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The Polar Decomposition And Vector Parametrization Of The Mueller Matrices

V.A. Dlugunovich and Yu.A. Kurochkin

B.I.Stepanov Institute of Physics of the National Academy of Sciences of Belarus, 68 Nezalezhnasci Ave., Minsk, Belarus 220072

Abstract. It was demonstrated that presentation of the coherent matrix (polarization density matrix) of the electromagnetic beams as biquaternion corresponding to the four-vector of the pseudo Euclidean space with intensity and Stokes parameters as components gives the possibility for introducing of the group transformations of such values isomorphic to the S(3.1) group. These transformations are the subset of the set of polarization Mueller matrices creating algebraic structure of semigroup. Reduction of the semigroup of Mueller matrices to the group of transformations make it possible to use the vector parameterization of transformations of the group SO(3.1) for interpretation of polar decomposition of the Mueller matrices. In this approach in particular the elements of Mueller matrices corresponding to retarders and polarizers are more simple and natural connected with there eigenpolarizations.

Keywords: Stokes parameters, Mueller matrices, biquaternions, vector-parameter, polar decomposition, group, semigroup, transformations

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INTRODUCTION

There are different methods of the description of the polarization of the electromagnetic waves that are connected to each other anyhow [1–5]. One of the more popular is the method based on the Stokes vector parameters of radiation and the Mueller matrixes of the investigated objects [5, 6]. This method becomes very important recently due to the wide application of Stokes and Mueller polarimetry in medicine [7, 8], biology [9], nondestructive testing [10, 11] and remote sensing of materials and objects [12, 13].

The polarization measurements play very important role in the astronomy and astronomies has been contributed to the creation of the instruments and devices for polarizing experiments [14-17].

As well known the four-by-four Mueller matrix **M** is defined as the matrix which transforms an incident Stokes vector **S** into the exiting (reflected, transmitted, or scattered) Stokes vector **S'**

$$S^{'} = MS\,, \tag{1}$$

or in expanded form

$$\begin{pmatrix} S_0^{'} \\ S_1^{'} \\ S_2^{'} \\ S_3^{'} \end{pmatrix} = \begin{pmatrix} m_{00}m_{01}m_{02}m_{03} \\ m_{10}m_{11}m_{12}m_{13} \\ m_{20}m_{21}m_{22}m_{23} \\ m_{30}m_{31}m_{32}m_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}. \tag{2}$$

By its physical nature the Mueller matrix optionally possesses inverse matrix and therefore in general their set did not form a group, but form a semigroup. But application of the Mueller matrix in many cases shows the possibility and effectivity of restriction semigroup of this matrix to a group [18–22]. The methods of the theory group and its representations are the mathematical apparatus of symmetry physical theories that is thoroughly developed and very effective at the solving of many physical problems.

We think that efficiency of the reduction of a number of transformations (2) to a group

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appears particularly significant for the interpretation of polar decomposition of the Mueller matrix that is developed in detail for the optical problems [23] and is wider applied in the optical calculations. Such decomposition has evident physical interpretation as optical, and from the kinematic points of view as it will be shown below.

The aim of this work is the interpretation of the polar decomposition of the Mueller matrix based on the results of [24] in terms of the vector parameterization of the group of transformations SO(3.1) isomorphic to the Lorentz group that is valid for the problems appeared in the optic.

QUATERNION PRESENTATION OF COHERENCE MATRIX AND GROUP OF MUELLER AND JONES MATRICES

In the work [24] on the basis of presentation of the matrix

$$\gamma^0 = \begin{pmatrix} |E_1|^2 & E_1E_2^* \\ E_2E_1^* & |E_2|^2 \end{pmatrix}, \gamma^0_{ab} = E_aE_b^* \qquad (3)$$

in the form of decomposition over the matrix basis formed from the unitary 2×2 matrix and Pauli matrices

$$\gamma^0 = \frac{1}{2}S_0\sigma_0 + \frac{1}{2}S_1\sigma_1 + \frac{1}{2}S_2\sigma_2 + \frac{1}{2}S_3\sigma_3 \qquad (4)$$

where

$$y_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (5)$$

are the Pauli matrices, and coefficients S_0, S_1, S_2, S_3 are the Stokes parameters, the transition to the arbitrary basis of the quaternion algebra is realized

$$y_0 \to e_0 \quad 1, y_{\overline{k}} \to ie_k, \; k = 1,2,3, \qquad (6)$$

that is determined by the next relationships between the introduced algebraic units [20]

$$e_je_k = \varepsilon_{jkl}e_l - \delta_{jk}e_0, e_0e_k - e_ke_0 = 0, \quad e_0^{\,2} = e_0, (j,k,l = 1,2,3). \qquad (7)$$

In the basis (7) one may find for the matrix (4)

$$\gamma^0 = S_0e_0 - iS_ke_k.$$

In the sequel instead of γ^0 in definition (8) we will use the expression that is differed from γ^0 only by multiplying on imaginary unit and symbolized by S. The expression

$$S = iS_0e_0 + S_ke_k, \qquad (8)$$

is a biquaternion that is the vector of the four-dimensional pseudo-Euclidean space [25]. The Stokes parameters are the components of this vector. Such biquaternions are distinguished from number of biquaternions (in this case of biquaternions determined over set of complex numbers) of general form by condition

$$\overline{S}^* = -S. \qquad (9)$$

In the expression (9) the overline denotes the quaternion conjugate:

$$\overline{S} = iS_0 - S_ke_k, \qquad (10)$$

and asterisk denotes usual complex conjugation

$$S^* = -iS_0 + S_ke_k.$$

At this space biquaternions transformations are determined

$$S' = AS\overline{A}^*, \qquad (11)$$

that as well known forms the group of transformations isomorphic to the group of SO(3.1) [20] when biquaternions *A* are followed the condition

$$A\overline{A} = 1, A^*\overline{A}^* = 1. \qquad (12)$$

The expression (11) may be presented in matrix form in the four-dimensional space. For that it is enough the biquaternions *S* and *S'* represent as a column vector whose components are Stokes parameters [20]. Thus the transformation (11) is a form (special case) of the transformation (1), (2) that is prescribed by Mueller matrix. The fact that biquaternions (or matrices) are used is not contradicted to the reality of the Stokes parameters. As well known the formulation of the transformations of rotations at the pseudo-Euclidean space what kind are the transformations (11) may be simply changed in terms of objects and structures on real numbers. They are such substantially.

The transformation (11) is the representation of transformations of group separated in the semigroup of 2×2 Jones matrices. The Jones matrices are determined the transformation of the complex components of the electric-field vector $E_a (a = 1,2)$ in the phase plane [1, 5]

$$\vec{E}' = A\vec{E}, \begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \quad (13)$$

The 2×2 dimension matrix A may be decomposed on the basis (5), (7) and be represented as biquaternion, as it was done in the case of matrix (3) and (4).

At the reduction of the set of the Jones matrices to the group of $R_0^1 \otimes SL(2.C)$ its presentation (13) necessary to consider as fundamental «spinor» while the transformation (11) in the space of biquaternions (3), (4) is realized «vector» presentation of SL(2.C) group. The complex eigenvalues $\zeta\exp(i\delta)$ of matrix A also presented as a polar decomposition of a complex number conforms to the so eigenpolarizations of the optical elements described by this Jones matrix A that coefficient ζ determines the amplitude (its squared modulus is the intensity) of the wave transmitted by the optical element, and coefficient $\exp(i\delta)$ obviously determines the phase change or retardance of a device. The diattenuation of a device D characterizes the dependence of the intensity transmittance of an element from the incident polarization state and is determined by [6, 18]

$$D = \frac{\left|T_q - T_r\right|}{\left|T_q + T_r\right|} = \frac{\left|\left|\xi_q\right|^2 - \left|\xi_r\right|^2\right|}{\left|\xi_q\right|^2 + \left|\xi_r\right|^2}, \quad 0 \le D \le 1. \quad (14)$$

The retardance R is related with a phase change of the transforming waves and defined as

$$R = \left|\delta_q - \delta_r\right|, \quad 0 \le R \le 1. \quad (15)$$

Polarization elements as polarizers and retarders are characterized by diattenuation D and by phase change due to birefringence R. For ideal polarizer $D = 1$ meaning complete attenuation (by absorption or reflection) of one of the polarized component of the transmitted radiation.

VECTOR PARAMETERIZATION OF THE TRANSFORMATIONS OF SO(3.1) GROUP AND ITS POLAR DECOMPOSITION

For matrix A known polar decomposition means its presentation as a product of the unitary matrix U and Hermitian matrix H.

For vector representations using the same terminology as for the group of transformations of rotations of four-dimensional pseudo-Euclidean space SO (3.1) (Lorentz group isomorphic to a group of transformations (11)) one can said that this decomposition corresponds to the presentation of the transformation in the form of a product of the transformation of a pure rotation and the transformation of the «boost» type, in other words pure Lorentz transformation.

Polar decomposition has very clear physical meaning, because each of the factors in this decomposition there is a certain optical element, namely, for the factor corresponding to Jones unitary transformation (13) or rotation in the Muller transformation (11) there is a retarder, for factor corresponding to Hermitian factor in the Jones transformation or Hermitian (symmetric) factor in the Muller transformation there is a polarizer.

We will present the results of polar decomposition of the Mueller matrices and its physical interpretation obtained in [18] with the help of vector parameterization and noted the undeniable advantage of this approach.

Recall that a vector parameterization [25] of the transformations of group SO (3.1) (11) defined with biquaternions (12) is introduced by the following expressions

$$A = \frac{1 + \vec{q}}{\sqrt{1 + (\vec{q}^{\,2})}}, \vec{q} = \frac{A - \overline{A}}{A + \overline{A}},$$
$$A^* = \frac{1 + \vec{q}^{\,*}}{\sqrt{1 + (\vec{q}^{\,*2})}}, \vec{q}^{\,*} = \frac{A^* - \overline{A^*}}{A + \overline{A^*}}. \quad (16)$$

where $\vec{q} = \vec{a} + i\vec{b}, \vec{q} = q_k e_k;\ e_k (k = 1,2,3)$ are the basal quaternion elements. Here for the product of biquaternions

$$A'' = A(\vec{q}\,'') = A'A = A(\vec{q}\,')A(\vec{q}) \quad (17)$$

the formula of the composition of vector-parameters is correspond

$$\vec{q}\,'' = \langle \vec{q}\,', q\rangle = \frac{\vec{q}\,' + \vec{q} + \vec{q}\,'\times\vec{q}}{1-(\vec{q}\,'\vec{q})}. \qquad (18)$$

Relations similar to (16) – (18) obviously occurred for the complex-conjugate parameters.

The transformation (11) is introduced as a $\vec{q}$ and $\vec{q}^{\,*}$ dependent 4×4 matrix [26] in the following way

$$L(\vec{q},\vec{q}^{\,*}) = \frac{1}{\left|1+\vec{q}^{\,2}\right|}\times$$

$$\begin{pmatrix} 1+|q|^2 & -(\vec{q}-\vec{q}^{\,*})-[\vec{q}\vec{q}^{\,*}] \\ \vec{q}-\vec{q}^{\,*}-[\vec{q}\vec{q}^{\,*}] & 1-\left|q^2\right|+(\vec{q}+\vec{q}^{\,*})^{\times}+\vec{q}\cdot\vec{q}^{\,*}+\vec{q}^{\,*}\cdot\vec{q} \end{pmatrix} \qquad (19)$$

presented in a block form where the point between vectors denotes a matrix-dyad, and oblique cross over vector denotes a matrix dual to this three-dimensional vector $(\vec{q}^{\,\times})_{ik} = \varepsilon_{ijk} q_j$. In turn vector-parameter $\vec{q}$ is expressed through the matrix $L(\vec{q},\vec{q}^{\,*})$.

$$\frac{1}{2}(\vec{q}+\vec{q}^{\,*}) = \frac{L-\tilde{L}}{L_t} = \begin{pmatrix} \vec{a}^{\times} & i\vec{b} \\ -i\vec{b} & 0 \end{pmatrix}, \qquad (20)$$

where $\tilde{L}$ is a matrix transposed to a matrix L, and L_t is spur of matrix $\tilde{L}$.

VECTOR-PARAMETER OF THE MUELLER MATRICES AND EIGENPOLARIZATIONS OF POLARIZERS AND RETARDERS

In the work [23] which provides an interpretation of the polar decomposition of Mueller matrices are introduced the vector $\vec{D}$ specified the attenuation direction, and the vector $\vec{R}$ specified the retardation direction. The vector $\vec{D}$ is defined as the product of the diattenuation D and the Stokes vector characterizing the eigenpolarizations of a polarizer. This vector is a vector part of the Stokes 4-vector (biquaternion) normalized to the intensity of the incident radiation $s = (i, \vec{s})$. Here

$$\vec{D} = D\vec{s}, \quad (\vec{s}^{\,2}) = 1, \qquad (21)$$

and corresponding Mueller matrix of such optical device introduced in [23] is the «boost» multiplied on the transmittance T_u for unpolarized light. The function of vector $\vec{D}$ is similar to the role of speed in the transformations used in the special relativity. Purely imaginary vector-parameter $i\vec{u}$ of this transformation is expressed by standard practice through a vector $\vec{D}$ (see [26]), namely

$$\vec{u} = \frac{\vec{D}}{1+\sqrt{1-(\vec{D}^2)}}. \qquad (22)$$

The corresponding Mueller matrix can be expressed in accordance with the formula (19) when was adopted $\vec{q} = i\vec{u}$. And vice versa on the base of the measured Muller matrix for an ideal polarizer it is possible to find the corresponding vector-parameter by using (20), and hence the Stokes vectors corresponding to the eigenpolarizations of the polarizer.

The vector $\vec{R}$ characterizing the retardation effects (phase change) is determined for phase plates in which the axes of its eigenpolarizations are specified by the Stokes 4-vector $r = (i, \vec{r})$ as

$$\vec{R} = R\vec{r}, (\vec{r}^{\,2}) = 1. \qquad (23)$$

The corresponding Mueller matrix is the transformation matrix of a pure rotation forming the subgroup in the group of transformations SO(3.1). Real-valued parameter $\vec{c}$ is determined this 4×4 matrix of the transformations in the following way

$$\vec{c} = \vec{r}tg\frac{R}{2}. \qquad (24)$$

There is no difficulty in understanding that the Stokes 4-vectors $S_D = (i, \pm\vec{D})$ and $S_R = (i, \pm\vec{R})$ are the eigenvectors of the Mueller matrices determined by vector-parameters (22) and (24) respectively.

In the first of them symbol «plus» corresponds to the maximum transmittance and symbol «minus» corresponds to the minimum transmittance of a polarizer, in the second case symbol «plus» corresponds to the fast axis of the eigenpolarizations of a retarder and symbol «minus» corresponds to the slow axis.

The Mueller matrix of the optical element consisted with a retarder and polarizer is the product of

$$M = T_u L(\vec{c}) L(i\vec{u}) \quad T_u L(\langle \vec{c}, i\vec{u} \rangle), \qquad (25)$$

where $\vec{q} = \langle \vec{c}, i\vec{u} \rangle$ is a composition (18) of the vector-parameters (22) and (24).

Here are some consequences of the formulation of the polar decomposition of the Mueller matrices in the vector parameterization.

It is known that the Mueller matrix of a series of phase plates is the product of the Mueller matrices of the each of phase plate. Using vector parameterization with the composition (18) of the vector-parameters (24) one can find a common vector-parameter of the system independently of the mutual orientation of the axes correlated to the axes of the eigenpolarizations. In a view of the group properties of such an operation it should be that the beam transformation will be associated only with a phase change.

At the combination of a series of polarizers with the different orientation of the directions of the eigenpolarizations it should be observed not only the effect of the attenuation of the transmitted radiation but the phase change that is similar to the Thomas precession in relativistic mechanics. This is the consequence of the ungrouped properties of the composition of the pure imaginary vectors (22) corresponding to the product of the Mueller matrix of each polarizer. This thesis is in accordance with the well-known Jones theorem [27] claiming that for a light with the fixed wavelength a random set of partial polarizers (linear amplitude devices) and ratardes is equivalent to an optical system with one retarder and one rotated partial polarizer. As it was mentioned above this thesis is true for a set of partial polarizers too, moreover the developed above approach gives a simple method of calculation of the eigenpolarizations of the equivalent optical system. We will demonstrate this by the example of a system consisting of a two consistent partial polarizers with the eigenpolarizations prescribed by the Stokes vectors according to $S^{(1)} = (i, D^{(1)}\vec{s}^{(1)})$ and $S^{(2)} = (i, D^{(2)}\vec{s}^{(2)})$. Then the vector-parameters $-i\vec{u}_1$ and $-i\vec{u}_2$ for each of the polarizers is determined according to (22) as

$$\vec{u}_1 = \frac{\vec{D}^{(1)}}{1+\sqrt{1-(\vec{D}^{(1)})^2}}, \ \vec{u}_2 = \frac{\vec{D}^{(2)}}{1+\sqrt{1-(\vec{D}^{(2)})^2}}, \qquad (26)$$

where $\vec{D}^{(1)} = D^{(1)}\vec{s}^{(1)}$, $\vec{D}^{(2)} = D^{(2)}\vec{s}^{(2)}$, and $(\vec{s}^{(1)})^2 = (\vec{s}^{(2)})^2 = 1$.

The vector-parameter of the equivalent system according to the composition law (18) corresponding to the product of the Mueller matrix of two partial polarizers as described above will be

$$\vec{q} = \vec{a} + i\vec{b} \quad \langle i\vec{u}_1, i\vec{u}_2 \rangle = \frac{i\vec{u}_1 + i\vec{u}_2 - \vec{u}_1 \times \vec{u}_2}{1+(\vec{u}_1\vec{u}_2)}, \qquad (27)$$

where obviously

$$\vec{a} = -\frac{\vec{u}_1 \times \vec{u}_2}{1+(\vec{u}_1\vec{u}_2)}, \ \vec{b} = \frac{\vec{u}_1 + \vec{u}_2}{1+(\vec{u}_1\vec{u}_2)}, \qquad (28)$$

where $(\vec{a}\vec{b}) = 0$.

This vector-parameter is complex. It can be represented as a composition corresponding to the polar decomposition (25) of the Mueller matrix of the obtained equivalent system as

$$\vec{q} = \langle \vec{c}, i\vec{u} \rangle \quad \vec{c} + i\vec{u} + i(\vec{c} \times \vec{u}). \qquad (29)$$

It takes into account that as it follows from (27) $(\vec{q}^{\,2})^* = (\vec{q}^{\,2})$ and so $(\vec{c}\vec{u}) = 0$, where $\vec{c}$ and $\vec{u}$ characterize the equivalent optical system. Comparing (27) with (29) one can obtain

$$\vec{c} = \vec{a} \quad -\frac{\vec{u}_1 \times \vec{u}_2}{1+(\vec{u}_1\vec{u}_2)}, \ \vec{u} = \frac{1-\vec{a}^{\times}}{1+\vec{a}^2}\vec{b}, \qquad (30)$$

where $\vec{a}^{\times}$ as before is the operator of the vector product. After non-complicated but rather intricate transformations it follows that

$$\vec{c} = \frac{\vec{D}^{(1)} \times \vec{D}^{(2)}}{(1+\sqrt{1-(\vec{D}^{(1)})^2})(1+\sqrt{1-(\vec{D}^{(2)})^2}) + (\vec{D}^{(1)}\vec{D}^{(2)})} = \vec{r}tg\frac{R}{2},$$

where as in the case of (23), (24) the unit vector $\vec{r}$ that defines the eigenpolarizations of the resulting equivalent phase element. For the effective $\vec{D}$ connected with $\vec{u}$ (30) according to (22) one can obtain

$$\vec{D} = \frac{\vec{D}^{(1)}(\sqrt{1-(\vec{D}^{(2)})^2}) + \vec{D}^{(2)}\left[\frac{1+(\vec{D}^{(1)}\vec{D}^{(2)})}{(1+\sqrt{1-(\vec{D}^{(2)})^2})}\right]}{1+(\vec{D}^{(1)}\vec{D}^{(2)})}.$$

A distinctive feature of the calculations is their independence of the choice of the reference frame.

CONCLUSION

In the conclusion we note another significant feature that differ the approach based on the vector parameterization as the transformations realizing both the Jones and Mueller matrices approaches. This feature is in more simple and useful relation between the vector-parameters determining by the eigenpolarizations of a polarizers or phase plates and the experimentally measured elements of the Jones and Mueller matrices [23]. In the traditional approach this relation is realized via exponent. In the presented approach this relationship is determined by formula (16) for the Jones matrices and (19), (20) for the Mueller matrices. On basis of the measured elements of the Jones and Mueller matrices the eigenpolarizations of polarizers are defined according to formulas (21), (22) and for retarders according to formulas (23), (24) respectively.

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