The free oscillations of an anelastic aspherical earth

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Summary. Some century-old results, due to Rayleigh and Routh, have been adapted to investigate the normal mode eigenfrequencies and eigenfunctions of an earth with laterally variable anelasticity and to determine the transient response of such an earth to earthquakes. Using degenerate perturbation theory, the eigenfrequencies are found to first order and the associated eigenfunctions to zeroth order in the small deviations of the Earth away from a spherical perfectly elastic reference earth model. Both the eigenfrequencies and the eigenfunctions are complex and, in addition, the latter are not mutually orthogonal, reflecting the non-Hermitian character of the normal mode eigenvalue problem. The effect of laterally heterogeneous attenuation on the shape of an unresolved multiplet spectrum has been investigated in the surface-wave geometrical-optics limit. Singlet cancellation leads in that limit to the appearance of a single resonance peak whose decay rate or apparent $Q^{-1}$ depends only on the average attenuation structure underlying the source—receiver great-circle path.

1 Introduction

Past theoretical treatments of the splitting of the Earth’s free oscillations have generally either ignored anelasticity entirely (Woodhouse & Dahlen 1978) or assumed that it is spherically symmetric (Woodhouse 1980). In this paper we shall consider the effect of laterally heterogeneous anelasticity. Although much of the discussion will be general, our principal interest is in the surface wave-equivalent multiplets $nS_l$ and $nT_l$, having $n < 1$. Those multiplets are believed to be split predominantly by lateral variations in the shear velocity $\beta$ of the crust and upper mantle. Variations in $\beta$ are known to be of order $\delta\beta/\beta \sim 10^{-3} - 10^{-2}$, which is the same as a typical average value of the reciprocal shear quality factor $Q_s^{-1}$. The recent attenuation study of mantle Rayleigh waves by Nakanishi (1979) suggests that the corresponding lateral variations in $Q_s^{-1}$ may be as large as 100 per cent, which is not surprising in view of its strong dependence compared to the relatively weak dependence of $\beta$ on temperature. If $\delta Q_s^{-1}$ and $\delta\beta/\beta$ are of the same order of magnitude, they can be expected to influence the nature of the splitting of the surface wave-equivalent multiplets by roughly...
equal amounts. In that case it is essential to consider them simultaneously in performing splitting calculations.

The Earth's rotation will be ignored in this paper, since it exerts a relatively minor influence on the surface wave-equivalent multiplets. The problem we shall consider is thus the oscillation of a non-rotating, slightly anelastic and slightly aspherical earth. For completeness, we shall consider the case of an arbitrary slight asphericity as well as both bulk and shear anelasticity even though $\delta \beta$ and $Q^{-1}_\mu$ are expected to be the dominant perturbations in the situation of principal interest. The method employed will be Rayleigh—Schrödinger perturbation theory, just as in the perfectly elastic case. As usual, we shall consider only a severely truncated version of that theory, which will be correct to zeroth order for the eigenfunctions and to first order for the (now complex) eigenfrequencies. We shall furthermore restrict attention to isolated multiplets $nS_1$ and $nT_1$; the coupling of quasi-degenerate multiplets (Woodhouse 1980) will not be considered. Extension of the results we obtain here to include both rotation and quasi-degeneracy should, however, be relatively straightforward.

A major difference between anelastic and purely elastic perturbation theory is that in the anelastic case the zeroth-order eigenfunctions are neither real nor, within any multiplet, mutually orthogonal. Despite this complication the transient response of an anelastic earth to earthquakes can be obtained in terms of zeroth-order eigenfunctions in closed form. This response is used here to investigate the shape of unresolved multiplet spectra on an earth whose lateral heterogeneity is smooth. On a smooth laterally heterogeneous earth whose attenuation structure is spherical, it is known that singlet cancellation gives rise to a single resonance peak which is not broadened by splitting (Dahlen 1979a, b). We shall demonstrate here that a similar result is true, again in the geometrical optics limit, on an earth whose attenuation structure is laterally variable. Dahlen (1980a) has conjectured that singlet cancellation should in that case give rise to a single resonance peak whose broadening depends only on the average attenuation structure underlying the great-circle path connecting the source and receiver, and we shall verify that conjecture here. As in the simpler case of spherical attenuation, this result is not valid near the source or its antipode due to the arrival of energy along other great-circle paths. We shall show, however, finally, that the uniformly valid asymptotic representation of spectra obtained by Dahlen (1980b) can easily be extended to account for smooth laterally heterogeneous anelasticity. This extension provides, in principle, a basis for the geographical interpretation of near-source and near-antipodal multiplet spectra on a laterally heterogeneous anelastic earth.

2 The eigenfrequency spectrum of an anelastic earth

Before pursuing perturbation theory, it is convenient to establish some general results regarding the eigenfunctions and complex eigenfrequencies of an anelastic earth. Throughout this paper we shall employ $\sigma$ to denote a complex frequency and $\omega$ to denote a real one. Let $V$ be the volume occupied by the earth and let $\rho_0$ and $\phi_0$ be its initial density and gravitational potential, respectively. The normal mode eigenvalue problem for a linear anelastic earth is of the form

$$\mathcal{H}(\sigma)s = \omega^2 s,$$

where $\mathcal{H}(\sigma)$ is a linear integro-differential operator acting on displacement eigenfunctions $s$. If anisotropic initial stresses and elastic anisotropy are ignored, the operator $\mathcal{H}(\sigma)$ is given by

$$\rho_0 \mathcal{H}(\sigma)s = \rho_0 \nabla \phi_0 + \rho_1 \nabla \phi_0 + \nabla (\rho_0 s \cdot \nabla \phi_0) - \nabla \cdot \mathbf{T}$$

(2)
Oscillations of an anelastic earth

\[ \rho_1 = - \nabla \cdot (\rho_0 s) \]  
\[ \phi_1 = - G \int_V \rho_0 s \cdot \nabla' (1/|r - r'|) dV' \]  
\[ T = \mathcal{K}(\nabla \cdot s) I + 2 \mathcal{M} \delta. \]

The quantities \( \rho_1, \phi_1 \) and \( T \) are, respectively, the changes in density, gravitational potential and stress associated with the deformation \( s \), and

\[ \delta = \frac{1}{2} [ \nabla s + (\nabla s)^T ] - \frac{1}{3} (\nabla \cdot s) I \]

in equation (3c) is the deviatoric strain. Equations (1)-(3) are to be solved subject to the boundary condition that

\[ \hat{n} \cdot T = 0 \]

where \( \hat{n} \) is the unit outward normal on the surface of the earth \( \partial \mathcal{V} \).

The only difference between the eigenvalue problem (1)-(4) and the corresponding problem on a perfectly elastic earth is that the bulk and shear moduli \( \mathcal{K}(\sigma) \) and \( \mathcal{M}(\sigma) \) are now both complex and frequency dependent. They can be written in terms of the associated time-domain creep relaxation functions \( \mathcal{K}(t) \) and \( \mathcal{M}(t) \) in the form (Nowick & Berry 1972)

\[ \mathcal{K}(\omega) = i \omega \int_0^\infty \mathcal{K}(t) \exp(-i \omega t) \, dt \]  
\[ \mathcal{M}(\omega) = i \omega \int_0^\infty \mathcal{M}(t) \exp(-i \omega t) \, dt. \]

Since \( \mathcal{K}(t) \) and \( \mathcal{M}(t) \) are real, \( \mathcal{K}(\omega) \) and \( \mathcal{M}(\omega) \) satisfy

\[ \mathcal{K}^*(\omega) = \mathcal{K}(\omega^*) \]  
\[ \mathcal{M}^*(\omega) = \mathcal{M}(\omega^*) \]

where an asterisk denotes complex conjugation. As a consequence of this symmetry, equation (1) implies that

\[ \mathcal{K}(-\sigma^*) s^* = (-\sigma^*)^2 s^*, \]

i.e. if \( \sigma \) and \( s \) are a complex eigenfrequency and eigenfunction of the earth, then so are \(-\sigma^* \) and \( s^* \).

If equation (1) is dotted with \( \rho_0 s^* \) and integrated over \( \mathcal{V} \), we find upon applying Gauss' theorem and the boundary condition (4) the result

\[ \sigma^2 \mathcal{F} = \mathcal{G} + \mathcal{E}(\omega) \]

where

\[ \mathcal{F} = \int_V \rho_0 s \cdot s^* \, dV \]  
\[ \mathcal{G} = \int_V [\rho_0 s \cdot \nabla \phi_1^* + \rho_0 s \cdot \nabla \nabla \phi_0 s^* + \rho_0 (\nabla s : \nabla s^* - |\nabla \cdot s|^2)] \, dV \]  
\[ \mathcal{E} = \int_V [\mathcal{K} |\nabla \cdot s|^2 + 2 \mathcal{M}(\delta \delta^* )] \, dV. \]
The quantity $p_0$ in equation (9b) is the initial hydrostatic pressure present in the earth, i.e.
\[ \nabla p_0 + p_0 \nabla \phi_0 = 0. \tag{10} \]
Suppose now we write $\sigma$ as
\[ \sigma = \omega + i\alpha \tag{11} \]
where $\omega$ and $\alpha$ are real. As we have seen, eigenfrequencies $\sigma$ must occur in pairs of the form $\pm \omega + i\alpha$. The imaginary part of equation (8) is easily seen to be
\[
2\omega \alpha \mathcal{F} = \int_V [(\text{Im } \mathcal{V}) |\nabla \cdot s|^2 + 2(\text{Im } \mathcal{U})(\delta : \delta^*)] \, dV, \tag{12}
\]
and equations (6) imply that
\[
\text{Im } \mathcal{V}(-\omega + i\alpha) = - \text{Im } \mathcal{V}(\omega + i\alpha) \tag{13a}
\]
\[
\text{Im } \mathcal{U}(-\omega + i\alpha) = - \text{Im } \mathcal{U}(\omega + i\alpha). \tag{13b}
\]
Since the quantities $\mathcal{F}$, $|\nabla \cdot s|^2$ and $\delta : \delta^*$ are all positive, equation (12) shows that if
\[
\text{Im } \mathcal{V}(\sigma) > 0 \tag{14a}
\]
\[
\text{Im } \mathcal{U}(\sigma) > 0 \tag{14b}
\]
for $\text{Re } \sigma > 0$ and $\sigma$ in the vicinity of the real axis, then
\[ \alpha > 0. \tag{15} \]

With the Fourier transform sign convention adopted in equations (5), oscillations behave as $\exp(i\omega t - \alpha t)$ and the condition (15) implies damping.

In summary, if the conditions (14) are satisfied (and if the density and elastic structure are such that the earth is stable) the seismic eigenfrequency spectrum will have the form illustrated in Fig. 1. Furthermore the eigenfunction associated with an eigenfrequency $-\omega + i\alpha$ is the complex conjugate $s^*$ of the eigenfunction $s$ associated with $\omega + i\alpha$. Because of this relation between left and right half-plane eigensolutions, we only need to consider positive frequencies $\omega$ in doing perturbation theory below.

Figure 1. Schematic diagram of the seismic eigenfrequency spectrum of a general anelastic earth. Eigenfrequencies occur in pairs $\sigma = \omega + i\alpha$ and $-\sigma^* = - \omega + i\alpha$ in the complex $\sigma$-plane. If the eigenfunction associated with $\sigma$ is $s$, that associated with $-\sigma^*$ is $s^*$. 

3 Perturbation theory

Since the anelasticity of the earth is slight it suffices to consider the complex moduli \( \mathcal{K} \) and \( \mathcal{M} \) to be functions of real frequency \( \omega \) rather than complex frequency \( \sigma \). In that case it is conventional to write \( \mathcal{K} \) and \( \mathcal{M} \) in the form (O’Connell & Budiansky 1978)

\[
\mathcal{K} = \kappa (1 + iq_{\kappa}) \quad (16a)
\]
\[
\mathcal{M} = \mu (1 + iq_{\mu}) \quad (16b)
\]

where \( \kappa (\omega), \mu (\omega) \) are the real elastic bulk and shear moduli and

\[
q_{\kappa}(\omega) = Q_{\kappa}^{-1}(\omega) \quad (17a)
\]
\[
q_{\mu}(\omega) = Q_{\mu}^{-1}(\omega) \quad (17b)
\]

are the corresponding reciprocal Qs.

The starting point of perturbation theory is a perfectly elastic spherical earth model whose properties are considered to be an appropriate description of the spherically averaged earth at some reference frequency \( \omega_r \). We shall throughout this paper use an overbar to label quantities or properties which are spherical, and we shall use \( \bar{\mathcal{R}} \) to denote this reference spherical earth model. We shall use \( \bar{\mathcal{B}} \) to denote the spherical volume occupied by the reference model and \( a \) to denote its radius. The model \( \bar{\mathcal{R}}(\omega_r) \) consists of the triplet of functions

\[
\bar{\mathcal{R}}(\omega_r) = \{ \bar{\rho}_0, \bar{\kappa}(\omega_r), \bar{\mu}(\omega_r) \},
\]

all of which are functions only of \( r = |\mathbf{r}| \).

We shall now focus attention on a single multiplet \( _{n}S_l \) or \( _{n}T_l \) whose degenerate eigenfrequency calculated using model \( \bar{\mathcal{R}}(\omega_r) \) is \( \omega_0 \). The associated \( 2l + 1 \)-dimensional eigenspace will be denoted by \( \mathcal{S}_0 \). Let \( \hat{x}, \hat{y}, \hat{z} \) be an arbitrary geocentric Cartesian axis system in \( \bar{\mathcal{B}} \) and let \( Y_l^m, -l < m < l \), be the associated canonical surface spherical harmonics defined by Edmonds (1960). A spheroidal eigenspace \( \mathcal{S}_0 \) has a basis of eigenfunctions \( s_m, -l < m < l \), of the form

\[
s_m = U \hat{x} Y_l^m + V \hat{y} Y_l^m
\]

and a toroidal eigenspace has a basis \( s_m, -l < m < l \), of the form

\[
s_m = W \hat{z} \nabla Y_l^m.
\]

The scalars \( U, V \) and \( W \) are functions only of \( r \) and \( \nabla = \nabla - \hat{r} (\hat{r} \cdot \nabla) \) is the surface gradient operator on the unit sphere \( \Omega \). Each \( s_m \) satisfies a zeroth-order Hermitian eigenvalue equation of the form

\[
\hat{\mathcal{M}}(\omega_r) s_m = \omega_0^2 s_m
\]

where \( \hat{\mathcal{M}}(\omega_r) \) denotes the spherical perfectly elastic operator defined by equation (2) for model \( \bar{\mathcal{R}}(\omega_r) \).

The spherical harmonics \( Y_l^m \) are orthonormal in the sense (Edmonds 1960)

\[
\int_{\Omega} Y_l^m \ast Y_{l'}^{m'} \ dA = \delta_{ll'} \delta_{mm'}.
\]
To be specific we shall suppose throughout this paper that the scalars, $U$, $V$ and $W$ have been normalized as in Dahlen (1979a), namely

$$
\omega_0^2 \int_0^a \rho_0 [U^2 + l(l + 1)V^2] r^2 \, dr = 1
$$

(23a)

$$
\omega_0^2 \int_0^a \rho_0 [l(l + 1)W^2] r^2 \, dr = 1.
$$

(23b)

It follows from (22) and (23) that the eigenfunctions $s_m$ constitute an orthonormal basis for the space $\mathcal{H}_0$ in the sense

$$
\omega_0^2 \int_{\mathcal{H}} \rho_0 s_m^* s_{m'} \, dV = \delta_{mm'}.
$$

(24)

Since the splitting induced by asphericity is slight, it suffices to consider the properties of the actual Earth at the degenerate frequency $\omega_0$. Let us denote the aspherical anelastic earth at that frequency by $\Theta(\omega_0)$. It consists of five functions of position within $\mathcal{Y}$; namely

$$
\Theta(\omega_0) = \{ \rho_0, \kappa(\omega_0), \mu(\omega_0), q_{k}(\omega_0), q_{\mu}(\omega_0) \}.
$$

(25)

The spherically averaged version of $\Theta(\omega_0)$ will be denoted by $\bar{\Theta}(\omega_0)$. It consists of the five functions of radius

$$
\bar{\Theta}(\omega_0) = \{ \bar{\rho}_0, \bar{\kappa}(\omega_0), \bar{\mu}(\omega_0), \bar{q}_k(\omega_0), \bar{q}_\mu(\omega_0) \}.
$$

(26)

The spherically averaged moduli $\bar{\kappa}(\omega_0)$ and $\bar{\mu}(\omega_0)$ will in general differ from the reference moduli $\kappa(\omega_r)$ and $\mu(\omega_r)$ because of physical dispersion. The departures from sphericity of $\Theta(\omega_0)$ will be denoted by $\delta \Theta(\omega_0)$, and we shall write symbolically

$$
\Theta(\omega_0) = \bar{\Theta}(\omega_0) + \delta \Theta(\omega_0).
$$

(27)

As well as volumetric perturbations in the five properties $\rho_0, \kappa, \mu, q_k$ and $q_{\mu}$, the symbol $\delta \Theta(\omega_0)$ might include perturbations $\delta h$ in the shape of internal boundaries of $\bar{\Theta}(\omega_0)$, i.e.

$$
\delta \Theta(\omega_0) = \{ \delta \rho_0, \delta \kappa(\omega_0), \delta \mu(\omega_0), \delta q_{k}(\omega_0), \delta q_{\mu}(\omega_0), \delta h \}.
$$

(28)

By definition the perturbation $\delta \Theta(\omega_0)$ is assumed to be strictly aspherical, i.e. symbolically,

$$
\overline{\delta \Theta}(\omega_0) = 0.
$$

(29)

It is convenient to expand the various constituents of $\delta \Theta(\omega_0)$ in terms of spherical harmonics $Y_s^t$, namely

$$
\delta \rho_0 = \sum_s \sum_t \delta \rho_s^t Y_s^t
$$

(30a)

$$
\delta \kappa = \sum_s \sum_t \delta \kappa_s^t Y_s^t
$$

(30b)

$$
\delta \mu = \sum_s \sum_t \delta \mu_s^t Y_s^t
$$

(30c)

$$
\delta q_k = \sum_s \sum_t \delta q_{k_s}^t Y_s^t
$$

(30d)
Because of (29), every sum over \( s \) in equations (30a–30f) begins at \( s = 1 \). For brevity in what follows we shall rewrite these equations in the symbolic form

\[
\delta \Phi(\omega) = \sum_s \sum_t \delta \Phi_s^t Y^t_s.
\]

The difference between the actual Earth \( \Phi(\omega_0) \) and the reference earth \( \Phi_{el}(\omega_r) \) is now seen to be two-fold. In addition to the aspherical perturbation \( \delta \Phi(\omega_0) \) there is a spherical perturbation \( \Phi(\omega_0) - \Phi_{el}(\omega_r) \) which takes into account the frequency dependence of \( \tilde{\kappa} \) and \( \tilde{\mu} \) and the spherical part of the attenuation \( q_k \) and \( q_\mu \). Symbolically we may write

\[
\Phi(\omega_0) = \Phi_{el}(\omega_r) + [\Phi(\omega_0) - \Phi_{el}(\omega_r)] + \delta \Phi(\omega_0).
\]

In what follows we shall assume that both perturbations \( \Phi(\omega_0) - \Phi_{el}(\omega_r) \) and \( \delta \Phi(\omega_0) \) are small, but we shall not assume that \( \delta q_k \) or \( \delta q_\mu \) is small compared to \( q_k \) or \( q_\mu \). The attenuation is thus assumed to be slight but allowed to be arbitrarily laterally heterogeneous since, as discussed in Section 1, this is the case of interest. The decomposition of \( q_k \) and \( q_\mu \) into a spherical and an aspherical part is convenient even though the asphericity of attenuation is not assumed slight.

Each of the perturbations \( \Phi(\omega_0) - \Phi_{el}(\omega_r) \) and \( \delta \Phi(\omega_0) \) will give rise to a perturbation of the elastic operator \( \mathcal{H}_{el}(\omega_r) \). Using an obvious notation we shall denote these two perturbations by \( \mathcal{H}(\omega_0) - \mathcal{H}_{el}(\omega_r) \) and \( \delta \mathcal{H}(\omega_0) \) respectively; the first of these is non-Hermitian and spherical and the second is non-Hermitian and aspherical. Let us denote the eigenfrequencies arising from \( \omega_0 \) by \( \sigma_j \) and their associated eigenfunctions by \( s_j \). These are solutions of the perturbed eigenvalue problem

\[
\mathcal{H}(\omega_0) s_j = \sigma_j^2 s_j
\]

where

\[
\mathcal{H}(\omega_0) = \mathcal{H}_{el}(\omega_r) + [\mathcal{H}(\omega_0) - \mathcal{H}_{el}(\omega_r)] + \delta \mathcal{H}(\omega_0).
\]

Rayleigh–Schrödinger perturbation theory (see, e.g. Schiff 1968) can in principle be used to solve equation (32) to any desired order of accuracy in the perturbations \( \Phi(\omega_0) - \Phi_{el}(\omega_r) \) and \( \delta \Phi(\omega_0) \), but, as mentioned above, we shall be content here with zeroth-order eigenfunctions \( s_j \) and first-order complex eigenfrequencies \( \sigma_j \). The latter are conveniently written in the form

\[
\sigma_j = \omega_0 + \delta \omega_0 + \delta \omega_j + i(\alpha_0 + \delta \alpha_j)
\]

where \( \delta \omega_0 + i \alpha_0 \) is due to the spherical perturbation \( \Phi(\omega_0) - \Phi_{el}(\omega_r) \) and \( \delta \omega_j + i \delta \alpha_j \) is due to the aspherical perturbation \( \delta \Phi(\omega_0) \). The quantities \( \delta \omega_0 \) and \( \alpha_0 \) are common to every eigenfunction \( s_j \) and are given by

\[
\delta \omega_0 = \int_0^a \left[ K_0(\tilde{\kappa}_0 - \tilde{\kappa}_r) + M_0(\tilde{\mu}_0 - \tilde{\mu}_r) \right] r^2 \, dr
\]

and

\[
\alpha_0 = \int_0^a \left[ \tilde{\kappa}_r K_0 q_\kappa + \tilde{\mu}_r M_0 q_\mu \right] r^2 \, dr
\]
where \( \bar{\kappa}_0 = \bar{\kappa}(\omega_0) \), \( \bar{\kappa}_t = \bar{\kappa}(\omega_t) \), \( \bar{\mu}_0 = \bar{\mu}(\omega_0) \) and \( \bar{\mu}_t = \bar{\mu}(\omega_t) \). The functions \( K_0(r) \) and \( M_0(r) \) are the spherical (\( s = 0 \)) Fréchet kernels for model \( \bar{n}_{el}(\omega_t) \) defined by Woodhouse & Dahlen (1978).

The zeroth-order eigenfunctions \( s_j \) are elements of the degenerate eigenspace \( \mathcal{Q}_0 \) of model \( \bar{n}_{el}(\omega_t) \), i.e. they may be written in the form

\[
s_j = \sum_m b_m^j s_m.
\]

The coefficients \( b_m^j \) and the perturbations \( \delta \omega_j + i \delta \alpha_j \) are, respectively, the eigencolumns and associated eigenvalues of a \( 2l+1 \) by \( 2l+1 \) perturbation matrix \( H \). To find them we must solve the eigenvalue problem

\[
\sum_{m'} H_{mm'} b_{m'}^j = (\delta \omega_j + i \delta \alpha_j) b_m^j.
\]

The matrix elements \( H_{mm'} \) are given, according to the rules of degenerate perturbation theory, by

\[
H_{mm'} = E_{mm'} + iQ_{mm'}
\]

where

\[
E_{mm'} = \sum_s \sum_t \left[ \int_0^a (K_s \delta \kappa_s^t + M_s \delta \mu_s^t + R_s \delta \rho_s^t) r^2 dr + \sum_d \Gamma_{ds} h_s^t \right] \left[ \int_\Omega Y_{l}^{m} \ast Y_{l}^{m'} dA \right]
\]

and

\[
Q_{mm'} = \sum_s \sum_t \left[ \int_0^a (\bar{\kappa}_t \delta q_{ks}^t + \bar{\mu}_t \delta q_{ks}^t + \bar{\mu}_t \delta q_{ks}^t) r^2 dr \right] \left[ \int_\Omega Y_{l}^{m} \ast Y_{l}^{m'} Y_{l}^{m'} dA \right].
\]

The Fréchet kernels \( K_s(r), M_s(r), R_s(r) \) and \( \Gamma_{ds} \) are all defined by Woodhouse & Dahlen (1978); the sum labelled \( d \) in equation (40) is over all discontinuities in model \( \bar{n}_{el}(\omega_t) \). Because of selection rules associated with the spherical harmonic triple product integrals, the sums over \( s \) in both (40) and (41) include only even values of \( s \) between \( s = 2 \) and \( s = 2l \).

The spherical harmonics \( Y_{l}^{m} \) satisfy

\[
Y_{l}^{-m} = (-1)^m Y_{l}^{m*}.
\]

Since the perturbations comprising \( \delta \omega(\omega_0) \) are all real the coefficients \( \delta a_s^t \) satisfy

\[
\delta a_s^t = (-1)^t \delta a_s^{t*}.
\]

Using the relations (42) and (43) it is easily verified that each of the matrices \( E \) and \( Q \) is Hermitian, i.e.

\[
E_{mm'} = E_{m'm}^* \tag{44a}
\]

\[
Q_{mm'} = Q_{m'm}^*. \tag{44b}
\]

The matrix \( H = E + iQ \) is not in general Hermitian but as a consequence of (43) it does possess a symmetry we shall employ below, namely

\[
H_{mm'} = (-1)^{m+m'} H_{-m'-m}.
\]
In general, the $2l + 1$ eigenvalues $\delta \omega_j + i\delta \alpha_j$ of $H$ will be distinct and the degeneracy of $\omega_0$ will be removed completely. If the eigenvalues $\delta \omega_j + i\delta \alpha_j$ are distinct the associated eigencolumns $b_j^m$ must be linearly independent (Halmos 1958). The zeroth-order eigenfunctions $s_j$ must therefore be linearly independent as well and, since there are $2l + 1$ of them, they must comprise a basis for $\mathcal{S}_0$. Each degenerate positive eigenfrequency $\omega_0$ with eigenspace $\mathcal{S}_0$ thus gives rise, when perturbed, to $2l + 1$ split complex eigenfrequencies $\sigma_j$ and to a basis of $2l + 1$ associated zeroth-order eigenfunctions $s_j$. The corresponding results for the negative eigenfrequency $-\omega_0$ associated with $\mathcal{S}_0$ can be easily deduced from the general considerations outlined in Section 2. In addition to $\sigma_j$, $s_j$ the multiplet $nS_l$ or $nT_l$ must have a set of eigenfrequencies

$$-\sigma_j^* = -\omega_0 - \delta \omega_0 - i(\alpha_0 + \delta \alpha_j)$$

with associated zeroth-order eigenfunctions $s_j^*$. The eigenfunctions $s_j^*$ must be linearly independent if $s_j$ are and hence they too must comprise a basis for the space $\mathcal{S}_0$. In general it will not be true that $s_j^* = s_j$ so the bases associated with $\sigma_j$ and $-\sigma_j^*$ will be different; they are, however, related as we shall show below.

First let us verify that perturbation theory, if it is applied to $-\omega_0$, leads to $-\sigma_j^*, s_j^*$. The frequency-dependent parameters of the anelastic earth $\delta (-\omega_0)$ satisfy, according to equation (6),

$$\kappa (-\omega_0) = \kappa (\omega_0)$$

$$\mu (-\omega_0) = \mu (\omega_0)$$

and

$$q_\kappa (-\omega_0) = -q_\kappa (\omega_0)$$

$$q_\mu (-\omega_0) = -q_\mu (\omega_0).$$

Since the Fréchet kernels $K_s(r)$ and $M_s(r)$ for $-\omega_0$ are the negative of those for $\omega_0$ (Woodhouse & Dahlen 1978), it is clear from (35) and (36) that the spherical perturbation $\delta (\omega_0) - \delta \omega_0(\omega_0)$ will give rise to a perturbation $-\delta \omega_0 + i\alpha_0$ in agreement with (46). Let us now denote the adjoint of the perturbation matrix $H$ by $H^{adj}$; by definition we have

$$H_{mm'}^{adj} = H_{m'm}^*. $$

Since $E$ and $Q$ are both Hermitian we may write $H^{adj}$ in the form

$$H^{adj} = E - iQ. $$

It is now clear that the application of degenerate perturbation theory to $\delta \theta (-\omega_0)$ will lead not to (38) but rather to the corresponding eigenvalue problem for the matrix $-H^{adj}$. By employing the symmetry (45) the original problem (38) may, however, be converted to

$$\sum_{m'} H_{mm'}^{adj} (-1)^m b_j^m s_j^{*m'} = (\delta \omega_j - i\delta \alpha_j) (-1)^m b_j^m s_j^{*m}. $$

The eigenvalues of $-H^{adj}$ are thus $-\delta \omega_j + i\delta \alpha_j$, as expected, and the associated eigencolumns are $(-1)^m b_j^m s_j^{*m}$. Equation (42) implies that

$$s_j^* = \sum_m (-1)^m b_j^m s_j^{*m}$$

so the zeroth-order eigenfunctions associated with $-\sigma_j^*$ are indeed $s_j^*$ as they must be.
It is generally true of course that the eigenvalues of adjoint matrices are complex conjugates (Halmos 1958). The additional relationship in this case that the eigencolumns of $H$ are $b_j^\mu$ while those of $H^{adj}$ are $(-1)^m b_j^{-m*}$ is a consequence of the symmetry (45). There is also a general relationship between eigencolumns of adjoint matrices which $b_j^m$ and $(-1)^m b_j^{-m*}$ must satisfy, namely they must be dual to each other in the sense (Halmos 1958)

$$\sum_j b_j^m (-1)^m b_k^{-m} = \delta_{jk}. \quad (53)$$

In setting the sum in (53) equal to one when $j = k$ we have specified the normalization of $b_j^m$ and $(-1)^m b_j^{-m*}$. The dual relationship between the two bases $s_j$ and $s_j^*$ implied by (53) is

$$\omega_0^2 \int \rho_0 s_j \cdot s_k \, dV = \delta_{jk}. \quad (54)$$

In spite of its appearance, equation (54) is not a statement that two eigenfunctions $s_j$ and $s_k$ are orthogonal since the natural inner product in the complex space $H_0$ must include a complex conjugation as in equation (24). Neither of the two bases $s_j$ or $s_j^*$ is in general orthonormal; they are, however, dual or bi-orthonormal in the sense (54).

An obvious generalization of the diagonal sum rule (Gilbert 1971) can be derived quite easily. An argument identical to Gilbert’s shows in fact that

$$\sum_j (\delta \omega_j + i \delta \alpha_j) = 0. \quad (55)$$

If we now define $\omega_j$ and $\alpha_j$ by

$$\omega_j = \omega_0 + \delta \omega_0 + \delta \omega_j \quad (56a)$$
$$\alpha_j = \alpha_0 + \delta \alpha_j \quad (56b)$$

equation (55) shows that

$$\frac{1}{2l+1} \sum_j \omega_j = \omega_0 + \delta \omega_0$$
$$\frac{1}{2l+1} \sum_j \alpha_j = \alpha_0. \quad (57a)$$

The average of the $2l+1$ real eigenfrequencies $\omega_j$ is thus the degenerate eigenfrequency $\omega_0 + \delta \omega_0$ of the spherically averaged earth $\Omega(\omega_0)$ and the sum of the $2l+1$ decay rates $\alpha_j$ is the decay rate $\alpha_0$ of $\Omega(\omega_0)$. This is of course an intuitively reasonable result.

There are three cases in which the above results can be simplified. The first case is that the lateral heterogeneity in attenuation is slight, so that

$$\|Q\| < \|E\| \quad (58)$$

where $\|\cdot\|$ denotes the matrix norm (Halmos 1958). In this case it is permissible to ignore $Q$ entirely in solving for $b_j^m$. Since $E$ is Hermitian we will then find that

$$b_j^m = (-1)^m b_j^{-m*}, \quad (59)$$

which in turn implies that

$$s_j = s_j^*. \quad (60)$$
i.e. the eigenfunctions \( s_j \) will be real. Solution of the eigenvalue problem for \( E \) alone determines \( s_j \) and the associated real split eigenfrequencies \( \omega_0 + \delta \omega_0 + \delta \omega_j \). The attenuation rate of every mode in the multiplet will be very nearly \( \alpha_0 \). The small perturbations \( \delta \alpha_j \) can be determined \textit{a posteriori} by the application of non-degenerate perturbation theory to the eigenfunctions \( s_j \). Since the natural inner product in a real space \( \mathcal{C}_0 \) does not include a complex conjugation, equation (54) is in this case a statement that the real basis functions \( s_j \) are orthonormal. In addition the two bases associated with \( \alpha_j \) and \( -\alpha_j^* \) will now be the same. These features are of course familiar from the perfectly elastic case.

The second case is that the lateral heterogeneity of density and elasticity is slight, i.e.

\[
\| E \| < \| Q \|. 
\]

If that is so then \( E \) can be ignored in solving for \( b_j^m \) and, since \( iQ \) is anti-Hermitian, the eigenfunctions \( s_j \) will again be real and orthonormal in the sense (54). In this case every mode in the multiplet will have very nearly the same real eigenfrequency \( \omega_0 + \delta \omega_0 \) but a different attenuation rate \( \alpha_0 + \delta \alpha_j \). The small perturbations \( \delta \omega_j \) which constitute ‘splitting’ \textit{per se} could be computed by the \textit{a posteriori} application of non-degenerate perturbation theory to \( s_j \). The importance of considering aspherical attenuation in doing splitting calculations is particularly clear in this example. If \( \delta q_k(\omega_0) \) and \( \delta q_\mu(\omega_0) \) were ignored and degenerate perturbation theory applied naively to the density and elastic perturbations alone, the zeroth-order eigenfunctions \( s_j \) and thus the first-order split eigenfrequencies \( \omega_0 + \delta \omega_0 + \delta \omega_j \) so calculated would be spurious. The correct eigenfunctions \( s_j \) are determined in this case not by the slight perturbations in density and elastic structure but by the lateral variations in attenuation. As noted in Section 1, the actual situation in the Earth is likely to be neither (58) nor (61) but, rather,

\[
\| E \| \approx \| Q \|. 
\]

In that case the two perturbations must be considered simultaneously as we have done above, and they will then play a roughly equal role in determining \( s_j \) and \( \delta \omega_j + i \delta \alpha_j \).

As the third and final special case let us suppose now that the various aspherical perturbations \( \delta p_\alpha(\omega_0) \), \( \delta k(\omega_0) \), \( \delta q_k(\omega_0) \) and \( h \) are negligible, and that the remaining two significant perturbations \( \delta \mu(\omega_0) \) and \( \delta q_\mu(\omega_0) \) are exactly correlated throughout the Earth, in the sense

\[
\delta \mu / \mu = \chi \delta q_\mu, 
\]

where \( \chi \) is a constant. Alternatively we could suppose that only \( \delta k(\omega_0) \) and \( \delta q_k(\omega_0) \) were significant and that they were correlated in the same way, namely

\[
\delta k / k = \chi \delta q_k. 
\]

In either case it is clear that the two matrices \( E \) and \( Q \) will commute or, equivalently, that the matrix \( H \) will be normal, i.e.

\[
HH^\text{adj} = H^\text{adj}H. 
\]

Normality in turn is a necessary and sufficient condition that the columns \( b_j^m \) and \((-1)^mb_j^{m*}\) are equal (Halmos 1958) and thus that the eigenfunctions \( s_j \) are real and orthonormal in the sense (54). The eigenvalues of a normal matrix \( H \) will in general be complex, of the form \( \delta \omega_j + i \delta \alpha_j \). Nakanishi (1979) found a strong correlation between the \( Q \) and the phase velocity of mantle Rayleigh waves, and Sipkin & Jordan (1980) found a similar correlation between the \( Q \) and the travel-time of \( ScS \) waves. In view of this, equation (63a)
may be a fairly reasonable approximation. On the other hand the simplification resulting from this approximation is slight and there is little point in employing an approximation if one does not have to. The only difference between this and the more general case is that the eigenfunctions $s_j$ are real and orthonormal. The duality (54) of the complex eigenfunctions $s_j$ and $s_j^*$ is, however, all that is required to determine the transient response of the Earth to an earthquake, as we shall now show.

4 Excitation by earthquakes

Let us consider first the response $s(t)$ to an impulsive applied force $f\delta(t)$ where $\delta(t)$ is the Dirac delta function. Introducing the Fourier transform

$$s(\omega) = \int_{-\infty}^{\infty} s(t) \exp(-i\omega t) dt,$$

the equation we must solve in the frequency domain is

$$\mathcal{H}(\omega)s(\omega) = \sigma^2 s(\omega) + f.$$  \hfill (66)

We shall find an approximate solution to (66) in terms of zeroth-order eigenfunctions. The zeroth-order eigenspaces $\mathcal{S}_0$ associated with different multiplets $nS_i$ and $nT_i$ are mutually orthogonal, so to that order we are entitled to consider the excitation of each multiplet separately. Henceforth in this paper we shall use $s(t)$ and $s(\omega)$ to refer to the displacement associated with a single multiplet $nS_i$ or $nT_i$ with split eigenfrequencies $\sigma_j$ and $-\sigma_j^*$ and zeroth-order eigenfunctions $s_j$ and $s_j^*$.

To find the spectrum $s(\omega)$ for positive frequencies $\omega$, it is convenient to consider an expansion in the form

$$s(\omega) = \sum_j a_j(\omega)s_j$$  \hfill (67)

This expansion is permissible since the $2l + 1$ $s_j$ are a basis for $\mathcal{S}_0$. If (66) is dotted by $\rho_0 s_j$ and integrated over $V$, we find upon employing the duality relation (54) that

$$a_j(\omega) = \omega_0^2 \left[ \int_V \rho_0 f \cdot s_j \, dV \right] [\Sigma_j^2(\omega) - \omega^2]^{-1}$$  \hfill (68)

where

$$\mathcal{H}(\omega)s_j = \Sigma_j^2(\omega)s_j.$$  \hfill (69)

For negative frequencies $-\omega$ we consider instead the expansion

$$s(-\omega) = \sum_j a_j(-\omega)s_j^*$$  \hfill (70)

and to find $a_j(-\omega)$ we dot (66) by $\rho_0 s_j^*$ and integrate over $V$. This leads to the result

$$a_j(-\omega) = \omega_0^2 \left[ \int_V \rho_0 f \cdot s_j^* \, dV \right] [\Sigma_j^2(-\omega) - \omega^2]^{-1}$$  \hfill (71)

where

$$\mathcal{H}(-\omega)s_j^* = \Sigma_j^2(-\omega)s_j^*.$$  \hfill (72)
Oscillations of an anelastic earth

Since $\Sigma_j(\omega)$ satisfies

$$\Sigma_j(-\omega) = -\Sigma_j^*(\omega)$$  \hspace{1cm} (73)

we see from (68) and (71) that

$$s(-\omega) = s^*(\omega),$$  \hspace{1cm} (74)

which is an expected result if $s(t)$ is to be real.

To find $s(t)$ we must evaluate the inverse transform

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\omega) \exp(i\omega t) d\omega.$$  \hspace{1cm} (75)

The integrand in (75) has simple poles at each of the eigenfrequencies $\sigma_j$ and $-\sigma_j^*$ since

$$\Sigma_j(\omega_0) = \sigma_j$$  \hspace{1cm} (76a)

$$\Sigma_j(-\omega_0) = -\sigma_j^*.$$  \hspace{1cm} (76b)

Application of the residue theorem thus leads to the result

$$s(t) = \text{Re} \sum_j \frac{\omega_j^2}{\omega_0^2} \left[ \int_V \rho_0 f \cdot s_j \, dV \right] \frac{1}{(i\sigma_j)^{-1}} \exp(i\sigma_j t) H(t)s_j$$  \hspace{1cm} (77)

where $H(t)$ is the Heaviside unit step function. The response to an arbitrary force $f(t)$ which vanishes for $t < 0$ can now be found by convolution. Upon integrating by parts, the general response $s(t)$ can be written in the approximate form

$$s(t) = \text{Re} \sum_j \left\{ \int_0^t \left[ \int_V \rho_0 \partial_\tau f(\tau) \cdot s_j \, dV \right] \left[ 1 - \exp(i\sigma_j(t - \tau)) \right] \, d\tau \right\} s_j.$$  \hspace{1cm} (78)

If the force $f(t)$ is given by

$$\rho_0 f(t) = -M(t) \cdot \nabla \delta(r - r_s)$$  \hspace{1cm} (79)

where $M(t)$ is the moment tensor of an earthquake, the response (78) takes the form

$$s(t) = \text{Re} \sum_j \left\{ \int_0^t \left[ \partial_\tau M(\tau) : \epsilon_{fs} \right] \left[ 1 - \exp(i\sigma_j(t - \tau)) \right] \, d\tau \right\} s_j$$  \hspace{1cm} (80)

where

$$\epsilon_{fs} = \frac{1}{2} \left[ \nabla s_j(r_s) + \nabla s_j^T(r_s) \right]$$  \hspace{1cm} (81)

is the strain evaluated at the source. Finally, if the source is a step, i.e.

$$M(t) = MH(t),$$  \hspace{1cm} (82)

equation (80) reduces to

$$s(t) = \text{Re} \sum_j \left[ M : \epsilon_{fs} \right] \left[ 1 - \exp(i\omega_j(t - \alpha_j t)) \right] H(t)s_j$$  \hspace{1cm} (83)

where $\alpha_j = \omega_j + i\sigma_j$. 
Equation (83) bears a strong resemblance to the corresponding result due to Gilbert (1970) for an earth whose attenuation structure is spherical. In that case the eigenfunctions $s_j$ are real and (83) reduces to Gilbert's result, which is

$$s(t) = \sum_j [M : \epsilon_{j\delta}] [1 - \cos \omega_j t \exp (-\alpha_0 t)]H(t)s_j. \quad (84)$$

The interpretation of equation (84) is well-known. Each mode begins to oscillate with its own characteristic frequency $\omega_j$ at $t = 0$ and each oscillation decays at the same rate $\alpha_0$; the limit $t \to \infty$ is non-zero, corresponding to the final static offset of the Earth. Since the eigenfunctions $s_j$ are real, the phase of every oscillation is at all times either 0 or $\pi$ throughout the Earth.

The interpretation of the more general result (83) is very nearly the same. Every mode again commences oscillation at $t = 0$ and the $t \to \infty$ limit once again corresponds to the Earth's final static response. The principal difference is that each mode now decays with its own characteristic rate of decay $\alpha_j$. In addition, since the eigenfunctions $s_j$ are no longer real, the phase of a single oscillation is no longer either 0 or $\pi$ everywhere; instead the phase is variable throughout the Earth.

5 The shape of unresolved multiplet spectra

Let us return now to the case of a general moment tensor source (79) and consider the multiplet spectrum $s(\omega)$. Henceforth we shall use the symbol $M$ to denote the Fourier transform of $\partial_\tau M(t)$ evaluated at $\omega_0$, i.e.

$$M = \int_{-\infty}^{\infty} \partial_\tau M(t) \exp (-i\omega_0 t) dt. \quad (85)$$

For frequencies $\omega$ near $\omega_0$ the spectrum $s(\omega)$ due to the source (79) can be written in the approximate form

$$s(\omega) = \sum_j A_j c_j(\omega) \quad (86)$$

where

$$c_j(\omega) = -\frac{1}{2} [\alpha_j + i(\omega - \omega_j)]^{-1} \quad (87)$$

and

$$A_j = [M : \epsilon_{j\delta}]s_j. \quad (88)$$

In writing (86)–(88) it has been assumed that $\alpha_j < \omega_j$. The corresponding expression for $s(\omega)$ on an earth with spherically symmetric attenuation is given by Dahlen (1979a); it differs from (86)–(88) only in the replacement of $\alpha_j$ by $\alpha_0$. If the representation (37) is employed in (88) we may write it in the form

$$A_j = \sum_m \sum_{m'} \left[(-1)^m b_j^{-m}b_j^{m'} \right] [M : \epsilon_{ms}]s_{m'}. \quad (89)$$

where $\epsilon_{ms}$ is the strain associated with $s_m$ evaluated at $r_s$. If the attenuation structure is spherical the quantity $(-1)^m b_j^{-m}$ in (89) can be replaced by $b_j^{-m*}$ by virtue of (59), and in
that case equation (89) agrees with equation (28) of Dahlen (1979a). Since the differences with Dahlen (1979a) are so slight most of the discussion of unresolved spectra given there can be carried over quite easily.

A quantity of interest in the retrieval of source mechanisms is the apparent complex amplitude $A_*$ defined by

$$A_* = -\frac{2}{\pi} \int_{-\infty}^{\infty} s(\omega) \, d\omega.$$  \hspace{1cm} (90)

Since the integral of each unit resonance peak $c_j(\omega)$ is $-\pi/2$, the amplitude $A_*$ is simply

$$A_* = \sum_j A_j.$$  \hspace{1cm} (91)

The sum in (91) can be evaluated by making use of

$$\sum_j (-1)^m b_j^m b_j^{m'} = \delta_{mm'},$$  \hspace{1cm} (92)

which follows directly from (53). The result is that

$$A_* = A$$  \hspace{1cm} (93)

where

$$A = \sum_m [M : \epsilon_{ms}^*] s_m$$  \hspace{1cm} (94)

is the spectral amplitude of $s(\omega)$ on the spherical earth $\delta_{e1}(\omega)$. That $A_* = A$ on an earth with spherical attenuation was first shown by Jordan (1978). Since the integral of $c_j(\omega)$ is independent of $\alpha_j$ (Gilbert & Dziewonski 1975) it is not surprising that $A_* = A$ on an earth with laterally heterogeneous attenuation as well. Correct to zeroth order in the eigenfunctions, free-oscillation source mechanism determinations based on equation (90) should not be affected by any aspect of the Earth's lateral heterogeneity including anelasticity.

Let us now use $s(\omega)$ to denote some particular component of $s(\omega)$ and $A_j$ to denote the same component of $A_j$ so that

$$s(\omega) = \sum_j A_j c_j(\omega).$$  \hspace{1cm} (95)

The apparent central frequency $\omega_*$ and half-width $\alpha_*$ of $s(\omega)$ will be defined as in Dahlen (1979a) by

$$\int_{-\infty}^{\infty} (\omega - \omega_*) |s(\omega)|^2 \, d\omega = 0.$$  \hspace{1cm} (96a)

$$\pi \alpha_* \int_{-\infty}^{\infty} |s(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} s(\omega) \, d\omega.$$  \hspace{1cm} (96b)

In evaluating $\omega_*$ and $\alpha_*$ upon an earth with spherically symmetric attenuation a minor error was committed in Dahlen (1979a). In particular, use was made of the relation $A_j A_k^* = A_j^* A_k$ in deriving equations (46) and (49) of that paper, although that relation is not generally true unless $M^* = M$. The quantities $\omega_*$ and $\alpha_*$ were called $\omega_{app}$ and $\alpha_{app}$ in Dahlen.
(1979a), and the correct form of equations (46) and (49) in the general case $M^* \neq M$ is

$$\omega_{\text{app}} = \frac{1}{2} \sum_{j} \sum_{k} A_j A_k^* (\omega_j + \omega_k) \left[1 - i(\delta \omega_j - \delta \omega_k)/2\alpha_0 \right]^{-1}$$

(97a)

$$\alpha_{\text{app}} = \frac{\alpha_0 \sum_{j} \sum_{k} A_j A_k^*}{\sum_{j} \sum_{k} A_j A_k^* \left[1 - i(\delta \omega_j - \delta \omega_k)/2\alpha_0 \right]^{-1}}$$

(97b)

Upon evaluating the integrals in equations (96) we obtain instead of (97) the more general results

$$\omega_* = \frac{1}{2} \sum_{j} \sum_{k} A_j A_k^* \left[(\omega_j + \omega_k) + i(\delta \omega_j - \delta \omega_k)\right] \left[1 - i(\delta \omega_j + i\delta \omega_j - \delta \omega_k + i\delta \omega_k)/2\alpha_0 \right]^{-1}$$

(98a)

$$\alpha_* = \frac{\sum_{j} \sum_{k} A_j A_k^*}{\sum_{j} \sum_{k} A_j A_k^* \left[1 - i(\delta \omega_j + i\delta \omega_j - \delta \omega_k + i\delta \omega_k)/2\alpha_0 \right]^{-1}}$$

(98b)

As they must, equations (98) reduce to (97) when $\delta \alpha_j = 0$.

In Dahlen (1979a) weighted moments $\langle \delta \omega^P \rangle$ were defined by

$$\langle \delta \omega^P \rangle \sum_j A_j = \sum_j (\delta \omega)^P A_j$$

(100)

Equations (69) and (70) of that paper were derived under the assumption that $\langle \delta \omega^P \rangle = \langle \delta \omega \rangle^P$. This, however, is also only true if $M^* = M$ and the correct general form of (69) and (70) if $M^* \neq M$ is

$$\omega_{\text{app}} = \omega_0 + \frac{1}{2} \sum_{p=0}^{\infty} (i/2\alpha_0)^p \sum_{q=0}^{p} (-1)^q \binom{p}{q} \langle \delta \omega^q \rangle^* \langle \delta \omega^{p-q+1} \rangle + \langle \delta \omega^{q+1} \rangle^* \langle \delta \omega^{p-q} \rangle$$

(101a)

$$\alpha_{\text{app}} = \sum_{p=0}^{\infty} (i/2\alpha_0)^p \sum_{q=0}^{p} (-1)^q \binom{p}{q} \langle \delta \omega^q \rangle^* \langle \delta \omega^{p-q} \rangle$$

(101b)

where

$$\binom{p}{q} = \frac{p!}{q! (p-q)!}$$

(102)
The natural quantity in the anelastic case is not $\langle \omega^\rho \rangle$ but $\langle (\omega + i\alpha)^\rho \rangle$, defined by

$$\langle (\omega + i\alpha)^\rho \rangle \sum_j A_j = \sum_i (\omega_j + i\alpha_j)^\rho A_j. \quad (103)$$

If the definition (103) is employed in equations (98) we obtain instead of (101) the expressions

$$\sum_{p=0}^\infty \sum_{q=0}^p (-1)^q \binom{p}{q} \langle (\omega + i\alpha)^q \rangle^* \langle (\omega - i\alpha)^p - q + 1 \rangle$$

$$\omega_* = \omega_0 + \delta\omega_0 + \frac{1}{2} \frac{\sum_{p=0}^\infty \sum_{q=0}^p (-1)^q \binom{p}{q} \langle (\omega + i\alpha)^q \rangle^* \langle (\omega + i\alpha)^p - q \rangle}{\sum_{p=0}^\infty \sum_{q=0}^p (-1)^q \binom{p}{q} \langle (\omega + i\alpha)^q \rangle^* \langle (\omega + i\alpha)^p - q \rangle}$$

$$\alpha_* = \frac{\sum_{p=0}^\infty \sum_{q=0}^p (-1)^q \binom{p}{q} \langle (\omega + i\alpha)^q \rangle^* \langle (\omega + i\alpha)^p - q \rangle}{\sum_{p=0}^\infty \sum_{q=0}^p (-1)^q \binom{p}{q} \langle (\omega + i\alpha)^q \rangle^* \langle (\omega + i\alpha)^p - q \rangle}$$

(104a)

In addition to the dispersive correction $\delta\omega_0$, equations (104) differ from (101) only by the replacement of $\langle \omega^p \rangle$ by $\langle (\omega + i\alpha)^p \rangle$. Once again the two sets of equations clearly coincide when $\delta\alpha = 0$.

As in Dahlen (1979a) we can eliminate $A_j$ from (103) and express $\langle (\omega + i\alpha)^p \rangle$ in terms of the matrix elements of $H^p$ ($H$ raised to the $p$th power). The eigencolumns of $H^p$ are the same as those of $H$ and the associated eigenvalues are $(\omega_j + i\alpha_j)^p$, i.e.

$$\sum_{j,m} H_{mm}' b_j^{m'} = (\omega_j + i\alpha_j)^p b_j^m.$$ 

(106)

Making use of (89), (92) and (106) we find for any component $s(\omega)$ of $s(\omega)$ that

$$\langle (\omega + i\alpha)^p \rangle \sum_m [M : \epsilon_{ms}^*] s_m = \sum_{m'} \sum_m H_{m'm} [M : \epsilon_{ms}^*] s_{m'}.$$ 

(107)

This has exactly the same form as equation (66) of Dahlen (1979a) for $\langle \omega^p \rangle$. If $\delta\alpha = 0$ the matrix $H^p$ is Hermitian and equation (107) then shows that $\langle \omega^p \rangle$ is real if $M^* = M$. More generally $H^p$ will not be Hermitian and so $\langle (\omega + i\alpha)^p \rangle$ will be complex.

Equations (104) are valid regardless of the relative magnitude of splitting and broadening due to attenuation. If both $|\delta\omega_j|$ and $|\delta\alpha_j|$ are small compared to $\alpha_0$ then we may approximate both (104a) and (104b) by the first two terms

$$\omega_* \approx \omega_0 + \delta\omega_0 + \text{Re} \langle \omega + i\alpha \rangle$$

$$\alpha_* \approx \alpha_0 + \text{Im} \langle \omega + i\alpha \rangle.$$ 

(108a)

(108b)

The specific form (89) of $A_j$ has not been used in the derivation of either (104) or (108) although it has been used to obtain (107). Equation (108) is thus valid for any set of amplitudes $A_j$ provided $\langle (\omega + i\alpha)^p \rangle$ is defined by (103) and provided $|\delta\omega_j| < \alpha_0$ and $|\delta\alpha_j| < \alpha_0$. The interference between two closely spaced resonance peaks has been examined by Buland.
(1979, private communication), and equations (108) are the obvious generalization of his results to the case of more than two peaks. If weighted averages \( \langle \delta \omega \rangle \) and \( \langle \delta \alpha \rangle \) are defined by

\[
\langle \delta \omega \rangle \sum A_j = \sum \delta \omega_j A_j
\]

\[
\langle \delta \alpha \rangle \sum A_j = \sum \delta \alpha_j A_j,
\]

equations (108) take the form

\[
\omega_s \approx \omega_0 + \delta \omega_0 + \text{Re} \langle \delta \omega \rangle - \text{Im} \langle \delta \alpha \rangle
\]

\[
\alpha_s \approx \alpha_0 + \text{Re} \langle \delta \alpha \rangle + \text{Im} \langle \delta \omega \rangle.
\]

Equations (72) and (73) of Dahlen (1979a) are correct if \( M^* = M \) but more generally they should be replaced by (110) with \( \langle \delta \alpha \rangle = 0 \).

6 The geometrical optics limit

Equations (104) and (107) are true for any isolated multiplet \( g_i S_i \) and \( n_i T_i \) and any slight lateral heterogeneity \( \delta \Phi(\omega_0) \). We shall now restrict attention to the surface wave-equivalent multiplets \( n < l \) in the geometrical optics limit \( s < l \), where \( s \) is the maximum significant degree in the expansion of \( \delta \Phi(\omega_0) \). Now that the spadework has been done the asymptotics is easy. As noted, the only sensible difference between equations (104) and (101) is the replacement of \( \langle \delta \omega \rangle \) by \( \langle (\delta \omega + i \delta \alpha) \rangle \). Equation (107) for \( \langle (\delta \omega + i \delta \alpha) \rangle \) is, however, exactly the same as equation (66) of Dahlen (1979a) for \( \langle \delta \omega \rangle \); the only difference is that the matrix \( H^* \) is no longer Hermitian. Since that property was nowhere invoked in sections 7–9 of Dahlen (1979a), all the results obtained there can be generalized in an obvious way immediately.

At every point \( \theta, \phi \) on the surface of the Earth \( r = a \) let us now let \( \delta \omega_{\text{local}} \) and \( \delta \alpha_{\text{local}} \) be the perturbations to \( \omega_0 + \delta \omega_0 \) and \( \alpha_0 \) which would result if the spherical earth \( \Phi(\omega_0) \) were to suffer a spherical perturbation equal to that underlying \( \theta, \phi \). Correct to first order, \( \delta \omega_{\text{local}} \) and \( \delta \alpha_{\text{local}} \) are given by

\[
\delta \omega_{\text{local}} = \int_0^a [K_0 \delta \kappa + M_0 \delta \mu + R_0 \delta \rho_0] r^2 dr + \sum_d \Gamma_d h
\]

\[
\delta \alpha_{\text{local}} = \int_0^a [\bar{\kappa}_r K_0 \delta q_\kappa + \mu_r M_0 \delta q_\alpha] r^2 dr.
\]

Let us denote the source–receiver great-circle path by \( P \) (see Fig. 2) and define the great-circular averages \( \mathcal{W} \) and \( \mathcal{A} \) by

\[
\mathcal{W} = \frac{1}{2\pi a} \oint_P \delta \omega_{\text{local}} ds
\]

\[
\mathcal{A} = \frac{1}{2\pi a} \oint_P \delta \alpha_{\text{local}} ds.
\]

It is clear that equation (121) of Dahlen (1979a) stating that \( \langle \delta \omega \rangle \sim \mathcal{W} \) is more generally replaced by

\[
\langle (\delta \omega + i \delta \alpha) \rangle \sim (\mathcal{W} + i \mathcal{A})^P.
\]
Figure 2. The great-circle path connecting the source $r_s$ and the receiver $r$ is denoted by $P$. The great-circle path whose take-off azimuth at $r_s$ is $\psi$ is denoted by $P(\psi)$ so that $P(0) = P$.

where $\sim$ denotes asymptotic equality in the limit $n \ll l$ and $s \ll l$. When the result (113) is substituted into equations (104) and the binomial identity

$$\sum_{q=0}^{p} (-1)^q \binom{p}{q} = \delta_{p0}$$

is employed we obtain the simple asymptotic results

$$\omega_* \sim \omega_0 + \delta \omega_0 + \mathcal{M}$$

$$\alpha_* \sim \alpha_0 + \mathcal{A}.$$  

These are the generalization of equations (124) and (125) of Dahlen (1979a) to the case of an aspherical anelastic earth.

The asymptotic form of the spectrum $s(\omega)$ is evidently

$$s(\omega) \sim -\frac{1}{2} A \left[ \alpha_* + i(\omega - \omega_*) \right]^{-1}.$$  

This is the result conjectured by Dahlen (1980a). It is the same as equation (126) of Dahlen (1979a) except that $\alpha_0$ is replaced by the great-circle average $\alpha_*$. The corresponding result in the time domain in the case $M(t) = M(t)$ is

$$s(t) \sim A \left[ 1 - \cos \omega_* t \exp (-\alpha_* t) \right].$$

This is a single standing wave of frequency $\omega_*$ and decay rate $\alpha_*$. The mechanism responsible for the appearance of a single resonance peak is cancellation of adjacent singlets as discussed by Dahlen (1979b).

Near the epicentre and its antipode the approximation (116) breaks down for reasons discussed in Dahlen (1979b, 1980b). The uniformly valid results obtained in the latter may, however, be generalized easily. In particular let us now label the various great circles passing through $r_s$ by $P(\psi)$ where $P(0) = P$ (see Fig. 2), and let us define $\omega_*(\psi)$ and $\alpha_*(\psi)$ by

$$\omega_*(\psi) = \omega_0 + \delta \omega_0 + \frac{1}{2\pi} \oint_{P(\psi)} \delta \omega_{\text{local}} \, ds$$

$$\alpha_*(\psi) = \alpha_0 + \frac{1}{2\pi} \oint_{P(\psi)} \delta \alpha_{\text{local}} \, ds.$$
An expression for \( s(\omega) \) which is uniformly valid on \( r = a \) is

\[
s(\omega) \sim \int_0^{2\pi} a(\psi)c_*(\psi) d\psi
\]  

(119)

where \( a(\psi) \) is given by equations (75)–(84) of Dahlen (1980b) and

\[
c_*(\psi) = -\frac{i}{2} [\alpha_*(\psi) + i(\omega - \omega_*(\psi))]^{-1}.
\]  

(120)

This differs from equation (74) of Dahlen (1980b) only by the substitution of \( \alpha_*(\psi) \) for \( \alpha_0 \) in (120), and all the remarks regarding its validity and its properties are still applicable.

7 Conclusions

The problem we have considered is a generalization of a well-known problem in classical mechanics, namely the determination of the small oscillations of a non-rotating system with a finite number of generalized coordinates and with frictional forces proportional to the general velocities. More-or-less exhaustive treatments of this problem have been given by Rayleigh (1887), Routh (1884) and Lamb (1932) among others. It is known that the problem can be reduced to a secular equation of the form

\[
\det [\sigma^2 T + i\sigma F + V] = 0
\]  

(121)

where \( T, F \) and \( V \) are real symmetric matrices. In the absence of dissipation the Rayleigh dissipation function \( F = 0 \) and it is known that \( T \) and \( V \) can be diagonalized simultaneously. In that case, if both \( T \) and \( V \) are positive definite it is known that the eigenfrequencies \( \sigma \) are all real. If \( F \) is positive definite as well it is known that roots \( \sigma \) occur in pairs \( \omega + i\alpha \) and \( -\omega + i\alpha \) where \( \alpha > 0 \). A special case which, as Rayleigh remarks, ‘occurs frequently, in books at any rate’ arises if the matrix \( F \) happens to be diagonalized by the same transformation diagonalizing \( T \) and \( V \). In that case the real normal modes of the non-dissipative system defined by \( T \) and \( V \) are modes of the dissipative system as well, and the phase of every freely decaying oscillation is always either 0 or \( \pi \). This obviously corresponds to the normal case \( HH^\text{adj} = H^\text{adj}H \) considered in Section 3. More generally, if \( F \) cannot be diagonalized with \( T \) and \( V \), the non-dissipative normal modes become coupled and the phase of freely decaying oscillations becomes variable. This corresponds to the complex eigenfunctions \( s_j \) found above.

The most immediately useful result obtained here is the asymptotic form for an isolated multiplet spectrum \( s(\omega) \) in the limit \( n \ll l \) and \( s \ll l \). In that limit away from the epicentre and its antipode singlet cancellation leads to a single-peak spectrum whose complex amplitude \( A \) is the same as on the spherical reference earth model \( \tilde{e}_\text{el}(\omega_\tau) \). The equivalent time-domain signal \( s(t) \) is a single decaying sinusoid of apparent frequency \( \omega_* \) and apparent decay rate \( \alpha_* \). The apparent frequency of oscillation \( \omega_* \), as discussed by Dahlen (1979a, b), depends only on the average density and elastic structure underlying the great-circle path \( P \) connecting the source and receiver, i.e.

\[
\omega_* = \omega_0 + \frac{1}{2\pi a} \int_P \delta\omega_\text{local} \, ds.
\]  

(122)

The degenerate eigenfrequency \( \omega_0 \) in (122) is that of the reference earth model \( \tilde{e}_\text{el}(\omega_\tau) \) and \( \delta\omega_0 \) is the correction due to dispersion \( \delta(\omega_0) - \tilde{e}_\text{el}(\omega_\tau) \). The local quantity \( \delta\omega_\text{local} \) is due to the aspherical perturbation \( \delta e(\omega_0) \) underlying each point \( \theta, \phi \) on \( P \). The new result
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obtained here has been the demonstration that the apparent decay rate \( \alpha_* \) likewise depends only on the average attenuation structure underlying \( P \); in fact

\[
\alpha_* = \alpha_0 + \frac{1}{2\pi a} \oint_P \delta\alpha_{\text{local}} \, ds. \tag{123}
\]

The quantity \( \alpha_0 \) is the multiplet decay rate due to the spherical part of the attenuation structure \( q_e(\omega_0), q_\mu(\omega_0) \) and \( \delta\alpha_{\text{local}} \) is due to the aspherical structure \( \delta q_e(\omega_0), \delta q_\mu(\omega_0) \) underlying \( \theta, \phi \).

The importance of these results is that they tell us explicitly how to interpret two readily measured quantities, namely \( \omega_* \) and \( \alpha_* \), in terms of the lateral heterogeneity of the Earth. Silver & Jordan (1981) have in a recent study measured over 2000 apparent frequencies \( \omega_* \) and used them to infer \( \delta\omega_{\text{local}} \) for each of six different tectonic provinces by means of regression analysis. The geological reasonableness of their final local estimates is a convincing demonstration of the promise of using asymptotic methods to interpret normal mode measurements in the range \( n < 1 \). A few measurements of \( \alpha_* \) have likewise been collected recently by Geller & Stein (1979), but they did not have enough data to attempt a geographical interpretation. As more \( \alpha_* \) data are collected a geographical interpretation based on the asymptotic result (124) will become possible. Testing spectra to see if they actually do have the shape of a single resonance peak should strictly precede any interpretation. The narrow-band time-domain method of Geller & Stein (1979) is one straightforward way of performing such a test.

The asymptotic version of the diagonal sum rule guarantees that the average of a large number of randomly selected \( \omega_* \) measurements tends to \( \omega_0 + \delta\omega_0 \) and that the average of a large number of \( \alpha_* \) measurements tends to \( \alpha_0 \). This provides a means of determining the spherically averaged elastic and anelastic structure of the Earth \( \bar{\delta}(\omega_0) \) and provides a justification for the averaging of \( \alpha_* \) values performed by Geller & Stein (1979).

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