Anomalous Fluctuations near Nonequilibrium Soft Transitions. II
—— Symmetry-Breaking Transitions ——

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Critical fluctuations of a nonequilibrium soft intrinsic symmetry-breaking transition are investigated by taking the Rayleigh-Bénard instability in a nematic liquid crystal as an example. This system is particularly important because critical fluctuations have been observed experimentally due to the strong orientational fluctuations. In this actual system, there appear not only fluctuations of the thermal origin but also those of the surroundings which cause multiplicative stochastic forces. These fluctuations are taken into account by deriving a stochastic Ginzburg-Landau type equation for an order-parameter density with the aid of the projector method for the contraction of state variables developed in previous papers. Then, in addition to anomalous fluctuations of the order parameter generated by the coupling to irrelevant variables, various new behaviors such as a shift of the critical point due to a multiplicative stochastic force and a smoothing of the transition due to an inhomogeneous term are illustrated.

§ 1. Introduction

In a previous paper,1) a critical behavior of a soft homogeneous transition in a nonequilibrium system has been investigated. It has been shown that there appears a new kind of fluctuating force acting on the order parameter through the mode coupling to irrelevant macrovariables, leading to four different critical regimes with different scalings and different critical dimensionalities. In particular, the variance of the order parameter due to this fluctuating force does not diverge even near the critical point, and exhibits a remarkable difference from equilibrium critical phenomena.2)

Fluctuations are usually small except in the case of unstable systems. They, however, give important information on many aspects of the system. Fortunately, there exists a good candidate in which anomalous fluctuations near nonequilibrium soft transitions are observable with present-day experimental techniques. That is the Rayleigh-Bénard instability in a nematic liquid crystal. A nematic liquid crystal is known to exhibit strong orientational fluctuations at the microscopic level,3) and these fluctuations couple to hydrodynamic fluctuations.4)

In fact, in para-azoxyanisole, the hydrodynamic fluctuations were recently studied by using long time-scale counting in a neutron scattering experiment.5) Critical enhancement was observed and a similarity to critical scattering of
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Equilibrium systems is demonstrated. In addition, a rounding of the transition has been observed with external magnetic field of 70 ersted. Therefore the nematic liquid crystal would be important for investigating nonequilibrium critical behavior, in particular, not only to show the similarity to the equilibrium critical phenomena but also to demonstrate characteristic features of the nonequilibrium transitions.

In this paper, we shall investigate such critical fluctuations of a soft intrinsic symmetry-breaking transition. As actual systems, the Rayleigh-Bénard instabilities in an isotropic liquid and in a nematic liquid crystal are studied. In these systems, there exist fluctuations of the thermal origin and of the surroundings, for example, the temperature fluctuations at the boundary and the fluctuations of the applied magnetic field in the nematic liquid crystal. They cause an additive or a multiplicative stochastic force.\(^6\)\(^7\) In order to take account of such fluctuations, we use a projector method for the contraction of state variables in stochastic equations of motion developed in a previous paper.\(^8\) Then, in addition to the anomalous fluctuations investigated in I, various behaviors such as a shift of the critical point due to a multiplicative stochastic force and a smoothing of the transition due to an inhomogeneous term will be illustrated.

In §2, we study a reaction-diffusion system which exhibits an intrinsic symmetry-breaking transition. In §§3 and 4, the Rayleigh-Bénard instabilities in the isotropic liquid and in the nematic liquid crystal are investigated as actual examples. Section 5 is devoted to a short summary.

§2. Reaction-diffusion systems

In order to proceed in parallel to I,\(^1\) let us first consider reaction-diffusion system which exhibits a soft intrinsic symmetry-breaking transition. The starting equation is

\[
\dot{x}(r, t)/\dot{t} = -\overline{Q}(p)x + \overline{D} \nabla^2 x + N(x, p) + S(r, t),
\]

(2.1)

where the notations are the same as in I. In this paper, notations that are not explained are the same as in I. In this case the eigenvalues and eigenvectors are those of the matrix \(\tilde{F}(p, k^2) = \tilde{Q} + \tilde{D}(ik)^2\),\(^9\)

\[
\tilde{F}(p, k^2)\alpha^l = \lambda^l(p, k^2)\alpha^l.
\]

(2.2)

The intrinsic symmetry-breaking transition is characterized by the vanishing of at least one root, \(\lambda^l\), at the critical point \(p = p_c\) for \(k = k_c \neq 0\). If we assume that the spatial order is one dimensional with \(k = 0\), then it is useful to expand \(x\) as

\[
x(r, t) = \sum_l \sum_k u^l_k(t) e^{ikr}\alpha^l.
\]

(2.3)
where \( |a^i_r\rangle = |a^i, p, k_c^2\rangle \) and \( u_{ar} \equiv (u_{ar})^* \). Then (2.1) is transformed into

\[
\frac{\partial u_{ar}}{\partial t} = -\lambda^i_s(p) u_{ar} + \sum_{a} D_{a\beta}(p) [p^2 + 2i k_c \cdot \mathbf{P}] u_{ar}^* + \sum_{\lambda} \sum_{n} N^{m}(\eta, \lambda, n) u_{ar}^n u_{ar}^{*n} + \cdots + F_p^i(\mathbf{r}, t)
\]  

(2.4)

with

\[
D_{a\beta}(p) = \langle a^i | \hat{D} | \beta^j \rangle ,
\]

(2.5)

\[
N^{m}(\eta, \lambda, n) = \sum_{\alpha} \sum_{\lambda} \sum_{\beta} \langle a^\alpha | N^m(p) | \beta^\eta \rangle | \beta^\lambda \rangle | \beta^m \rangle ,
\]

(2.6)

\[
F_p^i(\mathbf{r}, t) = \frac{1}{L'} \int_{-L'/2}^{L'/2} d\mathbf{r} e^{-i \mathbf{k_c} \cdot \mathbf{r}} \langle a^i | \hat{S} \rangle ,
\]

(2.7)

where \( L' \) characterizes the size of a region which is large compared to \( 1/k_c \), but small compared to the variation of \( u_{ar}^* \). The order parameters are the complex variables \( u_{ar} \equiv \{ u_{ar}^i, u_{ar}^\beta \} \) with a soft eigenvalue \( \lambda_0 = \lambda_1^1 \), and other variables are the stable modes. The spectral density matrix \( Q^i_\alpha^\beta \) is given by

\[
\langle F_p^i(\mathbf{r}, t) F_p^\alpha(\mathbf{r}', t'); \psi_0 \rangle = 2 Q^i_\alpha^\beta(u_{ar}) \delta(\mathbf{r} - \mathbf{r}') \delta(t - t')
\]

(2.8)

with

\[
Q^i_\alpha^\beta(u_{ar}) = \sum_{\alpha} \frac{1}{L'} \int_{-L'/2}^{L'/2} d\mathbf{r} e^{i \mathbf{k_c} \cdot \mathbf{r}} \langle a^i | \hat{U} \rangle | a^\alpha \rangle \delta(\mathbf{r} - \mathbf{r}') \delta(t - t')
\]

(2.9)

where \( \hat{U} \) is the spectral density matrix of \( S(\mathbf{r}, t) \) and may depend on the order parameter through the state variables as in the case of the Oregonator described in I. The coupled equations (2.4) are quite similar to those of (2.5) of I. For the Rayleigh-Bénard instability in the isotropic liquid and in the nematic liquid crystal, the basic hydrodynamic equations can be transformed into the form of (2.4) so that they can be investigated by the same procedure.

There also exists a critical region characterized by the critical scaling described in I. In this case, however, a scaling for the mode amplitudes is

\[
u_{ar}^n \sim L^{-n},
\]

(2.10a)

\[
u_{ar}^1 \sim (a \pm 1), \quad \nu_{ar}^0 \sim L^{-2},
\]

(2.10b)

\[
u_{ar}^n \sim L^{-n}, \quad \text{otherwise}.
\]

(2.10c)

With this scaling, (2.4) is reduced, to the lowest order in \( 1/L \), to the equations which are essentially equivalent to the spatially-inhomogeneous Haken-Zwanzig model (1.1) of I in the neighborhood of the critical point. They are written as

\[
\frac{\partial u_{ar}^i}{\partial t} = -\lambda^i_s u_{ar}^i + \sum_{\alpha} D_{a\beta}(p^2 + 2i k_c \cdot \mathbf{P}) u_{ar}^i
\]

(2.10)
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\[\begin{align*}
  &+ \sum_a [\bar{N}_{a(1)}(r_0) u_{1r}^0 u_{ar}^0 + \bar{N}_{a(1)}(r_0) u_{1r}^{-1} u_{ar}^2 ], \\
  \partial u_{1r}/\partial t = -\lambda_1 u_{1r} + D_1 \partial^2 \partial^r u_{1r} = 0, \quad (\alpha \neq 1) \\
  \partial u_{0r}/\partial t = -\lambda_0 u_{0r} + \bar{N}_{a(1)}(r_0) u_{ar}^0 \equiv 0, \\
  \partial u_{ar}/\partial t = -\lambda_a u_{ar}^2 + N_{a(1)}(r_0) u_{1r}^2 = 0, \\
  \partial u_{0r}/\partial t = 0, \quad \text{otherwise,}
\end{align*}\]  

and those complex conjugates, with \( \bar{N}_{a(1)}(r_0) = N_{a(1)}(r_0) + N_{a(1)}^*(r_0) \). Instead of \( 2N_{a(1)}(r_0) \), the notation \( \bar{N}_{a(1)}(r_0) \) is used, since, in the Rayleigh-Bénard instability, the nonlinear part includes the differential operator so that \( \bar{N}_{a(1)}(r_0) \equiv N_{a(1)}(r_0) \) in some cases. It is easily seen that the stable modes effectively coupled to the order parameter are \( u_{0r}^0 \) and \( u_{ar}^2 \), while \( u_{0r}^{-1} \) gives a contribution to the diffusion part.

First we consider the fluctuating forces such that \( F_{\alpha r}^2 \) and \( F_{\alpha r}^4 \) for \( (\alpha, l) \neq (1, \pm 1) \) are not cross-correlated. For such fluctuations, the elimination of the stable modes is performed quite parallel to that of \( I \), including a contribution to the diffusion part through \( u_{1r}^1 \). Then the reduced equation of motion for \( u_{1r}^1 \) turns out to be

\[\begin{align*}
  \partial u_{1r}/\partial t = &\langle Mu_{1r} \rangle + I(u_{0r}) + R^\text{col}(r, t) \\
  \text{with} \\
  &\langle Mu_{1r} \rangle + I(u_{0r}) = D \partial^2 \partial^r u_{1r} - \lambda_1 u_{1r}^2 - g |u_{1r}|^2 u_{1r} = 0, \\
  &I(u_{0r}) = (1/\omega) Ku_{1r}^0, \\
  &R^\text{col}(r, t) = R(r, t) + F_{\alpha r},
\end{align*}\]  

where

\[\begin{align*}
  K = &\sum_a [\bar{N}_{a(1)}(r_0) + \bar{N}_{a(1)}(r_0) \chi^0_a/\lambda_a^0 + \bar{N}_{a(1)}(r_0) \chi_0^a/\lambda_0^0 + \bar{N}_{a(1)}(r_0) \chi_0^a/\lambda_0^a], \\
  R(r, t) = &e^{i\omega t} \sum_a [2iD_a \chi^0_a \partial^r u_{ar}^0 + \bar{N}_{a(1)}(r_0) u_{1r}^0 u_{ar}^0 + \bar{N}_{a(1)}(r_0) u_{ar}^2], \\
  \langle u_{0r}^m u_{0r}^n \rangle = &2\chi_0^0 \delta_{m0} \delta(r - r'),
\end{align*}\]  

and \( D \) and \( g \) agree with those given by the conventional theories. Therefore, if \( d > d_c \) and \( d > 2 \), the stochastic Ginzburg-Landau type equation for the order parameter is obtained as before.

Let us investigate the fluctuations in the steady state outside the Ginzburg subregion. If we assume that the renormalization term is negligible and the most probable value in the steady state is written as \( u_* = ((u_1)_*, (u_0)_* \rangle \) with \( (u_1)^* = e^{i\theta} \sqrt{-\lambda_0/g} \), then the variance in the steady state is given by
\[ Q \langle \tilde{u}_q \rangle^2 = \frac{1}{D} \frac{Q_{tq}^{(0)}(u_0^*)}{q^2 + \xi^2}, \]  
(2.19)

where \( \tilde{u}_q = (e^{-i\theta} \tilde{u}_1^{(1)} + e^{i\theta} \tilde{u}_1^{(-1)})/\sqrt{2} \) and

\[ Q_{tq}^{(0)}(u_0^*) = \begin{cases} 0, & \text{if } \lambda_0 > 0, \\ (Q_{tq}^{(1)}(u_0^*) + Q_{tq}^{(0)}(u_0^*)), & \text{if } \lambda_0 < 0 \end{cases} \]  
(2.20a)

(2.20b)

with \( Q_{tq}^{(1)} \) being its value at the critical point and

\[ \langle R_r(r, t)R_r(r', t') \rangle = 2Q(u_{0r})\delta(r - r')\delta(t - t') \]  
(2.21)

for \( R_r = (e^{-i\theta} R + e^{i\theta} R^*)/\sqrt{2} \). This spectral density \( Q(u_{0r}) \) is due to the fluctuations generated by the mode coupling and strongly depends on the order parameter so that the variance due to these fluctuations becomes small or remains constant as the critical point is approached in the ordered phase, and does not diverge even near the critical point.

When the correlations between the fluctuating forces of the unstable modes and the stable modes exist, we take account of them by means of the projector elimination developed for the contraction of the variables in the previous paper.\( ^9 \)

Then corresponding to the terms including \( Q_{tq}^{(1)} \) of (1.4) and (1.5) of I, there appear additional terms in the renormalization term and the spectral density. The additional renormalization term is

\[ I_{add} = 2 \sum_s [\hat{N}_{(i)}^{(0)}(u_0^*) Q_{s}^{(1)}(u_0^*) / \lambda_s^0 + \hat{N}_{(i)}^{(2)}(u_0^*) Q_{s}^{(2)}(u_0^*) / \lambda_s^2)] / \omega, \]  
(2.22)

to the lowest order, which does not depend on the order parameter. It should be noted that this term \( I_{add} \) gives an inhomogeneous term to the Ginzburg-Landau type amplitude equation, which causes a smoothing of the transition.

In addition, let us consider the situation such that the externally controlled parameter fluctuates around \( p_0 \) so that the eigenvalue is replaced by the fluctuating one; \( \lambda_i'(p_0) + f_p(r, t) \). Then the multiplicative stochastic force \( (F_{\epsilon t})_p \) acting on \( \dot{u}_{p1}^{(1)} \) appears:

\[ (F_{\epsilon t})_p = -u_{p1}^{(1)} f_p(r, t). \]  
(2.23)

If \( f_p \) is assumed to be a Gaussian white noise with mean value zero and correlation

\[ \langle f_p(r, t) f_p(r', t') \rangle = 2Q_p \delta(r - r') \delta(t - t'), \]  
(2.24)

then, with the aid of the projector method,\( ^9 \) it turns out that they contribute to the spectral density as

\[ (Q_p(u_0^*))_p = 2Q_p |u_0^*|^2 \]  
(2.25)
and to the drift term for $u_{1r}$ as

$$I_p(u_{1r}) = (Q_p/\omega)u_{1r}^1,$$  

which shifts the critical point.

§ 3. The Rayleigh-Bénard instability in isotropic liquids

As an actual example, we next consider the Rayleigh-Bénard instability in the isotropic liquid. For an infinitely extended horizontal fluid layer of the depth $d_0$ with a free-free boundary condition and no Boussinesq effect, a roll pattern is expected to appear with a horizontal wave vector $k_z$. Let us consider two dimensional disturbances only and take $k_z$ along the $x$ direction. The hydrodynamic variables are the $z$ component of the velocity fluctuation $v$ and the deviation $\theta$ of the temperature from a linear gradient. The $x$ component of the velocity fluctuation $u$ is related to $v$ by the continuity equation. The starting equations are well-known hydrodynamic equations in the Boussinesq approximation and in dimensionless units:

$$\frac{\partial}{\partial t} \begin{pmatrix} P \nabla^2 v \\ \nabla^2 \theta \end{pmatrix} = \begin{pmatrix} PP^4, & PP^2 \\ 0 & \nabla^2 \end{pmatrix} \begin{pmatrix} v \\ \theta \end{pmatrix} + \begin{pmatrix} \vec{d} \cdot (v \cdot \nabla) v \\ -(v \cdot \nabla) \theta \end{pmatrix} + S,$$  

where $P$ and $R$ are the Prandtl number and the Rayleigh number which are given by $\nu/\kappa$ and $\alpha \beta \kappa d^4_0/\nu \kappa$, respectively, and $\vec{d} \equiv (P_x \nabla x, 0, -P_z^2)$, and $S$ is a fluctuating force. Notations in $P$ and $R$ are as follows: $\beta$ is the mean temperature gradient, $a$ is the thermal expansion coefficient, $\kappa$ is the gravitational acceleration, $\nu$ is the kinematic viscosity and $x$ is the thermal diffusivity. The externally controlled parameter is the Rayleigh number $R$.

As will be shown in Appendix A, the coupled equations for the mode amplitudes $\nu_1^{\alpha \beta}$ and $\theta_{11}^{\alpha \beta}$ take the same form as (2.4). With the use of $N^{(1)} = 0$, $N^{(2)}_{11} = \gamma_3 \varepsilon_0 \delta_{12} \pi \xi$, and $C^{(1)}_{11} = -\sqrt{2} i \lambda P$, it turns out that the stable mode which gives the efficient mode coupling to the unstable modes is $u_{2z}^{\alpha \beta}$ only, while $u_{2z}^{\alpha \beta}$ is expected to contribute to the diffusion part. The mode $u_{2z}^{\alpha \beta}$ causes $\theta^{\alpha \beta}$ which contributes to the Nusselt number. Thus $\nu_1^{\alpha \beta}$, $\theta_{11}^{\alpha \beta}$ and $\theta^{\alpha \beta}$ play an important role.

Corresponding to this, there are fluctuating forces $F_{11}^{\alpha \beta}$ acting on the order parameter $u_{11}^{\alpha \beta}$ directly, and $F_{3z}^{\alpha \beta}$ for $u_{3z}^{\alpha \beta}$ mode. The spectral density matrix of these fluctuating forces are calculated as in deriving (2.9) from (2.7). Therefore, let us first derive $L_{ij}$, the spectral density matrix of $S(r, t)$.

For the fluctuations, we take the thermal fluctuating force $S_{th}$ and the temperature fluctuations at the lower plate. As is well-known, the Landau-Lifshitz fluctuating forces are written as $S_{th}^{ij} = -\sum_i \delta_i \cdot \nabla_s s_{ij}$ and $S_{th}^{ij} = -\sum_i \delta_i \cdot \hat{q}_s$.
whose correlations are

$$\langle s_{ab}(r, t)s_{a'}(r', t') \rangle = 2L_0 [\delta_{ab}\delta_{a'b'} + \delta_{ab}\delta_{a'b'}] \delta(r - r') \delta(t - t'),$$

(3.2a)

$$\langle q_{a}(r, t)q_{a'}(r', t') \rangle = 2L_1 \delta_{ab} \delta(r - r') \delta(t - t').$$

(3.2b)

with $L_0 = (\nu/\kappa' \rho)k_B T \delta^{-d}$ and $L_1 = (a g/\nu')^2 \delta^{-d} k_B T^2 / \rho c_p$. Here $c_p$ is the specific heat, $k_B$ is the Boltzmann constant and $T$ is the mean temperature of the liquid. It should be noted that $T$ is a sum of the linear gradient part and its deviation $\theta$ whose mean value is zero and would depend on the statistical state of the order parameter in the convection state. Then $(Q_{a' a}^{m n'})_h$ is calculated from (A·10) and (3.2), and it is well-known that the contribution from $L_1$ is negligible. For $(Q_{a' a}^{m n'})_h$, however, $L_1$ cannot be neglected since $S_{ab}(x)$ gives no contribution to it.

On the other hand, if we assume that the temperature at the lower plate does not maintain a constant value but fluctuates around a mean value $T_0$, then the temperature of the liquid is written as

$$T(r, t) = T_0 - \beta z + \beta(t, r, t) + \beta'(r, t)(1 - z/d_0),$$

(3.3)

where $\beta(t, r, t)$ represents the fluctuations of the temperature at the bottom. Since the external disturbances which cause the temperature fluctuation at the bottom is given at $z = 0$, the fluctuation $\beta(t, r, t)$ is assumed to be a Gaussian white noise with its correlation

$$\langle \beta(x, t)\beta(x', t') \rangle = 2L_0 \delta(x - x') \delta(t - t') / \omega^{1/2}.$$ 

(3.4)

This fluctuation contributes to the fluctuating force of (3.1) as

$$S_{ab}(r, t) = \left( \begin{array}{c} \nu \frac{a}{\kappa'} f_{ab}(r, t)(1 - z) \\ \nu \frac{a}{\kappa'} f_{ab}(r, t)(1 - z) - \nu \cdot \nu \cdot f_{ab}(r, t)(1 - z) \end{array} \right).$$

(3.5)

where $f_{ab}$ is the dimensionless form of $\beta(t, r, t)$. Then (3.5) and (A·10) with (3.4) lead to

$$
(Q_{a' a}^{m n'})_h = \langle a_{a' a} \beta_{m n'}^{(t)}(2\pi)^3 n n' I'' L_{ab}^{(t)} 
$$

(3.6)

with

$$L_{ab}^{(t)} = \frac{1}{L} \int_{-L/2}^{L/2} dx \nu \nu \nu \nu \nu \nu L_{ab}(x),$$

(3.7)

where $I''$ comes from the use of the dimensionless units and given by $\gamma z a^2 \delta^{-d} / \kappa z^2$. It should be noted that the multiplicative stochastic force from $-\nu \cdot \nu \cdot f_{ab}(1 - z)$ does not appear since the fluctuations $f_{ab}$ given at $z = 0$ do not couple to the velocity.

First we consider the case in which there are the thermal fluctuations and the
temperature fluctuations at the lower plate such that $L_s(x)$ is constant. In this case, $L_b^{-\lambda}$ vanishes except for $l = l'$. Then $F_1^{11}$ and $F_2^{12}$ are not cross-correlated. With the aid of the projector method, we obtain a reduced equation of motion with a streaming term of the well-known form and a renormalization term $I(u_b)$ and a fluctuating force. The total fluctuating force consists of $F_1^{11}(r, t) = (F_1^{11})_b + (F_1^{11})_a$ and $R(r, t)$ generated by the mode coupling. If $I(u_b)$ is assumed to be negligible, then the behavior of the fluctuations in the steady state can be investigated as in § 2. For a silicon oil of $P=57$ and $d_b=1$ cm, $Q_{\omega n}(u_b^a)$ is evaluated by using $|u_b^a|^2 = 3\pi^2 \tilde{R}$ with $\tilde{R} = (R - R_c)/R_c$ as

$$Q_{\omega n}(u_b^a) = \begin{cases} \frac{Q_n + Q_b}{Q_n(1 + 2.635 \tilde{R} + 2.251 \tilde{R}^2)} + \frac{Q_b(1 + 2.169 \tilde{R})}{(\lambda_b > 0)}, \\ \frac{Q_n}{Q_n(1 + 2.635 \tilde{R})} + \frac{Q_b(1 + 2.169 \tilde{R})}{(\lambda_b < 0)} \end{cases} \ (3.8a)$$

with $Q_n = 8.95 \times 10^{-9}$ and $Q_b = 9.30 \times 10^5 L_a^9$. In this estimation, smaller terms such as those proportional to $R^{12}, R^{32}$ and diffusion originated terms are dropped.

From (3.8), the crossover between the different critical regimes is easily shown. When the temperature fluctuations of the bottom plate are sufficiently suppressed so that $Q_{ln} \gg Q_n$, then the critical behaviors of the fluctuations are determined by the thermal fluctuations. In this situation, the $\tilde{R}$ term is dominant for $R > 0.38$. Even in the regime where the zeroth term is dominant, for example at $R = 0.2 < 0.38$, the part which is $R$ dependent gives 40% of the total spectral density. This part becomes small as $\tilde{R} \to 0$ and the variance due to this part never diverges even near the critical point. These behaviors, however, would be difficult to observe since $Q_{ln}$ is very small. On the other hand, if $Q_{ln} \ll Q_n$, then the $\tilde{R}$ dependent part gives 30% of the total spectral density at $\tilde{R} = 0.2$.

It should be noted that as the critical point is approached, another type of the crossover may occur. In the region where the correlation length $\xi$ is small enough as compared with the depth of the layer, the fluctuations are three dimensional and the renormalization term is negligible. On the other hand, in the region where $\xi$ is larger than $d_b$, i.e., $\tilde{R} < 0.135$ in the ordered phase, the correlation in the horizontal direction would develop and the fluctuations are expected to be two dimensional. Then, in this region, the renormalization term must be added. In the case of the silicon oil studied above, $I(u_b)$ is calculated with the choice of cutoff $q_c = d_b^{-1}$ as $I = I_a + I_b$ with $I_a = 1.11 \times 10^{-14} u_b^2$ by the thermal fluctuations, and $I_b = 3.51 \times 10^7 L_b u_b^2$ by the temperature fluctuations at the lower plate. The shift of the critical point due to $I_a$ is negligibly small.

Finally, when $L_s(x)$ is not constant, $Q_{\omega n}^{102}$ appears and contributes to the renormalization term as $I_{aa} = 9.41 \times 10^8 L_s^{-1}$, with the choice of cutoff $q_c = l_1^{-1}$. Then the Ginzburg-Landau type amplitude equation has this inhomogeneous term and the transition is no longer sharp but becomes smooth.111 This smoothing of the transition comes from the fact that the flow in the horizontal direction may occur even if $R < R_c$, when $L_s(x)$ is not constant and depends on the position in
the lower plate. Since the existence of the horizontal flow is inconsistent to the conduction picture, the sharp transition from the conduction state to the convection state is broken and the transition is rounded. When $L_0(x)$ is constant, the inhomogeneous term $J_{inh}$ vanishes so that no horizontal flow occurs.

§ 4. The Rayleigh-Bénard instability in nematic liquid crystals

The Rayleigh-Bénard instability in the nematic liquid crystals shows a great variety of behaviors as compared to that in the isotropic liquids. Due to a coupling between the flow induced by an initial temperature fluctuation and a distortion of the director created by the gradients of this flow field, and due to an anisotropic heat conductivity, there is a destabilizing heat focusing effect, apart from a buoyancy force mechanism which controls the isotropic liquid case. For the director, the diffusive relaxation of its fluctuation is very slow compared to those of the velocity fluctuation and the temperature fluctuation. This orientational relaxation time, however, can be changed very widely by applying an external magnetic field $H$. In addition, the nematic liquid crystals exhibit a strong orientational fluctuation. Pretransition effects are experimentally observed in the case of the Rayleigh-Bénard instability by neutron scattering and in the case of the electrohydrodynamic instability. A shift of the critical point induced by an external white noise is also experimentally observed in the electrohydrodynamic instability. Thus the nematic liquid crystals are very useful to investigate the nonequilibrium phase transitions.

In the case of the Rayleigh-Bénard instability in the nematic liquid crystals, let us start from the equations of the nematodynamics in the Boussinesq approximation. An infinitely extended thin horizontal fluid layer with the free-free boundary condition is considered. In addition, we investigate the planer configuration in which the instability occurs when heated from below. In this configuration, a roll pattern is expected to occur with a horizontal wavevector $k_x$ along the direction of the director. We shall limit ourselves to the analysis of two dimensional disturbances only. The director is $(1, 0, 0)$, and the corresponding disturbances are $n=(0, 0, n)$ and $v=(u, 0, v)$ for the director and the velocity fields, respectively. The nematodynamic equations are

\[
\begin{align*}
\frac{\partial}{\partial t} & \begin{pmatrix} \varphi_x n \\ \theta \\ \varphi_z v \end{pmatrix} = \begin{pmatrix} \frac{K_2}{\gamma_1} \varphi_x^2 \varphi_x \\ \varphi_z \varphi_z \\ \varphi_z \varphi_z \end{pmatrix} \\
&+\begin{pmatrix} 0 \\ \frac{-\alpha_2}{\gamma_1} \varphi_z^2 \\ \frac{-\alpha_2}{\gamma_1} \varphi_z^2 \end{pmatrix} \\
&+\begin{pmatrix} (\alpha_2/\gamma_1) \varphi_z^2 \\ \beta \\ (\gamma_3/\rho) \varphi_z^2 \end{pmatrix} \begin{pmatrix} n \\ \theta \\ v \end{pmatrix} \\
&+\begin{pmatrix} 0 \\ \sigma \sqrt{\gamma_1} \\ \sigma \sqrt{\gamma_1} \end{pmatrix} \begin{pmatrix} n \\ \theta \\ v \end{pmatrix} + N + S,
\end{align*}
\]

(4.1)

where $\varphi_z = \varphi_z^2 + (K_3/\gamma_1) \varphi_z^2$, $\varphi_z = \varphi_z^2 - (\alpha_2/\gamma_1) \varphi_z^2$, $\varphi_z = \varphi_z^2 + (\gamma_3/\rho) \varphi_z^2$. 
and \( \mathcal{F}_v^* \equiv \mathcal{F}_v^* + 2(\nu_1 + \nu_2 - \nu_3) / \nu_2 \cdot \mathcal{F}_v^* \mathcal{F}_v^* + \mathcal{F}_v^* \). For other material constants, we follow the notations used in Ref. 3; \( \alpha_i \) (\( i = 1 \sim 6 \)) are viscosity coefficients usually called the Leslie coefficients and \( \gamma_i \equiv \alpha_3 - \alpha_2 \); \( \chi_2 \) and \( \chi_\perp \) are the thermal conductivities, parallel and orthogonal to the director, respectively and \( \chi_a \equiv \chi_2 - \chi_\perp \); \( K_1 \) and \( K_3 \) are Frank's constants; \( \nu_i \) (\( i = 1 \sim 3 \)) are the viscosity coefficients used by Harvard group; \( N \) is a nonlinear term and \( S \) is a fluctuating force. The pressure term is eliminated as is usual in the hydrodynamic equations of isotropic liquids. The relevant variables are \( n, v \) and \( \theta \). It should be noted that we take the director disturbance \( n \) instead of an angle \( \psi \) between the director and the \( x \) direction. Therefore the above equations are valid where sin \( \psi \) can be approximated as \( \psi \), namely, near the critical point where the director disturbance is not too large.

In the presence of an external magnetic field applied parallel to the \( x \) direction, the term \( K_3 \gamma / \gamma_1 \mathcal{F}_v \) in (4·1) is modified as \( (K_3 \mathcal{F}_v^2 + K_1 \mathcal{F}_v^2 - \chi_a H^2) / \gamma_1 \) with \( \gamma_a \equiv \chi_2 - \chi_\perp \), where \( \chi_2 \) and \( \chi_\perp \) are the magnetic susceptibilities parallel and orthogonal to the director, respectively.9 Then we have \( (K_3 \gamma / \gamma_1 \mathcal{F}_v) / \mathcal{K} \), where \( \mathcal{K} \equiv K_3(1 + H^2 / H_c^2) \) with \( H_c^2 = \pi^2 K_3 / \chi_a d_0^2 \), and \( \mathcal{K} \equiv K_3 \mathcal{F}_v^2 + K_1 \mathcal{F}_v^2 - \chi_a H^2 / \mathcal{K} \), which depends on \( H \) and reduces to \( \mathcal{F}_v \) when \( H = 0 \). This external magnetic field applied parallel to the director stabilizes the director and reduces the relaxation time constant of the orientational fluctuations.

Let us introduce the dimensionless units as

\[
\begin{align*}
& t \rightarrow t / \gamma_{nh}, \\
& r \rightarrow d_0 r, \\
& n \rightarrow n, \\
& v \rightarrow (\alpha \phi x_0 / \gamma_0 \gamma \rho d_0) v, \\
& \theta \rightarrow (\phi x_0 / \gamma_0 \rho d_0) \theta,
\end{align*}
\]

(4·2)

where \( \gamma_{nh} \equiv K / \gamma_1 d_0^3 \), \( \gamma_2 \equiv \chi_2 / d_0^2 \), \( \gamma_3 \equiv \nu_3 / \rho d_0^4 \) with \( K \equiv K_3(1 + H^2) \). By using these units, (4·1) is transformed into

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix}
\mathcal{F}_{v, n} \\
\mathcal{F}_{v, v}
\end{pmatrix} &=
\begin{pmatrix}
\mathcal{F}_{v, n} \mathcal{F}_v & 0 & R \mathcal{F}_v^2 \\
-\mathcal{P}_v \mathcal{F}_v & P_v \mathcal{F}_v^2, & -R \mathcal{P}_v \chi_2 / \gamma_a d_0 \\
-(\alpha_2 / \mathcal{K}) \mathcal{F}_{v, n} \mathcal{F}_v^2 & P_v \mathcal{F}_v^2, & P_v \mathcal{F}_v^2
\end{pmatrix} \begin{pmatrix}
\eta \\
\theta
\end{pmatrix}
\end{align*}
\]

(4·3)

where \( R \equiv -\alpha_2 / \gamma_a d_0 / \gamma_1 \gamma_3 \gamma / \gamma_{nh} \) is the Rayleigh number, \( \mathcal{F}_{v, n} \equiv (K_3 \mathcal{F}_v^2 + K_1 \mathcal{F}_v^2 - K_3 H^2 / \mathcal{K} / K, P_v = \gamma_2 / \gamma_{nh}, P_v = \gamma_3 / \gamma_{nh} \) and \( \mathcal{P}_v = \gamma_3 \gamma / K \). The nonlinear term \( N \) consists of two parts; a part common in the isotropic liquid \( N_I \) and a part characteristic of the nematic liquid crystals \( N_{nc} \). Both expressions are listed in Appendix B.

The coupled equations of the mode amplitude \( u_{n}^{*} \) are obtained from (4·3) with the same procedure as in the case of the isotropic liquids shown in Appendix A. First, as will be shown in Appendix C, the eigenvalues are obtained by the linear problem. The soft eigenvalue is given by
\[ \lambda^4 = (\bar{\kappa}_n^2)^2 (P_3(\bar{k}_c)^4/P_2(\bar{k}_c)^4) \bar{R} + \mathcal{O}(\bar{R}^2), \quad (4.4) \]

with the critical Rayleigh number

\[ R_c = \frac{P_3}{P_2} \frac{(\bar{k}_n^2 \bar{k}_c)^2}{k_c^2} \frac{k_c}{\gamma \alpha \alpha_2 (\bar{k}_n^2)^2 + P_2(\bar{k}_c)^2}, \quad (4.5) \]

where \( k_c \) is determined so as to minimize \( R_c \), and \( P_3 = \eta_c/\gamma n, \) \((\bar{k}_c)^4 = k_c^4 + (b + \beta')/\eta_c \cdot k_c^{-2} \pi^2 + (\bar{\kappa}_c/\eta_a) \pi^4, \) \( \eta_c = \alpha_2 + \alpha_1 + \alpha_3 \)/2, \( \eta_a = (\alpha_3 + \alpha_1 + \alpha_2)/2, \) \( b = (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6)/2 \) and \( \beta' = (\alpha_4 - \alpha_5 - \alpha_3)/2. \) It should be noted that the term \( (x_c \gamma_1/\alpha_2 \alpha_3)(\bar{k}_n^2)^2 \) comes from the buoyancy force effect, while \( P_2(\bar{k}_c)^2 \) represents the heat focusing effect. Then by introducing the corresponding eigenvectors and expanding \( x(r, t) = \text{Col}(u, \theta, v) \) in terms of them, (4.3) can be transformed into the form of (A.9). The eigenvectors are chosen in such a way that the matrix \( (U^{\text{in}})^{-1} = \text{Col}(1, 2, 3) \) has diagonal elements with \( \langle 1^u \rangle = \langle 2^u \rangle = \langle 3^u \rangle = 1/3 \), where \( A^u \) are determined so as to satisfy \( \det(U^{\text{in}})^{-1} = 1. \)

The fluctuating forces \( F_{\text{th}}^{\text{in}} \) are obtained from \( S \) by a similar transformation to (A.10). In order to obtain their spectral density matrix \( Q_{\text{th}}^{\text{in}} \), let us investigate \( S(r, t) \) first. There are the thermal fluctuations, the temperature fluctuations at the boundary and the fluctuations of the external magnetic field: \( S = S_{\text{th}} + S_{\text{b}} + S_{H}. \) For the fluctuations of the thermal origin, it is well known that due to the intrinsic properties of the orientational order, there exist strong orientational fluctuations. In the case of the Rayleigh-Bénard instability, the fluctuations at the long wavelength cutoff, given by the dimension of vessel, are most strongly excited. This fluctuating force is written as \( S_{\text{th}}^{\text{in}} = P_3 S_{\text{th}}^{\text{in}} \) so that

\[ \langle S_{\text{th}}^{\text{in}}(r, t) S_{\text{th}}^{\text{in}}(r', t') \rangle = 2L_n \delta(r-r') \delta(t-t'), \quad (4.6) \]

with \( L_n = k_b T d_a^d/\gamma n \rho \gamma n. \) \( L_n \) is easily reduced by the external magnetic field.\(^{31a)}\)

There are also \( S_{\text{b}}^{\text{in}} \) and \( S_{H}^{\text{in}}, \) but the orientational fluctuating force plays a dominant role except for the case of the large magnetic field.

The temperature fluctuations at the lower plate have the same feature as those in the isotropic liquids. On the other hand, if the external magnetic field fluctuates around a mean value \( H, \) then it has a fluctuating part \( f_H(r, t) \) with mean value zero. This is assumed to be a Gaussian white noise with

\[ \langle f_H(r, t) f_H(r', t') \rangle = 2L_n \delta(r-r') \delta(t-t'), \quad (4.7) \]

where \( L_n = (K_\alpha \gamma_2 \pi^2 / K_\gamma) d_a^d \) comes from the use of the dimensionless units. Then the fluctuations of the surrounding are

\[ S_{\text{b}}^{\text{in}} = \begin{pmatrix} P_1 \mathcal{P} \tau^2 f(1-z) \\ 0 \end{pmatrix}, \quad (4.8 \ a) \]
\[
S_n = 2H \left( \begin{array}{c}
-\nabla f \nabla n \\
0 \\
(a^2/RK_0) \left( \nabla_a \cdot \nabla f \nabla n + \frac{\tau_a}{a^2} \nabla_a \nabla \cdot \nabla f \nabla n \right) \right) + O(f_n^2). \tag{4.8b}
\]

As in the case of the isotropic liquids, the multiplicative part of \( S_n \) and \( S_n^{\text{ip}} \) gives no contributions so that they can be dropped. Then it turns out that \( F_n^{\text{ip}} \) consist of the additive stochastic forces due to \( S_n \), \( S_n^{\text{ip}} \), and of the multiplicative stochastic force given by \( S_n \) through \( n = \sum \alpha \Sigma_{\alpha} \sum_{\beta} \beta^2 \sin (n \pi \alpha) a_{\beta\alpha}^{(\alpha)} \). The correlations between these fluctuating forces are easily calculated as before. Then with these fluctuating forces, various critical behaviors are expected to occur.

In order to investigate these behaviors, we study MBBA. The various material coefficients at \( T = 20^\circ C \) are \( \chi_1 = 1.54 \times 10^{-3} \text{ cgs} \), \( \chi_2 = 0.93 \times 10^{-3} \text{ cgs} \), \( K_1 = 6 \times 10^{-7} \text{ dyne} \), \( K_2 = 7.5 \times 10^{-7} \text{ dyne} \), \( \alpha_1 = 0.065 \text{ poise} \), \( \alpha_2 = -0.785 \text{ poise} \), \( \alpha_3 = -0.012 \text{ poise} \), \( \alpha_4 = 0.832 \text{ poise} \), \( \alpha_5 = 0.463 \text{ poise} \), \( \alpha_6 = 0.334 \text{ poise} \), \( \chi_0 = 1.19 \times 10^{-7} \text{ cgs} \). We investigate the cases of \( H = 0 \), \( H = 40 \), and sufficient large \( H \), which exhibit different behaviors due to the reduction of the orientational relaxation time by the external magnetic field.

(A) \( H = 0 \)

In this case, \( P_1 = 1.59 \times 10^5 \text{ dyne}, P_2 = 2.3 \times 10^6 \text{ dyne} \), and hence \( \gamma_{nn} \ll \gamma_{1n} \ll \gamma_{0n} \), which shows the very slow relaxation of the director fluctuations. The eigenvalues are approximately given by \( \lambda_{11} \approx P_1 (\vec{k} \cdot \vec{\epsilon})^2 \) and \( \lambda_{11} \approx P_2 (\vec{k} \times \vec{\epsilon})^2 / (\vec{k} \cdot \vec{\epsilon})^2 \), corresponding to the temperature mode and the velocity mode. The critical Rayleigh number is approximated by neglecting \((\chi_1 \gamma_1 / \chi_2 \alpha_2) (\vec{k} \cdot \vec{\epsilon})^2 / (\vec{k} \cdot \vec{\epsilon})^2 \) in the denominator of (4.5), since \((\chi_1 \gamma_1 / \chi_2 \alpha_2) \sim 1 \) whereas \( P_1 \sim 10^3 \). This means that the mechanism of the instability is controlled by the heat focusing effect. Thus the director plays the important role.

As a result of calculation, we have \( R_c = 2738.4 \) with \( k_c = 0.7766 \pi \) leading to the temperature difference \( \Delta T_c = 1.605 \times 10^{-3} / d_0^3 \). The fluctuations are those of the thermal origin and the temperature fluctuations at the lower plate. If \( L_0(x) \) is constant and \( d > d_0 \) and \( d > 2 \), then a stochastic Ginzburg-Landau type equation is obtained with \( \lambda_{11} = -17.20 \vec{R}, D = 5.305 \) and \( \gamma = 6.484 \). It should be noted that, in addition to \( u_{22}^{(2)} \), the modes \( u_{01}^{(2)} \) are excited, but their magnitudes are not large enough to destroy the single mode picture of the convection. In addition, the contribution from \( N_c \) is negligibly small, and \( N_c \) plays a dominant role.

For the fluctuations, with the assumptions as before, we have

\[
Q_{\alpha n}^{(\alpha)}(u_{\alpha n}^{(\alpha)}) = \begin{cases} Q_{\alpha n} + Q_{\alpha n}, & (\lambda_0 > 0) \\ Q_{\alpha n} + 1.726 \times 10^{-3} \vec{R} + Q_{\alpha n} (1 + 5.00 \times 10^{-4} \vec{R}), & (\lambda_0 < 0). \end{cases} \tag{4.9a,b}
\]

where \( Q_{\alpha n} = 1.09 \times 10^{-7} \) and \( Q_{\alpha n} = 1.71 \times 10^{3} L_{\alpha} \) for \( d_0 = 1 \text{ cm} \). If the temperature fluctuations at the lower plate are sufficiently suppressed so as to obtain \( Q_{\alpha n} \gg Q_{\alpha n} \),
then the thermal fluctuations play the dominant role. In this situation and for
$R - R_c < 4$ where the analysis may be valid, the effect of the fluctuating forces
due to the mode couplings are at least $10^{-5}$ times smaller than $Q_{th}$. In the case $Q_{th} < Q_b$, this effect is also very small; $7 \times 10^{-4}$ times smaller than $Q_b$. Thus, in spite
of the strong orientational fluctuations which make it possible to observe the
critically-enhanced fluctuations experimentally, the effect of the fluctuations
generated by the mode couplings are very small. This smallness comes from the
clear separation of the time scales of the variables, which cannot make the
couplings effective in this case. The renormalization term is estimated as $I(t_{0x})$
$= (2.84 \times 10^{-10} + 2.11 \times 10^5 L_b) u_{1x}^1$.

On the other hand, if $L_b(x)$ is not constant, then the inhomogeneous term due
to $Q_{12}^{102}$ and $Q_{12}^{12}$ for $x=1 \sim 3$ must be added to the amplitude equation as $I_{inh} = 5.65 \times 10^3 i L_b^1$. This inhomogeneous term rounds the transition.

(B) $H = 40$

In this case, $P_1 = 0.992$, $P_2 = 156.4$, which leads to $\gamma_{nh} \approx \gamma \approx \gamma_s$. Namely the
relaxation of the director and the temperature fluctuations are of the same order.
Then the approximation for $R_c$ as the case of (A) cannot be used. This means
that both the heat focusing effect and the buoyancy force effect work.

As a result of calculations, $R_c = 320.8$ with $k_c = 0.536 \pi$, leading to the temperature
difference $\Delta T_c = 0.301/d_s^3$. In this case, the fluctuations are those from $S_b$, $S_0$ and $S_H$ due to the fluctuating magnetic field. First, if $L_b(x)$ is constant, then
a stochastic Ginzburg-Landau type equation is obtained with $\lambda_1^{11} = -8.59 \bar{R}$, $D = 0.866$, and $g = 8.467 \times 10^{-2}$. The modes $u_{22}^2$ are also excited, but $N_l$ becomes
important as well as $N_{lc}$.

For the fluctuations, with the assumptions as before, we have

$$Q_{10}^{10} (u_0) = \begin{cases}
Q_{th} + Q_b, \\
Q_{th}(1 + 2.28 \times 10^5 \bar{R}) + Q_b(1 + 3.935 \bar{R}) + Q_{bl}(\bar{R} + 0.12 \bar{R}^2),
\end{cases}$$

(4.10a) (4.10b)

with $Q_{th} = 3.86 \times 10^{-12}$, $Q_b = 1.23 \times 10^9 L_b^6$ and $Q_{bl} = 2.01 \times 10^{-9} L_b$ for $d_0 = 1$ cm. If the fluctuations of the surroundings are completely suppressed, then $S_b$ and $S_H$ vanish and there are the thermal fluctuations only. In this situation
and for $R - R_c < 4$ where the above analysis may be valid, the fluctuations due to
the mode coupling contribute to the total spectral density up to 22%. This
contribution is large compared with the case of (A), since there exist more
effective couplings between the modes which have the time scales of the same
order. Therefore the variance of the order parameter has not only the part
coming from $Q_{th}$ which causes the large critical fluctuations but also the part
which is independent of $\bar{R}$ and never diverges even near the critical point.
The renormalization term is calculated as \( I(u_{0x}) = (8.25 \times 10^{-13} + 2.81L_b^3)u_{1x}^4 \), while the fluctuating magnetic field contributes to the drift term as \( I_H = 1.32 \times 10^{-7}L_m u_{1x}^3 \). On the other hand, if \( L_b(x) \) depends on \( x \), then the inhomogeneous term must be added to the amplitude equation as \( I_{inh} = 2.12 \times 10^3 iL_m^3 \). The renormalization term \( I(u_{0x}) \) and \( I_H \) shift the critical point, while the inhomogeneous term \( I_{inh} \) rounds the transition.

(C) large \( H \)

If a large external magnetic field is applied such that it reduces the orientational relaxation time much smaller than others, then \( P_i \ll 1 \) and \( \gamma_{inr} \ll \gamma_\tau, \gamma_\nu \). In this case, the critical Rayleigh number is approximated by neglecting \( P_i(k_0^2) \gamma \) which represents the heat focusing effect characteristic to the nematic liquid crystals. This \( R_c \) is also obtained by assuming \( \frac{\partial}{\partial t} \varphi_{xh} = 0 \), which means that the director is adiabatically eliminated and the relevant variables are \( \nu \) and \( \theta \). In addition, the excitations of \( \nu_{05}^2 \) modes are negligibly small, as well as the contribution from \( N_{LC} \). Finally, the orientational fluctuations are also reduced, and \( F_{th}^{(5)} \) becomes dominant. Therefore, for the large magnetic field, the director plays no important role and the situation is qualitatively the same as the isotropic case.

§ 5. A short summary and remarks

Critical behaviors of a nonequilibrium soft transition have been studied in the case of an intrinsic symmetry-breaking transition. With the aid of the projector method for the contraction of state variables, the elimination of the stable modes with various fluctuating force has been performed.

The nematic liquid crystal is of interest since it exhibits strong orientational fluctuations and their critical fluctuations can be experimentally observed. In the case \( H = 0 \), there exist large fluctuations, but the contribution from the fluctuations generated by the mode coupling is still small due to the distinct separation of the time scales between the variables. In the case \( H = 40 \) in the dimensionless unit, however, the relaxation time for the director is reduced to the order of the temperature fluctuations and the mode-coupling contribution becomes much larger than that of \( H = 0 \). The detailed situation would be different, depending on the relative magnitude of \( Q_{th}, Q_H \) and \( Q_{th} \). For example, if \( Q_H \) is negligibly small and \( Q_{th} \gg Q_H \), then the contribution from the fluctuations through the mode coupling becomes about one fifth of the total spectral density at \( R = R_c + 4 \). Then the variance of the order parameter would show a difference from the critical fluctuations of the mean field type. Namely, it has a part which remains constant and never diverges as \( R \rightarrow 0 \). On the contrary, if \( Q_H \gg Q_{th} \), then this contribution is 4.7% of the total spectral density at \( R = R_c + 4 \). On the other hand, if \( Q_H \gg Q_{th}, Q_H \), then a multiplicative stochastic force dominated so that the
spectral density strongly depends on the order parameter and the variance never diverges except in the very vicinity of the critical point. Thus the crossover between the different critical regimes and the contribution from the above fluctuations would strongly depend on the situation of the experiment.

Due to these fluctuations, a shift of the critical point and a smoothing of the transition are expected. Experimentally, a shift induced by external white noises has been observed in the electrohydrodynamic instability\cite{14} and a smooth transition has been observed in the Rayleigh-Bénard instability of PAA with $H = 70$ ersted\cite{15} and in the isotropic liquid.\cite{16} It has been shown that the shift is induced since the external white noise works in the multiplicative way.\cite{17} This situation would be the same as in the Rayleigh-Bénard instability in the nematic liquid crystal in the fluctuating magnetic field. For the smoothing of the transition, many possible mechanisms are proposed: a deterministic forcing field due to the imperfection in the cell geometry or the lateral heat flow, a rounding by the fluctuations\cite{18} and in the zero dimensional case. In this paper, it has been shown that the temperature fluctuations at the plate could cause a deterministic forcing field. The smoothing has to be seen more clearly for a smaller magnetic field, since, by applying the magnetic field, the director is stabilized and the fluctuations are suppressed. In fact, the rounding observed with $H = 70$ ersted is no longer seen at $H = 200$ ersted.\cite{19}

In this series of papers, we have shown that, in order to investigate characteristic features of the nonequilibrium phase transition, it is necessary to take account of the fluctuations generated by the mode coupling to irrelevant macro-variables in addition to the external random noises directly acting on the order parameter from outside. These various fluctuating forces would be important not only in the case of the white noises but also in the case of the colored noises, and not only in the first transition but also in higher transitions.

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Appendix A

—— Derivation of the coupled equation in §3 ——

First, let us solve the linear problem. By expanding $\mathbf{x}(r, t) = \text{Col}(v, \theta)$ as

$$\mathbf{x}(r, t) = \sum_{\ell = -m}^{m} \sum_{n = 0}^{\infty} e^{i\theta \ell r} \sin(n \pi z) \mathbf{x}^{\ell n}, \quad (A\cdot1)$$

where $\mathbf{x}^{\ell n} = \text{Col}(v^{\ell n}, \theta^{\ell n})$ and $\mathbf{R}$ is a horizontal wave vector; (3.1) leads to

$$\partial \left[ \mathbf{B}_{\theta}^{\ell n} + \sum_{l=1}^{2} \mathbf{B}_{\ell'}^{\ell n} (\mathbf{F}_z)^{l'} \right] \mathbf{x}^{\ell n} / \partial t = \left[ \mathbf{F}_{\theta}^{\ell n} + \sum_{l=1}^{2} \mathbf{B}_{\ell'}^{\ell n} (\mathbf{F}_z)^{l'} \right] \mathbf{x}^{\ell n}, \quad (A\cdot2)$$
where

$$\tilde{B}_0^{\in} = \begin{pmatrix} -(\tilde{k}^n)^2 & 0 \\ 0 & 1 \end{pmatrix}$$

(A·3)

with \((\tilde{k}^n)^2 = (kl)^2 + (n\pi)^2\). If higher order terms including \(\mathcal{F}_x\) are neglected in both sides, then (A·2) leads to

$$\partial x^{in}/\partial t = -\tilde{I}^{in}(R, k^n)x^{in}$$

(A·4)

with

$$\tilde{I}^{in}(R, k^n) = \begin{pmatrix} P(\tilde{k}^n)^2 & -P(lk)^2/(\tilde{k}^n)^2 \\ -R & (\tilde{k}^n)^2 \end{pmatrix}.$$  

(A·5)

The eigenvalues are

$$\lambda^{i}_{0} = [1 + P](\tilde{k}^n)^2 - \sqrt{(1 + P)^2(\tilde{k}^n)^4 + 4P(\tilde{k}^n)^2(R - \tilde{R}^{i}_{n})/(\tilde{k}^n)^2}] / 2$$

(A·6)

with \(R^{i}_{n} = (\tilde{k}^n)^2/(lk)^2\). As is well-known, the soft eigenvalue is \(\lambda^1\), which takes zero value at \(R = R^1 = 27\pi^4/4\) with \(k_c = \pi/\sqrt{2}\). Next, with the eigenvectors \([|a^{in}\rangle\], let us expand \(x(r, t)\) as

$$x(r, t) = \sum_{\alpha} \sum_{i=-\infty}^{\infty} \sum_{n=0}^{\infty} u_{\alpha}^{in}(t) e^{i\alpha r} \sin(n\pi z) |a^{in}\rangle,$$

(A·7)

where \(u_{\alpha}^{in}(t)\) varies slowly in space and time. The eigenvectors are taken as

|\(a^{in}\rangle = \begin{pmatrix} 1 \\ R/(\lambda^{i}_{0} + (\tilde{k}^n)^2) \end{pmatrix}, \quad l \neq 0$$

(A·8a)

|\(1^{in}\rangle = \begin{pmatrix} 1 \\ R/(1 - P)(\pi n) \end{pmatrix}, \quad |\tilde{2}^{in}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(A·8b)

with \((\tilde{k}^n)^2 = (k_c l)^2 + (n\pi)^2\). Substitution of (A·6) into (3·1) leads to

$$\partial u_{\alpha}^{in}/\partial t = -\lambda^{i}_{0} u_{\alpha}^{in} + \sum_{\gamma}[C^{i}_{\alpha \gamma}(\mathcal{F}_x + D^{i}_{\alpha \gamma})(\mathcal{F}_x)] u_{\gamma}^{in}$$

$$+ \sum_{\alpha} \sum_{\gamma} \sum_{\mu} \sum_{\nu} \sum_{\eta} \sum_{\xi} \sum_{\gamma'} \sum_{\gamma''} \sum_{\gamma'''} \sum_{\gamma''''} \delta_{\gamma,\gamma'} \delta_{\gamma',\gamma''} \delta_{\gamma'',\gamma'''} \delta_{\gamma''',\gamma''''} \mathcal{F}_x \mathcal{F}_x \mathcal{F}_x \mathcal{F}_x$$

$$+ F_{\alpha}^{in},$$

(A·9)

where \(C^{i}_{\alpha \gamma}(R) = \langle a^{in} | \tilde{C}^{i}_{\alpha \gamma} | a^{in} \rangle\), \(D^{i}_{\alpha \gamma}(R) = \langle a^{in} | \tilde{D}^{i}_{\alpha \gamma} | a^{in} \rangle\), and

$$F_{\alpha}^{in} = \sum_{j} \langle a^{in} | (\tilde{B}_0^{in})_{j}^{-1} \rangle \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dz e^{-ik_c x} \sin(n\pi z) S_j$$

(A·10)
with

\[
\tilde{C}^{in} \equiv (\tilde{B}_2^{in})^{-1} \tilde{D}_1^{in} = 2i k_c l \begin{pmatrix} 2, & -P/(\tilde{k}_c^{in})^2 \\ 0, & 1 \end{pmatrix},
\]

\[
\tilde{D}_1^{in} = (\tilde{B}_1^{in})^{-1} \tilde{D}_2^{in} = \begin{pmatrix} 2P + 4P(k_c l)^2/(\tilde{k}_c^{in})^2, & -P/(\tilde{k}_c^{in})^2 \\ 0, & 1 \end{pmatrix}.
\]

(A.11)

(A.12)

The nonlinear term \(N_{k_c^{(0)}}^{(n)}(\nu^{(n')}\nu^{(n'')})\) consists of two parts which come from \(\delta \cdot (v \cdot \varphi) v\) and \((v \cdot \varphi) \theta\). The terms due to \((v \cdot \varphi) \theta\) are important and given by

\[
[N_{k_c^{(0)}}^{(n)}(\nu^{(n')}\nu^{(n'')})]^{(n)} = \begin{cases} -\pi \langle a^{(n')}_c \vert \beta^{(n'')}_c \rangle \ominus \delta a^{(n')}_c \ominus \nu \ominus \delta_{n-n'} \ominus \nu \ominus (n'' - n') \ominus \nu \ominus (n'') & \text{if } l' \neq 0, \\ + (\delta_{n-n'} \ominus \delta_{n-n'} \ominus \nu \ominus \nu \ominus \nu \ominus (n'' + n') \ominus \nu \ominus (n'') & \text{if } l' = 0, \end{cases}
\]

(A.13a)

(A.13b)

to the lowest order, with \(a^{(0)}_c = \langle a^{(0)}_c, R_c, k_c \rangle\). Other terms from \(\delta \cdot (v \cdot \varphi) v\) give no contribution to \(N_{k_c^{(0)}}^{(n)}(\nu^{(n')}\nu^{(n'')})\).

Appendix B

--- Nonlinear Terms \(N_i\) and \(N_{i,c}\) ---

\[
N_i = \begin{pmatrix} \gamma_i / \alpha \end{pmatrix} R(v \cdot \varphi) \theta,
\]

\[
N_{i,c} = (N_{i,c}^{(0)}),
\]

(B.1)

(B.2)

where

\[
N_{i,c}^{(n)} = (\gamma_i / \alpha) R \varphi \langle n \varphi_x v \rangle,
\]

\[
N_{i}^{(0)} = P_a [\varphi_x (n \varphi_x \theta) + \varphi_x (n \varphi_x \theta)] - P_l \varphi \langle n \varphi_x \theta \rangle^2,
\]

\[
N_{i,c}^{(0)} = -(\alpha_2 \gamma / \rho K R) \varphi_x^2 \varphi_x (n \varphi_x n) + (\gamma_i / K \rho) \langle \alpha_2 \varphi_x \varphi_x \varphi \varphi_x (n \varphi_x \theta) - \varphi_x (n \varphi_x \theta) \rangle - \alpha_3 \varphi_x^2 \langle [\varphi_x (n \varphi_x \theta) + \varphi_x (n \varphi_x \theta)] \}
\]

(B.3a)

(B.3b)

(B.3c)

with \(\alpha_2 = \alpha_1 + \alpha_2 \gamma / \gamma_i\). In \(N_{i,c}\), several terms in \(N_{i,c}^{(n)}\) and \(N_{i,c}^{(0)}\) such as \((\gamma_i / \alpha) R \varphi \langle v \cdot \varphi \rangle\) are neglected, since they give no contributions. In addition, the terms in \(N_{i,c}^{(0)}\) coming from \(c_p^2 [\alpha_1 (\varphi_x v)^2 + \cdots]\) are dropped, since they are negligibly small.

Appendix C

--- Derivation of (4.4) and (4.5) ---

The matrix corresponding to (2.2) or (A.5) is, for \(l \neq 0\),
Anomalous Fluctuations near Nonequilibrium Soft Transitions

\[ \tilde{F}^{1n} = (\gamma_{ij}^{1n}) \]

\[
\begin{pmatrix}
(\tilde{k}_{ln}^{1n})^2, & 0, & R(\tilde{k}_{ln}^{1n})^2/ikl \\
-\alpha^2/\nu K, & P_{ijkl}, & R(\tilde{x}_i \gamma_1/\nu \alpha x) \\
-\alpha^2/\nu K, & R(\tilde{x}_i \gamma_1/\nu \alpha x) & P_{ijkl}
\end{pmatrix}
\]

where \((\tilde{k}_{ln}^{1n})^2\) for \(i = nH, \alpha, T, v\) come from \(\nu^{1n}\) for \(i = nH, \alpha, T, v\), respectively. It should be noticed that for \(l = 0\), \(\gamma_{ij}^{1n}\) is not \((\gamma_{ij}^{1n})_{l=0}\) from (C·1) but zero, which is derived from the nematodynamic equations for \(n, \theta, \nu^{1n} v\) and \(\nu^{1n} u\).

The eigenvalues are obtained from

\[
(\lambda^{1n})^3 - \alpha^{1n}(\lambda^{1n})^2 + \Delta^{1n} \lambda^{1n} - \beta^{1n} = 0
\]

with \(\alpha^{1n} = tr\tilde{F}^{1n}, \Delta^{1n} = \gamma_1^{1n} \gamma_2^{1n} + \gamma_2^{1n} \gamma_3^{1n} + \gamma_3^{1n} \gamma_1^{1n} - \gamma_2^{1n} \gamma_3^{1n} - \gamma_3^{1n} \gamma_1^{1n}, \beta^{1n} = \det \tilde{F}^{1n}\). The critical Rayleigh number \(R_c\) is determined by \(\beta^{11}(R_c) = 0\), and near the critical point, the soft eigenvalue is given by \(\lambda_1^{1n} \approx \beta^{11}(R)/\Delta^{11}\).

References

1) K. T. Mashiyma, K. Takayoshi and H. Mori, Prog. Theor. Phys. 65 (1981), 1820. This will be referred to as I.
10) For example, Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. 54 (1975), 687.