Edge-Bonded Dissimilar Orthogonal Elastic Wedges Under Normal and Shear Loading

J. DUNDURS. The author has presented a most interesting paper on the adhering elastic quarter-planes that are subjected to specified surface tractions. It is especially refreshing to see that he was not intimidated by the great complexity of the solution in the physical parameters, and that explicit results are given. The problem considered belongs to the class of propositions in plane elastostatics for which the stress field shows a reduced dependence on elastic constants. It is possible, for this reason, to achieve a modest simplification in the results and, what may be more important, to put some of the interpretations on a physically more tangible basis.

When a composite body consisting of two isotropic and elastic phases is loaded by prescribed surface tractions, the stress field generally depends on three parameters formed from the elastic constants. For instance, the ratio of the shear moduli and the Poisson’s ratios of the two phases could be used for this purpose. However, when the geometry and the loading of the body is such that it is in a state of plane deformations, and when the vector sums of tractions formed as line integrals vanish on all individual holes in the body, the stress induced by prescribed surface tractions depends on only two combinations of the elastic constants. It is, of course, understood that the prescribed surface tractions are specified independently of the elastic constants. A remarkable feature of this result is that it holds regardless of the topological intricacies of the composite body, and that it matters not what mechanical conditions are imposed at the interfaces between the phases, be it full adhesion or frictionless slip. The result is noted, finally, that the conditions which must be imposed for the composite body are precisely the same as those under which the stress field in a homogeneous body is independent of Poisson’s ratio. Thus, in both cases, there is a reduction by one in the dependence on elastic constants.

The choice of the two composite parameters to be formed from the elastic constants is not unique, and the writer believes now that the parameters employed in the original paper were not the best possible. Consequently, new parameters will be proposed and used in this discussion.

It is expedient to treat both plane strain and generalized plane stress at the same time by introducing the constant \( \kappa = 3 - 4v \) for plane strain and \( \kappa = (3 - 2v)/(1 + v) \) for plane stress, where \( v \) denotes Poisson’s ratio. Using subscripts 1 and 2 on the elastic constants to distinguish between the two phases, and setting \( \Gamma = G_1/G_2 \), where \( G \) is the shear modulus, the composite parameters proposed here are

\[
\alpha = \frac{\Gamma(k_1 + 1) - (k_2 + 1)}{\Gamma(k_1 + 1) + k_1 + 1}, \quad \beta = \frac{\Gamma(k_1 - 1) - (k_2 - 1)}{\Gamma(k_1 + 1) + k_1 + 1}.
\]

The parameters \( \alpha \) and \( \beta \) admit a direct physical interpretation of sorts. Hooke’s law specialized for plane deformations is

\[
e_{ij} = (1/2G)\left[ \sigma_{ij} - \frac{1}{4}(3 - \kappa)\sigma_{kk}\delta_{ij} \right], \quad (i, j = 1, 2),
\]

and it follows that

\[
e_{ii} = (\kappa - 1)/(2G) \left( \frac{1}{2} \sigma_{ii} \right) = A \left( \frac{1}{2} \sigma_{ii} \right).
\]

Moreover, for \( \sigma_{ii} = 0 \), we have

\[
e_{ii} = (\kappa + 1)/2G\sigma_{ii} = C\sigma_{ii},
\]

The constants \( A \) and \( C \) defined by (3a) and (4a) could be called a real compliance and uniaxial compliance, respectively. For plane stress, the uniaxial compliance is equal to the inverse of Young’s modulus. Using the compliances,

\[
\alpha = (C_1 - C_2)/(C_1 + C_2), \quad \beta = (A_1 - A_2)/(A_1 + A_2).
\]

When the labeling of the two phases is inverted, and the new parameters denoted by \( \bar{\alpha} \) and \( \bar{\beta} \), it follows from (1a) that \( \bar{\alpha} = -\alpha \) and \( \bar{\beta} = -\beta \). Therefore such an inversion corresponds to a reflection through the origin in the \( \alpha, \beta \)-plane.

The \( \alpha, \beta \)-plane provides a convenient means for classifying composite materials according to their physical behavior and for exhibiting results such as, for example, stress-concentration factors that depend on the elastic constants. Because of the physical limits \( \Gamma > 0 \) and \( 1 \leq \kappa \leq 3 \), the admissible values of \( \alpha \) and \( \beta \) are restricted to a bounded region, or more specifically a parallelogram, in the \( \alpha, \beta \)-plane, as shown in Fig. 1. The case of equal shear moduli, or \( \Gamma = 1 \), corresponds to the straight line \( \alpha = \beta = 0 \), and identical materials are represented by \( \alpha = \beta = 0 \). It should be noted that, for generalized plane stress, the constant \( k \)
is restricted to the range \( \frac{1}{3} \leq \kappa \leq 3 \), and the corresponding parallelogram is contained in the parallelogram for plane strain shown in Fig. 1. This means that physical behavior is better studied for plane strain because, conceivably, there could be phenomena exhibited by plane strain that do not occur in plane stress. Also the extreme values of a physical quantity for plane strain are likely to fall outside the range for plane stress.

The three families of curves \( \Gamma = a = \text{const}, \kappa_1 = b = \text{const} (k_1 \text{ variable}) \); \( \Gamma = a, \kappa_2 = c = \text{const} (k_2 \text{ variable}) \) and \( \kappa_1 = b, \kappa_2 = c = \text{const} (\Gamma \text{ variable}) \) are sets of straight lines in the \( a, \beta \)-plane. The lines \( \Gamma = a, \kappa_1 = b \) radiate from \( \alpha = \beta = -1 \), and those corresponding to \( \Gamma = a, \kappa_2 = c \) from \( \alpha = \beta = 1 \). If the parallelogram is covered with a net of the three types of loci, one can visualize at a glance the possible values of \( \Gamma, \kappa_1, \) and \( \kappa_2 \) for given \( \alpha \) and \( \beta \). For a fixed value of \( \Gamma \), say, \( \Gamma = a \), the possible values of \( \alpha \) and \( \beta \) are restricted to a rather narrow quadrangle. The boundaries of such a quadrangle are formed by the combinations of \( \Gamma = a \) with the extreme values of \( \kappa_1 \) and \( \kappa_2 \). One diagonal of the quadrangle is the line \( \alpha = \text{const} \) corresponding to \( \Gamma = a, \kappa_1 = \kappa_2 \). The \( \kappa_1 = b, \kappa_2 = c \) lines for specific values of \( b \) and \( c \) are shown in Fig. 2.

The \( \Gamma \)-zones for \( \Gamma > 1 \) also are indicated in Fig. 2, where the particular values of \( \Gamma \) have been selected so that no overlapping occurs. Because of the property \( \alpha = -\alpha, \beta = -\beta \), the corresponding zones for the reciprocals of \( \Gamma > 1 \) also are indicated in Fig. 2, where the par­

Fig. 2

Fig. 3

Fig. 4

The nature of the singularity at the vertices of the quarter­planes is best discussed with the aid of the \( a, \beta \)-diagram. The transition from a \( (1/r^\gamma) \)-type singularity with \( \gamma > 0 \), to what is most likely a logarithmic singularity, is specified by (50) in the paper. This condition is equivalent to

\[ \alpha(\alpha - 2\beta) = 0. \]  

The \( (1/r^\gamma) \)-type singularities are restricted to combinations of materials for which \( \alpha(\alpha - 2\beta) > 0 \), or materials such that the \( \alpha, \beta \)-values do not fall within the shaded regions shown in Fig. 3. A display of results, preferable to Fig. 3 in the paper, would be the loci of \( \sigma_1 + 2 = \text{const} \) in the \( a, \beta \)-plane. Some of the necessary values for these loci can be extracted from Figs. 3 and 4 in the paper, and a rough sketch of the curves is shown in Fig. 3 here. Due to the fact that the author considered only plane stress in the numerical calculations, merely parts of the curves \( \sigma_1 + 2 = \text{const} \) can be obtained and indicated. As is seen from Fig. 3, values of \( \sigma_1 + 2 \) larger than the 0.311 quoted by the author can be expected for plane strain.

Condition (50) implies either \( \alpha = 0 \) or \( \alpha = 2\beta = 0 \). Combinations of materials for which this is true have a special signif­

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by stretching the materials perpendicular to the interface without having to load, say, material 2, parallel to the interface. Such materials could be called consonant in tension perpendicular to the interface. Thinking about a straight boundary perpendicular to the interface in Fig. 4, analogous statements about the consonant materials also could be made in the context of quarter-planes. It is precisely the consonant combinations of materials at which the transition from a $(1/r^3)$-type to a logarithmic singularity takes place. The combinations of materials for which $\alpha(\alpha - 2 \beta) > 0$ are characterized by the following facts: The material, which carries the higher tensile stress for tension parallel to interface, must be compressed parallel to the interface when both materials are subjected to tension perpendicular to the interface. In contrast, this material would have to be stretched for $\alpha(\alpha - 2 \beta) < 0$.

The writer hopes that the author could be induced to work out and report some additional results, either in the framework of this discussion or in a later publication, in order to throw more light on this interesting problem. One desirable new result ought to be the study of the nature of the singularity for materials with $\alpha(\alpha - 2 \beta) < 0$. The other would be to present more explicit results for some specific cases of loading. In particular, concentrated unit loads for which the fields are Green’s functions of the present problem could be suggested for such an investigation. Thus it would be interesting to see more than a one-term asymptotic expansion in place of (35) and (36) in the paper. Furthermore, an interesting question would also be to find the fractions of the unit loads that are transmitted by the individual quarter-planes to the “foundation at infinity.”

**Author’s Closure**

The author thanks Professor Dundurs for writing this very informative discussion and also for making available an early copy of his manuscript. The reduction in the algebraic complexity of the solution when cast in terms of his two composite parameters $\alpha$, $\beta$, which are related to the three parameters $k$, $m'$, $m''$, used in the paper by

$$\alpha = \frac{km'' - m'}{km' + m'}, \quad \beta = \frac{k(m'' - 2) - (m' - 2)}{km' + m'},$$

is quite significant. This simplification has been largely responsible for the author’s interest in pursuing the problem much further. As a result many additional investigations have been completed, including all of those suggested in the last paragraph of the discussion, and they will be presented in a forthcoming paper.

**Finite Deflections and Snap-Through of High Circular Arches**

D. A. DeDEPPO\(^1\) and R. SCHMIDT.\(^2\)

The author’s numerical technique yields remarkably accurate results. The writers have also been investigating the same problem and have calculated many exact values by means of elliptic integrals of the first and second kinds.\(^4\) For example, for an inextensional arch with $h/L = 0.25$, the exact value of the critical sidesway load $P$ is 13.0002 $EI_K$\(2\), as compared to the author’s 13.01 $EI_K$\(2\) ($Q = 33.3$) and Lind’s 13.4 $EI_K$\(2\) [2, 3]. It might also be of interest to note that the classical eigenvalue theory of buckling of arches at small deflections [4] yields the value 12.4 $EI_K$\(2\), which is only 4.6 percent lower than the exact value. This good agreement between the exact and “classical” values might be explained by the near constancy of the thrust $-N$ along the arch just before the sidesway.

Further calculations have indicated that the theory in [2] or [3] predicts higher critical loads than those obtained by the exact theory of inextensional elastic arches, for all height-to-span ratios. The classical eigenvalue theory is erratic: The error displays positive and negative signs. However, for arches shallower than the semicircle, both approximate theories yield fair estimates of the critical load.

The exact values of the critical load $P$ and of the corresponding vertical deflection of the crown of the arch are presented in Table 1 for different values of the subtending angle $2\phi$.

**Table 1**

<table>
<thead>
<tr>
<th>$\phi$, deg</th>
<th>$P/EI_K$</th>
<th>$K_\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>66.81</td>
<td>0.0035</td>
</tr>
<tr>
<td>20</td>
<td>33.62</td>
<td>0.0141</td>
</tr>
<tr>
<td>30</td>
<td>22.66</td>
<td>0.0227</td>
</tr>
<tr>
<td>40</td>
<td>17.15</td>
<td>0.0597</td>
</tr>
<tr>
<td>50</td>
<td>13.82</td>
<td>0.0953</td>
</tr>
<tr>
<td>60</td>
<td>11.44</td>
<td>0.1353</td>
</tr>
<tr>
<td>70</td>
<td>9.462</td>
<td>0.1711</td>
</tr>
<tr>
<td>80</td>
<td>7.617</td>
<td>0.1921</td>
</tr>
<tr>
<td>90</td>
<td>5.861</td>
<td>0.1946</td>
</tr>
<tr>
<td>100</td>
<td>4.287</td>
<td>0.1828</td>
</tr>
<tr>
<td>110</td>
<td>2.975</td>
<td>0.1607</td>
</tr>
<tr>
<td>120</td>
<td>1.944</td>
<td>0.1334</td>
</tr>
<tr>
<td>130</td>
<td>1.179</td>
<td>0.1026</td>
</tr>
<tr>
<td>140</td>
<td>0.617</td>
<td>0.0732</td>
</tr>
<tr>
<td>150</td>
<td>0.307</td>
<td>0.0454</td>
</tr>
<tr>
<td>160</td>
<td>0.113</td>
<td>0.0222</td>
</tr>
<tr>
<td>170</td>
<td>0.023</td>
<td>0.0062</td>
</tr>
</tbody>
</table>

The passage of the curve in Fig. 3 of the paper through the origin is an interesting phenomenon which is not just a coincidence. A plot of the same quantities for the arch with $\phi_0 = 10$ deg exhibited the same kind of behavior, i.e., for $P = 0$, $\delta = 0$, with $\Delta = 0.00214/K_\Delta$. One possible explanation of this phenomenon could be based on the following reasoning. Let us imagine that the crown point of the arch is constrained to move along the vertical line through point $C$, i.e., no sideways is permitted. As the load $P$ is increased, the crown moves downward and the arch remains symmetrical until the symmetrical configuration becomes unstable and the arch snaps into an asymmetrical equilibrium configuration, which can be maintained with no external load. It is conceivable that the same buckled, asymmetrical configuration may be reached along a different configuration-space path, when sideways is permitted.

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\(^2\) Department of the Theory of Materials, University of Sheffield, Sheffield, England.
