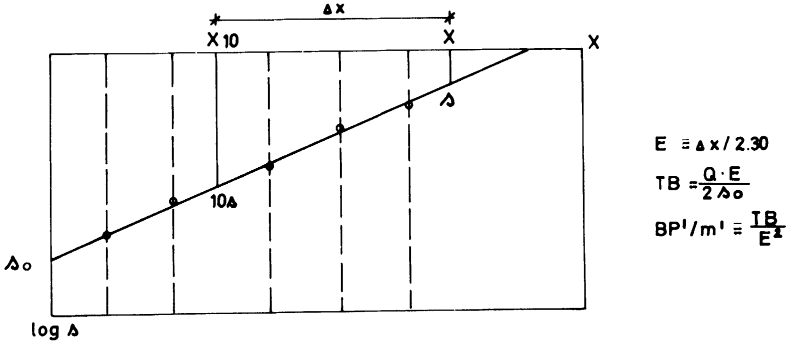


Fig. 9. Steady-state evaluation for a leaky channel aquifer.

Rearrangement of the equations give:

$$10\,e^{-x/E} \equiv e^{-x_{10}/E} \tag{25}$$

$$\ln 10 \equiv \frac{x-x_{10}}{E} = \frac{\Delta x}{E} \tag{26}$$

$$E \equiv \frac{\Delta x}{2.30} \tag{27}$$

Putting $x = 0$ gives:

$$s_0 = \frac{Q \bullet E}{2TB} \tag{28}$$

$$TB = \frac{Q \bullet E}{2s_0} \tag{29}$$

Validity of the Leakage Coefficient

The interpretation of the leakage coefficient, BP^1/m^1, for the channel aquifer is difficult to make as the leakage normally is not vertical. The original definition - vertical permeability divided by thickness of the semiconfining bed - is not fulfilled since the leakage normally takes place from the sides of the esker. Since the esker can be considered as a one dimensional system this will not affect the analysis. The leakage coefficient can therefore be considered as a constant characterizing the contact between the aquifer and the adjacent reservoir.

Pumping Test at Snöån, Smedjebacken, Kopparbergs County, Sweden

In order to illustrate the application of the above described theory a short description of a pumping test in a leaky esker will be given.

Hydrogeology

The investigated area is a part of the esker Malingsboåsen. The esker is located to the rather narrow valley of the lakes Saxen and Haggen, and further south it completely fills the lower parts of the valley of the small river Saxån.

Fig. 10 shows a geological map of the investigation area. The esker has a north-west south-east direction and forms the south west bank of the lakes Haggen and Saxen. The esker normally shows as a 5-10 m high ridge along the lake. Between the observation wells Rb 7402 and Rb 7405 it is covered by fine grained sediments. The drillings have shown that the thickness of the coarse sediments of the esker core is about 35 m in the vicinity of the well and that the esker is homogenous in the area.

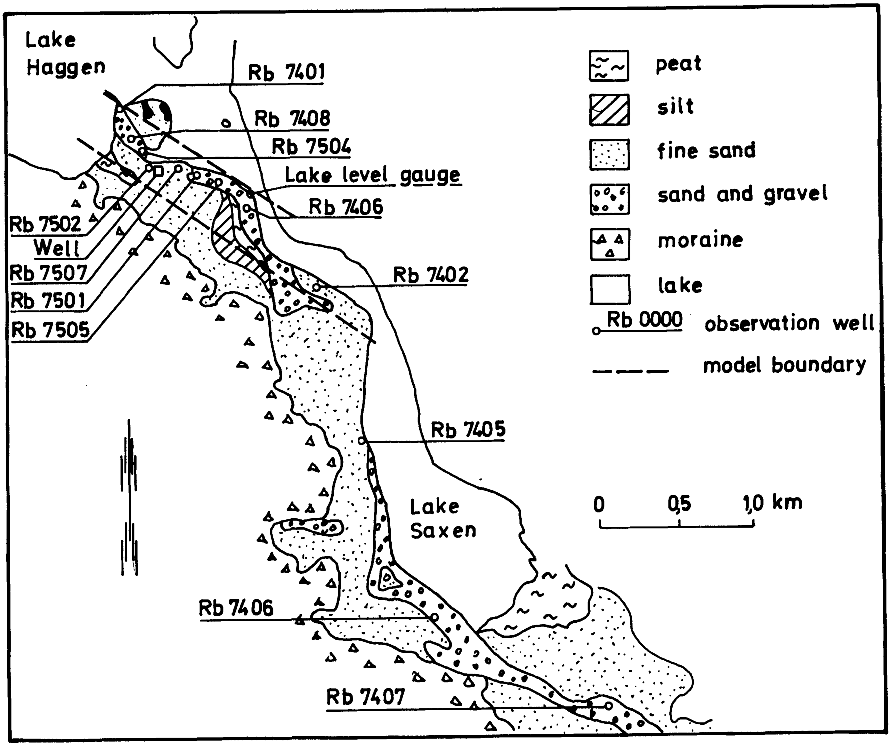


Fig. 10. Geological map of the Snöån area, scale 1:50.000. Mainly after SGU (1937).

The ground water level corresponds to the level of the lakes, about 150.4 m a.s.l., under undisturbed conditions. However the measured levels show a slight gradient towards north-west along the esker caused by the groundwater flow from the esker areas further south.

The pumping well was constructed in the spring 1975. It has a depth of 34 m and a total screen length of 14 m. The drilling diameter is 800 mm and the well has a gravel pack filter. The maximum yield of the well is about 0.130 m^3/s.

Pumping Test

In October 1975 a pumping test with a duration of 12 days was performed using the new well. The pumpage was held constant at 108.5 l/s during the test except for a two hour stop on the third day.

Observation Network

During the hydrogeological investigation of the area totally 15 drillings were made. Of these wells 12 are situated in the main aquifer and were used for the evaluation of pumping test data. The locations of these wells are shown in Fig. 10. The properties of the observation wells are given in Table 1.

Table 1

Obs well	Distance from well r(m)	Bench mark level +h(m)	Total depth of aquifer d(m)	Groundwater level +h(m)	Total draw down s(m)
Rb 7401	470	155.54	14.9	150.05	0.586
Rb 7402	1,420	152.61	27.2	150.25	0.179
Rb 7405	2,190	154.29	>29.0	150.52	0.069
Rb 7406	3,820	160.37	>20.8	154.18	0.069
Rb 7407	4,700	163.42	16.3	156.12	0.029
Rb 7408	230	152.49	>25.8	150.08	0.711
Rb 7501	270	154.58	>31.5	150.16	0.672
Rb 7502	22,1	156.71	34.0	150.12	0.949
Rb 7503	1,5	156.76	36.0	150.12	1.032
Rb 7504	160	152.38	>23.0	150.06	0.740
Rb 7505	450	157.53	>17.0	149.56	0.878
Rb 7506	620	152.41	>31.0	150.20	0.584
Rb 7507	152	154.71	>13.5	150.15	0.750
Well	0	156.17		150.12	4.313
Lake Haggen		152.12		150.44	

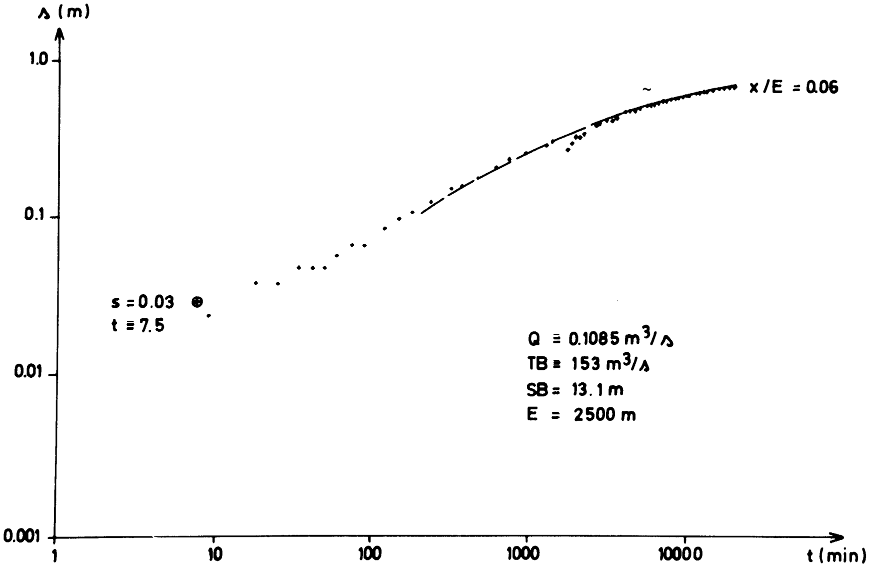


Fig. 11. Data curve for observation well Rb 7504.

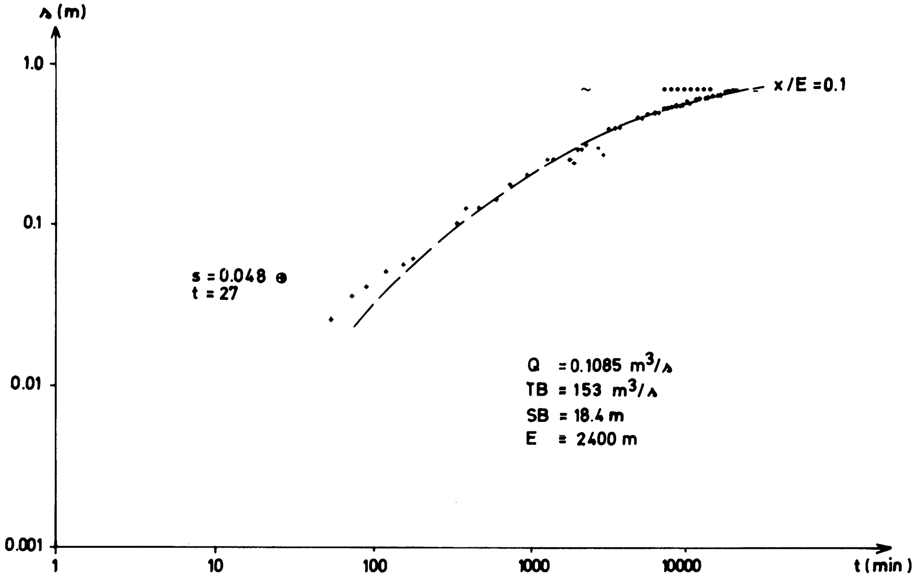


Fig. 12. Data curve for observation well Rb 7501.

The observation well Rb 7505 was bent during the drilling thus giving misleading data.

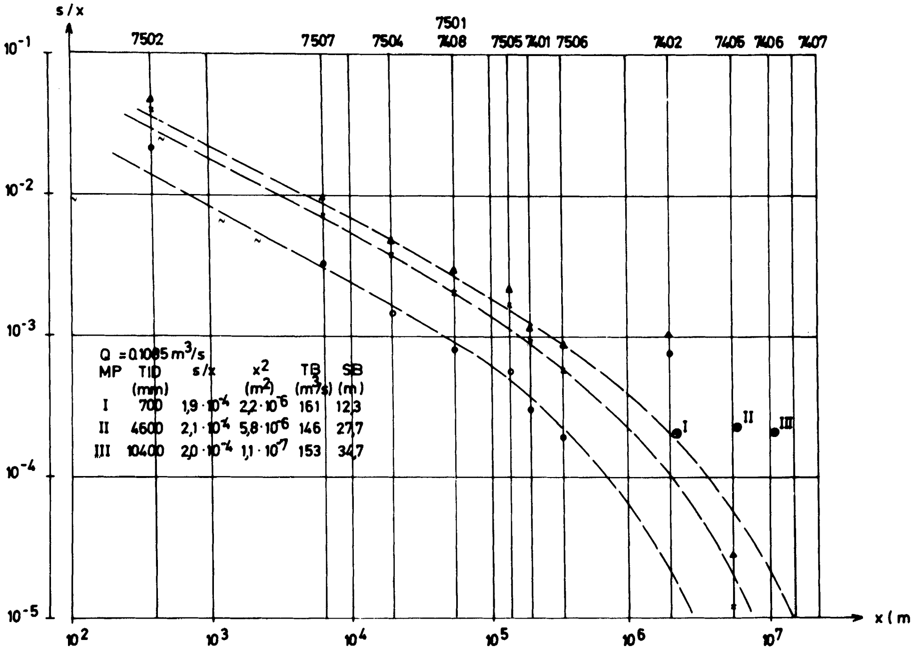
Evaluation of the Hydraulic Properties

The data curves for the draw down period follow the drain function for leaky aquifers except for a delayed yield period of about 200 minutes. Figs. 11 and 12 show two typical datacurves and the interpreted corresponding typecurves.

The evaluated hydraulic properties for the different observation wells give all practically the same results; *TB* = 150 m³/s, *SB* = 12 - 20 m, *E* ≡ 1,800 - 2,800 m, thus indicating a homogenous aquifer in the vicinity of the well.

The from the pumping well distant observation wells Rb 7405-7407 give higher values for the channel storage and Rb 7407 also gives a higher value of the channel conductivity. This is caused by the change in the hydraulic properties of the esker that takes place at the southern end of lake Saxen.

Fig. 13 shows a distance draw down analysis of pump test data for three different pumping times. As can be seen from the figure the channel storage *(SB)* virtually increases with time. This indicates that a leakage to the aquifer takes place. This can also be seen from the data curves, where data deviates from the drain function towards a steady state.



Q = 0.1085 m³/s

MP	TID (min)	s/x	x^2 (m²)	TB (m³/s)	SB (m)
I	700	$1{,}9 \cdot 10^{-4}$	$2{,}2 \cdot 10^{-6}$	161	12,3
II	4600	$2{,}1 \cdot 10^{-4}$	$5{,}8 \cdot 10^{-6}$	146	27,7
III	10400	$2{,}0 \cdot 10^{-4}$	$1{,}1 \cdot 10^{-7}$	153	34,7

Fig. 13. Distance draw-down analysis of pump test data.

However a steady state was not reached during the pumping test. If the leakage curves are extended to steady state conditions an evaluation of the final draw downs can be made, see Fig. 14.

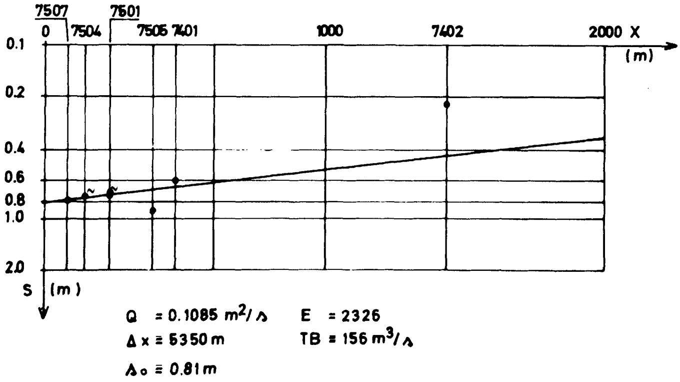


Fig. 14. Evaluation of the hydraulic properties at steady state.

As can be seen from Fig. 14 this analysis gives corresponding values of the hydraulic parametres. A summary of calculated values is given in appendix 2.

Mathematical Model

Since the hydraulic properties of the aquifer are known a check of the found values by a mathematical model can be made. The geometry of the final model is shown in Fig. 10. The draw downs have been calculated for 5.160, 13.350, and 20.155 minutes of pumpage and $TB = 150$ m^3/s, $SB = 13$ m and $E = 2.000$ m and a channel width of 375 m. The results are given in Table 2.

The results show a good agreement between calculated and measured draw downs except for the distant observation well Rb 7402. Therefore the found hydraulic properties of the aquifer can be used for prediction of future draw downs caused by pumpage.

Table 2

Obs well	Distance from well	5,160 min Draw down		13,350 min Draw down		20,115 min Draw down	
	r(m)	calculated s(m)	measured s(m)	calculated s(m)	measured s(m)	calculated s(m)	measured s(m)
Rb 7401	470	0.46	0.41	0.55	0.54	0.61	0.59
Rb 7408	230	0.53	0.50	0.65	0.65	0.69	0.71
Rb 7504	160	0.55	0.54	0.68	0.68	0.70	0.74
Rb 7507	152	0.60	0.53	0.73	0.68	0.34	0.76
Rb 7501	270	0.50	0.76	0.64	0.61	0.68	0.67
Rb 7506	620	0.40	0.39	0.53	0.52	0.54	0.58
Rb 7402	1,420	0.28	0.06	0.29	0.14	0.41	0.18

Summary of the Analysis

The analysis of pump test data from Snöån, Smedjabacken, has shown that the esker has a channel conductivity $TB = 150$ m^3/s and a channel storage $SB = 13$ m. The esker that forms the south west bank of the lakes Haggen and Saxen is hydraulically connected to the lakes by leakage, thus inducing a recharge from the lakes when pumping. The leakage factor is determined to $E = 2.000$ m. A calculation of future draw downs has shown that the future water demand for the community of Smedjebacken 5.500 m^3/d can be withdrawn from the area without difficulty.

Acknowledgment

The Author wishes to thank Mr. Björn Engqvist, Uppsala, who assigned the solution of the differential equation for leaky channel aquifers.

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Appendix 1

w/x/E	0	0.01	0.02	0.03	0.05	0.08	0.1	0.2	0.3	0.5	0.8	1.0
0.00001	314.46	170.85	86.87	57.33	33.72	20.45	16.04	7.25	4.38	2.15	0.996	0.620
2	221.84	155.26	86.54									
3	180.81	140.59	85.90									
4	156.35	128.75	84.56	57.15								
5	139.66	119.23	82.79	57.08								
6	127.33	111.44	80.82	56.90								
7	117.76	104.93	78.78	56.62								
8	110.04	99.41	76.76	56.25								
9	103.64	94.65	74.80	55.80								
0.00010	98.24	90.49	72.92	55.30								
0.0001	98.24	90.49	72.92	55.30	33.62							
2	68.95	66.11	58.75	49.43	33.26							
3	55.98	54.42	50.16	44.30	32.25	20.40						
4	48.25	47.23	44.38	40.27	30.98	20.33	16.01					
5	42.97	42.24	40.16	37.09	29.68	20.19	15.99					
6	39.08	38.52	36.92	34.51	28.44	19.98	15.96					
7	36.05	35.61	34.33	32.37	27.29	19.73	15.90					
8	33.61	33.25	32.19	30.57	26.23	19.44	15.81					
9	31.59	31.29	30.40	29.02	25.57	19.14	15.71					
0.0010	29.88	29.62	28.86	27.67	24.38	18.32	15.56					
0.001	29.88	29.62	28.86	27.67	24.38	18.82	15.56					
2	20.63	20.54	20.27	19.83	18.52	15.90	14.02	7.24				
3	16.54	16.49	16.34	16.10	15.36	13.78	12.56	7.17	4.38			
4	14.10	14.07	13.97	13.81	13.32	12.25	11.38	7.03	4.37			
5	12.44	12.42	12.35	12.23	11.88	11.09	10.43	6.85	4.36			
6	11.21	11.20	11.14	11.06	10.79	10.17	9.65	6.66	4.34			
7	10.26	10.25	10.21	10.14	9.92	9.43	9.01	6.45	4.31			
8	9.50	9.49	9.45	9.36	9.22	8.81	8.46	6.25	4.27			
9	8.86	8.85	8.83	8.78	8.63	8.29	7.99	6.06	4.23			
0.010	8.33	8.32	8.30	8.25	8.13	7.83	7.57	5.87	4.18			
0.01	8.33	8.32	8.30	8.25	8.13	7.83	7.57	5.87	4.18	2.15	0.996	
2	5.44	5.44	5.43	5.41	5.37	5.26	5.17	4.47	3.60	2.11	0.995	
3	4.17	4.17	4.17	4.16	4.14	4.08	4.03	3.64	3.10	2.01	0.993	
4	3.43	3.43	3.42	3.42	3.40	3.37	3.33	3.08	2.71	1.88	0.985	
5	2.92	2.92	2.92	2.92	2.91	2.88	2.86	2.67	2.40	1.76	0.971	
6	2.55	2.55	2.55	2.55	2.54	2.52	2.50	2.37	2.16	1.64	0.951	
7	2.27	2.27	2.27	2.27	2.26	2.24	2.23	2.12	1.96	1.53	0.927	
8	2.04	2.04	2.04	2.04	2.03	2.02	2.01	1.92	1.79	1.44	0.901	
9	1.86	1.86	1.86	1.86	1.85	1.84	1.83	1.76	1.65	1.35	0.872	0.620
0.10	1.70	1.70	1.70	1.70	1.70	1.69	1.68	1.62	1.52	1.27	0.844	0.610
0.1	1.701	1.701	1.701	1.701	1.699	1.688	1.680	1.619	1.525	1.266	0.844	0.610
2	0.896	0.896	0.896	0.896	0.895	0.893	0.890	0.872	0.843	0.758	0.591	0.476
3	0.575	0.575	0.575	0.575	0.574	0.573	0.572	0.564	0.551	0.510	0.425	0.361
4	0.402	0.402	0.402	0.402	0.402	0.401	0.401	0.396	0.389	0.366	0.316	0.277
5	0.295	0.295	0.295	0.295	0.295	0.295	0.294	0.292	0.287	0.273	0.242	0.216
6	0.224	0.224	0.224	0.224	0.224	0.224	0.223	0.223	0.219	0.209	0.189	0.171
7	0.174	0.174	0.174	0.174	0.174	0.174	0.174	0.172	0.170	0.164	0.149	0.137
8	0.137	0.137	0.137	0.137	0.137	0.137	0.137	0.136	0.135	0.130	0.120	0.111
9	0.110	0.110	0.110	0.110	0.110	0.110	0.110	0.109	0.108	0.105	0.097	0.091
1.0	0.089	0.089	0.089	0.089	0.089	0.089	0.089	0.088	0.088	0.085	0.080	0.075
1	0.089	0.089	0.089	0.089	0.089	0.089	0.089	0.088	0.088	0.085	0.080	0.075
2	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.014	0.014
3	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003
4	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Appendix 2

Calculated Hydraulic Properties for Different Observation Wells

Time-draw down analysis

Obs well	Distance to well r(m)	Channel Conductivity TB(m^3/s)	Channel Storage SB(m)	Leakage Factor E(m)
7401	470	153	12,7	2110
7402	1420	154	58,5	1787
7404	1760	141	107	2187
7405	2190	137	59,5	2150
7406	3820	135	37,6	4000
7407	4700	299	48,7	4900
7406	230	156	12	2350
7501	270	153	18,4	2400
7502	22,1	-	-	-
7503	1,5	-	-	-
7504	160	153	13,1	2500
7505		153	11,2	2800
7506	620	153	13,2	2000
7507	152	149	13,1	2500

Distance-draw down analysis

700 min		161	12,3	
4600 min		146	27,7	
10400 min		153	34,7	

Steady state analysis		156		2326

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