Gaussian Random Bond Problem for Annealed System in $1/n$ Expansion

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Critical phenomena for the annealed $n$-vector model are studied in $1/n$ expansion by assuming that the exchange interactions between spins are fluctuating variables obeying the Gaussian distribution. It is shown that for an overall fluctuation of the interaction between spins the susceptibility never diverges at any finite temperature except for $n\to\infty$ (spherical limit) and $n=1$ (Ising) cases.

On the other hand, if the fluctuation is restricted to only nearest-neighbor pairs, the susceptibility is proved to diverge at an appropriate transition temperature in the limit $n\to\infty$. The critical exponent $\gamma$ in this case is shown to be the same as the spherical model value. Also the dependence of transition temperature on the strength of fluctuation is discussed.

§ 1. Introduction

The purpose of this paper is to study the critical behavior of a certain random system via the $n$-vector model in $1/n$ expansion. One of the present authors has studied $1/n$ expansion for the weakly random site problem in the quenched case in the limit $n\to\infty$. Its calculation has been extended up to $O(1/n)$ recently. Another author has proved the instability of Mattis spin glass to a uniform external magnetic field in the limit $n\to\infty$ by applying the technique used in the above. In this paper we deal with the annealed system with a suitable random bond.

The outline of this paper is as follows: In § 2 we consider the partition function of the $n$-vector model in the presence of a magnetic field and present a precise definition of our random bond problem. The partition function is transformed into a form suitable for studying $1/n$ expansion in § 3. Section 4 is devoted to a discussion of the system having short-range interaction with an overall fluctuation between spins. Calculational details up to $O(1/n)$ are given in the Appendix. It is shown that the susceptibility $\chi$ never diverges at any finite

*) Part of this work was carried out by Abe in 1978 during his stay at the Theoretical Physics Institute, University of Alberta.
temperature in case of the continuous symmetry $n \geq 2$. On the other hand, we show in § 5 that if the fluctuation is restricted to only nearest-neighbor pairs, $\chi$ is divergent at an appropriate transition temperature. The critical exponent $\gamma$ and the behavior of transition temperature are studied in the limit $n \to \infty$. Finally § 6 is devoted to concluding remarks.

§ 2. Gaussian random bond

The partition function $Z$ of the $n$-vector model in the presence of a magnetic field $h$ is expressed as

$$Z = \int_{-\infty}^{\infty} d\sigma_k(m) \exp \left[ \frac{1}{2} \sum_{j,k} K_{jk} \sigma_j(m) \sigma_k(m) + h \sum_j \sigma_j(m) \right] \times \prod_j \delta \left[ n - \sum_m \sigma_j^2(m) \right],$$

where $\sigma_j(m)$ represents the $m$-th component ($m=1, 2, \cdots, n$) of a classical spin at the $j$-th site ($j=1, 2, \cdots, N$) and $K_{jk} = J_{jk}/k_B T$ ($k_B$: Boltzmann constant) is the exchange interaction between the $j$-th and the $k$-th lattice points. The spherical constraints are incorporated into $\delta$ functions. We assume that $K_{jk}$ is a random variable whose probability distribution is Gaussian. Thus we introduce a probability distribution function $P$ defined by

$$P = \prod_j \left( \frac{n}{4\pi a^2_{jk}} \right)^{1/2} \exp \left[ -\frac{1}{2} \sum_n \frac{n(K_{jk} - \bar{K}_{jk})^2}{2a^2_{jk}} \right].$$

Here $\bar{K}_{jk}$ is the average value of $K_{jk}$ and $4a_{jk}\sqrt{n}$ is the width of distribution where the factor $\sqrt{n}$ is put in order to simplify our procedure. It is assumed that $a_{jk}$ is a function of $|r_j - r_k|$ and that the system is ferromagnetic, i.e., $\bar{K}_{jk} > 0$. For the time being, we keep $n$ finite.

If we multiply $Z$ by $P$ and integrate over all $K_{jk}$, it is easily seen that the average value $\bar{Z}$ of partition function is expressed as

$$\bar{Z} = \int_{-\infty}^{\infty} d\sigma_k(m) \exp \left[ \frac{1}{2} \sum_{j,k} \bar{K}_{jk} \sigma_j(m) \sigma_k(m) + h \sum_j \sigma_j(m) \right] + \frac{1}{4n} \sum_{j,k} a^2_{jk} \left[ \sum_m \sigma_j(m) \sigma_k(m) \right]^2 \prod_j \delta \left[ n - \sum_m \sigma_j^2(m) \right].$$

In case $n=1$, the third term in the argument of exponential function is only a constant. Therefore, in this case the critical behavior of the present system is exactly the same as the Ising model which is described by the average interaction $\bar{K}_{jk}$. However such a simple situation is no longer valid in the case $n \neq 1$.

Applying an identity:
we obtain the following partition function:
\[
Z = \frac{1}{\pi^{L/2}} \int_{-\infty}^{\infty} \prod_{jKm} d\sigma_j(m) \int_{-\infty}^{\infty} \prod_{jK} dt_{jk} \exp \left\{ -\sum_{jk} t_{jk}^2 + \frac{1}{\sqrt{\mu}} \sum_{jk} a_{jk} t_{jk} \sigma_j(m) \sigma_k(m) \right\} \prod_{jK} \delta \left[ n - \sum_{m} \sigma_j^2(m) \right],
\]
where \( L = N^2 \). Making the change of variables \( t_{jk} \rightarrow \sqrt{n} t_{jk} \) and using an integral representation for \( \delta \) function, we have
\[
Z = \frac{n^{L/2}}{\pi^{L/2}(2\pi i)^{L}} \int_{-\infty}^{\infty} \prod_{jK} dt_{jk} \int_{a-i\infty}^{a+i\infty} \prod_{jK} dt_j \exp \left[ \left( -\sum_{jk} t_{jk}^2 + \sum_{j} t_j \ln f \right) \right],
\]
where \( a \) is an arbitrary real constant and a function \( f \) is defined as
\[
f = \int_{-\infty}^{\infty} \prod_{K} d\sigma_k \exp \left[ -\sum_{j} t_j \sigma_j^2 + \frac{1}{2} \sum_{jk} K_{jk} \sigma_j \sigma_k + h \sum_{j} \sigma_j \right].
\]
Here we introduce the effective interaction:
\[
\tilde{K}_{jk} = K_{jk} + 2a_{jk} t_{jk}.
\]
§ 3. 1/n expansion for the partition function

In this section we transform the partition function $Z$ in (2.6) into a more appropriate form for $1/n$ expansion. Calculations are based on the method of the steepest descent. Practical calculational techniques of $1/n$ expansion were developed by Abe. The only outline is shown in what follows.

Hereafter we represent the saddle points by putting superscripts $0$. Although original $Z$ defined by (2.1) has no translational invariance, $\bar{Z}$ restores it. Therefore every lattice point can be regarded equivalent if the periodic boundary condition is imposed on our system. Thus it follows that $t^0_1 = t^0_2 = \cdots = t^0_N = t^0 = t = a$. At the same time, $t^0_i$ should depend on $|r_i - r_k|$. Next we transform integral variables $t_i$ and $t_{jk}$ to $x_j$ and $\tau_{jk}$ respectively as follows; $t_i \rightarrow t + ix_j$ and $t_{jk} = l^0_k + \tau_{jk}$. Then $\bar{Z}$ is written as

$$\bar{Z} = \frac{n^{L/2}}{\pi^{L/2}(2\pi)^N} \exp\{n\left[N l - \sum_{jk} t^0_{jk} \sigma_j \sigma_k + \ln f_0\right]\}$$

$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_j \exp\{n\left[i \sum_j x_j - \sum_{jk}(\tau^0_{jk} + 2t^0_{jk} \tau_{jk}) + \ln G\right]\},$$

where

$$f_0 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\sigma_j \exp\left[-\frac{1}{2} \sum_{jk} \tilde{K}^0_{jk} \sigma_j \sigma_k - t \sum_j \sigma_j^2 + h \sum_j \sigma_j\right].$$

Here the effective interaction at the saddle points $\tilde{K}^0_{jk} = K^0_{jk} + 2a_{jk} t^0_{jk}$ and a function $G = f/f_0$ are introduced for simplicity.

In order to eliminate the linear term $h \Sigma \sigma_j$ from the argument of the exponential in (2.7), we translate spin variable $\sigma_j$ as $\sigma_j \rightarrow \sigma_j + a_j$. A matrix equation to determine $a_j$ is

$$Aa = \frac{h - 2ix_1 a_1}{h - 2ix_2 a_2} \cdots + 2Ba,$$

where $A_{jk} = 2t^0_{jk} - \tilde{K}^0_{jk}$ and $B_{jk} = a_{jk} \tau_{jk}$. An element of the inverse matrix $A^{-1}$ is proved to be

$$(A^{-1})_{jk} = g(j, k) = \langle \sigma_j \sigma_k \rangle_0,$$

where the average is given by

$$\langle \cdots \rangle_0 = \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\sigma_k \cdots \exp\left[-\frac{1}{2} \sum_{jk} \tilde{K}^0_{jk} \sigma_j \sigma_k - t \sum \sigma_j^2\right]}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\sigma \exp\left[-\frac{1}{2} \sum \tilde{K}^0_{jk} \sigma_j \sigma_k - t \sum \sigma_j^2\right]}.$$
From (3.3) and (3.4) the following equations for \(a_j\) are derived:

\[
 a_j = h \sum_k g(j, k) - 2i \sum_k g(j, k)x_j a_k + 2 \sum_k g(j, k) B_{jk} a_l
\]  

(3.6a)

and

\[
 a_j(0) = h \sum_k g(j, k) = h g(0).
\]  

(3.6b)

Definition of \(g(0)\) is given by (A.3b) in the Appendix. Equation (3.6b) is the value of (3.6a) at the saddle points \([x_j = \tau_{jk} = 0]\). The function \(G\) is composed of two factors:

\[
 G = \exp\left(\frac{h}{2} \sum_j [a_j - a_j(0)]\right) G_0.
\]  

(3.7)

Appealing to successive iterations by the use of (3.6a) and (3.6b), the first factor in (3.7) is estimated. The second factor \(G_0\) is defined as

\[
 G_0 = \langle \exp(-i \sum_j x_j \sigma_j^2 + \sum_{jk} B_{jk} \sigma_j \sigma_k) \rangle_0,
\]  

(3.8)

which is calculated by means of cumulant expansion.

The linear terms concerning \(x_j\) and \(\tau_{jk}\) are canceled out thanks to the saddle point equations:

\[
 g(j, j) + h^2 g^2(0) = 1
\]  

(3.9)

and

\[
 2 t_{jk}^0 = a_{jk} [g(j, k) + h^2 g^2(0)].
\]  

(3.10)

Scale transformations of integral variables \(x_j\) and \(\tau_{jk}\) \([x_j \rightarrow x_j/\sqrt{n} \text{ and } \tau_{jk} \rightarrow \tau_{jk}/\sqrt{n}]\) lead to the following expression for \(\tilde{Z}\):

\[
 \tilde{Z} = \pi^{-L^2/2} (2\pi)^{-N} n^{-N/2} \exp \left\{ n \left[ N t - \sum_{jk} t_{jk}^0 + \ln f_0 \right] \right\}
 \times \int_{-\infty}^{\infty} \prod_j d\tau_{jk} \int_{-\infty}^{\infty} \prod_j dx_j \exp \left\{ -\sum_{jk} \tau_{jk}^2 \right\}
 \times \sum_{j_1, j_2} x_{ji} x_{ji} [g^2(j_1, j_2) + 2h^2 g^2(0) g(j_1, j_2)]
 - 2i \sum_{j_1, j_2} x_{ji} x_{ji} \tau_{ji} \tau_{ji} [g(j_1, j_2) g(j_1, j_3) + 2h^2 g^2(0) g(j_1, j_2)]
 + \sum_{j_1, j_2, j_3, j_4} a_{j_1, j_2, j_3, j_4} \tau_{ji} \tau_{ji} \tau_{ji} \tau_{ji} [g(j_1, j_2) g(j_1, j_3) + 2h^2 g^2(0) g(j_1, j_2)]
 + O(1/n). 
\]  

(3.11)
§ 4. Case I. $a_{jk} = a/\sqrt{N}$ for all $j$ and $k$

We deal with a system having short-range interaction $K_{jk}$ with an overall fluctuation of the interaction between spins. For the same reason as the Husimi-Temperley model$^6$ or the Sherrington-Kirkpatrick model,$^7$ constant value $a$ is divided by $\sqrt{N}$. Calculations up to $O(1/n)$ are given in the Appendix.

In what follows we discuss the zero field susceptibility for $h = 0$:

$$\chi = g(0) = [r - 2T(0)]^{-1} = 2/[r + (r^2 - 4a^2/N)^{1/2}]. \quad (4.1)$$

In this case the equation of state (A.15) is reduced to

$$\tilde{K} = \frac{1}{N} \sum_q g(q) + \frac{1}{n} \sum_q \frac{J_a(q)}{\nu_a(q)} - a^2 J_a^*(0) + O(1/n^2). \quad (4.2)$$

Various functions in the Appendix are modified as follows:

$$g(q) = [r + K(0) - K(q) - 2T(q)]^{-1} = \frac{2}{[r + K(0) - K(q) + (r + K(0) - K(q))^2 - 4a^2/N]^{1/2}}, \quad (4.3)$$

$$\phi_a(q) = 1 - a^2 g^2(q)/N = [((r + K(0) - K(q))^2 - 4a^2/N)^{1/2}g(q), \quad (4.4)$$

while $\nu_a(q)$, $J_a(q)$ and $J_a^*(0)$ are expressed in terms of (4.3) and (4.4).

Since $\chi$ should be real, the relation $r \geq 2a/\sqrt{N}$ must be satisfied. At the point $r = 2a/\sqrt{N}$, $\chi$ takes its maximum value $m/a$. If we take the thermodynamic limit $(N \to \infty)$, $r = 2a/\sqrt{N} \to 0$. Accordingly

$$g(q) \to [r + K(0) - K(q)]^{-1}, \quad (4.5)$$

$$J_a(q) \to f(q) = g(q)N^{-1} \sum_k g^2(k) - N^{-1} \sum_k g^2(k)g(q - k), \quad (4.6a)$$

$$J_a^*(0) \to f(0) \quad (4.6b)$$

and

$$\nu_a(q) \to \nu(q) = N^{-1} \sum_k g(k)g(q - k). \quad (4.7)$$

Consequently by the use of (4.3)~(4.7), Eq. (4.2) simply reads at the critical point $T_c$:

$$\tilde{K}_c = \tilde{J}/k_B T_c = \left\{ \frac{1}{N} \sum_q g(q) + \frac{2}{n} \sum_q \frac{f(q)}{\nu(q)} - a^2 f(0) \right\}_{r=0}. \quad (4.8)$$
If the space dimensionality $d$ is restricted to the region $2 < d < 4$, it follows that for the $d$-dimensional hypercubic lattice

$$\frac{1}{N} \sum_{q} g(q) \bigg|_{r=0} = W_d/2 = \text{finite}, \quad (4.9)$$

$$\frac{1}{N} \sum_{q} \nu(q) \bigg|_{r=0} = \text{finite} \quad (4.10)$$

and

$$J(0) |_{r=0} = -\infty. \quad (4.11)$$

In case of the 3-dimensional simple cubic lattice, for example, Eqs. (4.9) and (4.10) have been enumerated.\(^{a)} N^{-1} \sum \left( g(q) \right) |_{r=0} = 0.2527 \text{ and } N^{-1} \sum \left( \nu(q) / \nu(q) \right) |_{r=0} = -0.059. \text{ As a result, } K_c = \infty \text{ even for } 2 < d < 4. \text{ Remember the statement in § 2 that the critical behavior of our model is exactly the same as the Ising model. Taking this into account, it is concluded that there exists no phase transition at any finite temperature (i.e., } T_c = 0 \text{) for the continuous symmetry } (n \geq 2). \text{ In this case zero field susceptibility } \chi \to \infty \text{ as } T \to 0 \text{ (} N \to \infty \text{). Therefore whole finite temperature region is paramagnetic. This result can be expected from the fact that the interaction is short-range while the fluctuation of the interaction between spins is infinite-range.}

Only the spherical limit is an exceptional case. Such a pathological behavior\(^{**} \text{ is due to the definition (2.2) of the distribution function } P. \text{ In the limits } n \to \infty \text{ and } N \to \infty, \text{ Eq. (2.2) should be written as}

$$P = \prod_{j,k} \delta(K_{jk} - K_{jk}). \quad (4.12)$$

Hence our model is the spherical model itself\(^{43,9} \text{ bearing the exchange interaction } K_{jk}. \text{ It goes without saying that the critical behavior is just the same as the spherical model.}

§ 5. Case II. $a_{jk} = a$ for nearest-neighbor pairs; 0 otherwise

In contrast to the case I, in this case we assume that the fluctuations take place only between nearest-neighbor pairs. In accordance with this, we consider the nearest-neighbor interactions. Then in the $d$-dimensional hypercubic lattice, taking the lattice constant as unity, we have

$$K(q) = 2 K \sum_{i=1}^{d} \cos q_i, \quad (5.1)$$

\(^{a)} \text{ The } d\text{-dimensional Watson integral } W_d \text{ is given by (5.11) in the following section.}

\(^{**} \text{ In the pure system critical temperature is known to decrease with the increase of } n \text{ (the number of spin components).}
where \( q_i(i=1, 2, \ldots, d) \) are components of wave vector \( \mathbf{q} \). We have assumed that the system under consideration has no anisotropy. Owing to mathematical difficulties, we cannot help restricting ourselves to the discussion in the limit \( n \to \infty \).

Since \( t_{jk}^0 \) is zero unless \( j \) and \( k \) are nearest-neighbors, if we put \( t_{jk}^0 = t^0 \) for nearest-neighbor pairs, we obtain

\[
T(\mathbf{q}) = 2at^0 \sum_{i=1}^{d} \cos q_i . \tag{5.2}
\]

In order to determine \( t^0 \), we consider Eq. (3·10). If the point \( j \) is chosen as an origin, this reads

\[
2t^0 = a \sum_{k} g(j, k) = a \sum_{k} \frac{e^{-i\mathbf{q} \cdot \mathbf{r}_k}}{2(t - \tilde{K} \sum_{i=1}^{d} \cos q_i)} , \tag{5.3}
\]

where \( \tilde{K} = \tilde{K} + 2at^0 \), the point \( k \) is the nearest-neighbor to the origin and we make use of (A·3a) as well as (5·1) and (5·2). Writing (5·3) for points lying on the axes 1, 2, \ldots, \( d \) and summing them up, we have

\[
2dt^0 = \frac{d}{N} \sum_{q} \frac{\sum_{i=1}^{d} \cos q_i}{2(t - \tilde{K} \sum_{i=1}^{d} \cos q_i)} . \tag{5.4}
\]

By applying the relation \( N^{-1} \sum g(\mathbf{q}) = 1 \), we find

\[
2dt^0 = a(2t - 1)/2\tilde{K} , \tag{5.5}
\]

whence \( t^0 \) is solved as

\[
t^0 = \left[ (d^2 + 2d(2t - 1)a^2)^{1/2} - d \right]/4da . \tag{5.6}
\]

Here we have scaled \( a \) as \( a \to \tilde{K}a \). Since \( t^0 \) should be real, the following relation must be satisfied

\[
t \geq 1/2 - d/4a^2 . \tag{5.7}
\]

In order to discuss the behavior of \( \chi \), we notice the following expression for \( g(\mathbf{q}) \)

\[
g(\mathbf{q}) = \left[ 2t - 2\tilde{K}X \sum_{i=1}^{d} \cos q_i \right]^{-1} , \tag{5.8}
\]

where \( X \) is given by

\[
2X = 1 + [d^2 + 2d(2t - 1)a^2]^{1/2}/d . \tag{5.9}
\]

If we put \( t = \tilde{K}s \), by the use of (5·8), the relation \( N^{-1} \sum g(\mathbf{q}) = 1 \) is written as

\[
\tilde{K} = \frac{1}{2N} \sum_{q} \frac{1}{s - X \sum_{i=1}^{d} \cos q_i} . \tag{5·10a}
\]
Let us now assume that $\chi$ diverges at some transition temperature and denote quantities there by putting subscripts $c$. It is clear that $s_c = X_{cd}$. Therefore, at the transition temperature (5.10a) takes the following form:

$$\bar{K}_c = \frac{1}{2N\bar{X}_c} \sum_q (d - \sum_{i=1}^d \cos q_i).$$  \hspace{1cm} (5.10b)

If we note that the sum over $q$ is the $d$-dimensional Watson integral $W_d$, i.e.,

$$W_d = N^{-1} \sum_q [d - \sum_{i=1}^d \cos q_i]^{-1},$$

it follows that

$$\bar{K}_c = \frac{W_d}{2X_c}.$$  \hspace{1cm} (5.12)

Combining (5.12) with the relation $t_c = \bar{K}_c X_{cd}$, we have

$$t_c = dW_d/2.$$  \hspace{1cm} (5.13)

Since $t$ is subject to the condition (5.7), we find

$$d/2a^2 \geq 1 - dW_d.$$  \hspace{1cm} (5.14)

The numerical values of $W_d$ at $d = 3$ is $W_3 = 0.50546\cdots$. Thus the right-hand side in (5.14) is negative, so that the condition (5.14) is satisfied. This means that $\chi$ diverges at $\bar{K}_c$ given by (5.12). Equations (5.10a) and (5.10b) have forms similar to those in the pure case, only difference being that the quantity $X$ appears in this case. As a result, usual way\(^9\) to calculate the susceptibility exponent $\gamma$ is applicable by expanding $X$ around $X_c$ in powers of $\bar{K}_c - \bar{K}$. It turns out that $\gamma$ is given by $\gamma = 2/(d - 2)$ for $2 < d < 4$. Thus $\gamma$ is the same as the spherical model value. This result is also consistent with the one obtained previously.\(^1\)

In closing this section, we would like to discuss the behavior of $\bar{K}_c$ as a function of $a$. From (5.9), (5.12) and (5.13), $\bar{K}_c$ is calculated to be

$$\bar{K}_c = W_d/(1 + [1 + 2a^2(dW_d - 1)/d]^{1/2}).$$  \hspace{1cm} (5.15)

If $a$ is zero, (5.15) is naturally reduced to the result of the spherical model. When $a$ is small, by expanding (5.15) in powers of $a$, we obtain

$$\bar{K}_c = W_d[1 - a^2(dW_d - 1)/2d]/2 + O(a^4). \hspace{1cm} (a \ll 1)$$  \hspace{1cm} (5.16)

On the other hand, for $a \gg 1$, we see from (5.15) that $\bar{K}_c \propto a^{-1}$. In any way, $\bar{K}_c$ is
a decreasing function when $a$ increases.\textsuperscript{10)} This behavior may be understood qualitatively in the following way. Keeping the method of mean field theory in mind, if we replace the term $\sum \sigma_j(m)\sigma_k(m)$ by its average value in (2.3), it turns out that the effective interaction between nearest-neighbor pairs is expressed as $K_{jk} + a^2 g(j, k)$. Since we are interested in ferromagnetic interaction, the pair correlation function $g(j, k)$ should be positive. Therefore, the larger $a$ is, we have the stronger effective interaction, so that the higher transition temperature.

§ 6. Concluding remarks

In this paper, we have discussed the critical properties of annealed system with Gaussian random distribution of exchange interactions in $1/n$ expansion. The results have been proved to depend on the behavior of $a_{jk}$. If $a_{jk}$ is constant and each spin has a continuous symmetry ($n \geq 2$), no phase transition is observed at finite temperature, apart from the pathological behavior in the limit $n \rightarrow \infty$. On the other hand, if $a_{jk}$ is zero except for nearest-neighbor pairs, essential features of phase transition in the limit $n \rightarrow \infty$ remain the same as the spherical model. Although we assumed that $K_{jk} > 0$ in our treatments, it would be of considerable interest, in connection with spin glasses, to see what is going on if all $K_{jk}$ are zero. Also the extension of the present theory to the quenched case is a challenging problem.

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Appendix

\textbf{Free Energy and Equation of State in the Case $a_{jk} = a/\sqrt{N}$}

In order to obtain the free energy and the equation of state in case of overall fluctuation, we Fourier-transform various quantities below into momentum space

\textsuperscript{10)} Thorpe and Beeman studied the annealed Ising model with random exchange interactions.\textsuperscript{11)} Their exact solutions show that critical indices $\alpha$, $\beta$, $\gamma$ and $\delta$ in two dimensions are unaltered. However, when random exchange interactions have Gaussian distribution with small variance, Thorpe and Beeman showed that the critical temperature of the disordered system, in comparison with that of the pure system, decreases with the increase of variance. This result is contrary to ours, in that $T_c$ increases in our case. These differences may be merely of apparent nature, arising from differences of the ways to define an annealed system. To the contrary, if this inconsistency is ascribed to the calculations in the limit $n \rightarrow \infty$, we may obtain a qualitatively consistent result by evaluating $K_c$ up to $O(1/n)$. Further elaborate studies are needed to clarify this ambiguity.
in the first Brillouin zone:

\[ \vec{K}(\mathbf{q}) = \frac{1}{N} \sum_{\mathbf{j}, \mathbf{k}} \vec{K}_{\mathbf{j}, \mathbf{k}} e^{-i\mathbf{q} \cdot (\mathbf{r}_\mathbf{j} - \mathbf{r}_\mathbf{k})}, \]  

\[ T(\mathbf{q}) = \frac{a t^{0}(\mathbf{q})}{\sqrt{N}} = - \frac{a}{\sqrt{N}} \frac{1}{N} \sum_{\mathbf{j}, \mathbf{k}} t^{0}_{\mathbf{j}, \mathbf{k}} e^{-i\mathbf{q} \cdot (\mathbf{r}_\mathbf{j} - \mathbf{r}_\mathbf{k})}, \]  

\[ g(\mathbf{q}) = \frac{1}{N} \sum_{\mathbf{j}, \mathbf{k}} g(\mathbf{j}, \mathbf{k}) e^{-i\mathbf{q} \cdot (\mathbf{r}_\mathbf{j} - \mathbf{r}_\mathbf{k})} 
= [2t - \vec{K}(\mathbf{q})]^{-1} = [\vec{K} + \vec{K}(0) - \vec{K}(\mathbf{q}) - 2T(\mathbf{q})]^{-1}. \]  

(\text{A-3a})

and

\[ g(0) = \sum_{\mathbf{k}} g(\mathbf{j}, \mathbf{k}) = [\vec{K}s - 2T(0)]^{-1}, \]  

(\text{A-3b})

where \( \vec{K}(\mathbf{q}) = \vec{K}(\mathbf{q}) + 2T(\mathbf{q}) \) and \( \vec{K}s = 2t - \vec{K}(0) \).

By means of (\text{A-1})~(\text{A-3b}), saddle point equations (3·9) and (3·10) are transformed into the following forms respectively:

\[ \frac{1}{N} \sum_{\mathbf{k}} g(\mathbf{k}) = \frac{1}{2} \sum_{\mathbf{k}} T(\mathbf{k}) = \frac{a^{2}}{2N}, \]  

(\text{A-4})

and

\[ 2T(\mathbf{q}) = 2a t^{0}(\mathbf{q})/\sqrt{N} = \left( a^{2} / N \right) \left[ g(\mathbf{q}) + \delta_{\mathbf{q}0} N h^{2} g^{2}(0) \right]. \]  

(\text{A-5})

Combining (\text{A-4}) with (\text{A-5}), a sum rule

\[ 2 \sum_{\mathbf{q}} T(\mathbf{q}) = a^{2} \]  

(\text{A-6})

holds. In the same way as in Ref. 4), we have the free energy:

\[ F = \frac{1}{N n} \ln Z = - \ln (2\pi)^{1/2} \left( \frac{1}{n} \ln (4\pi n)^{1/2} + t - \frac{1}{N} \sum_{\mathbf{k}} t^{0}_{\mathbf{k}}^{2} + \frac{h^{2} g(0)}{2} - \frac{1}{2 n N} \sum_{\mathbf{q}} \ln \nu_{\mathbf{k}}(\mathbf{q}) \right. \]

\[ \left. - \frac{1}{2 n} \sum_{\mathbf{q}} \ln \phi_{\mathbf{k}}(\mathbf{q}) + O(1/n^{2}). \right) \]  

(\text{A-7})

The following functions are used here:

\[ \nu_{\mathbf{k}}(\mathbf{q}) = \begin{cases} \frac{1}{N} \sum_{\mathbf{k}} \frac{g^{2}(\mathbf{k})}{\phi_{\mathbf{k}}(\mathbf{k})} + \frac{2h^{2} g^{2}(0)}{\phi_{\mathbf{k}}(0)} & \text{for } \mathbf{q} = \mathbf{0} \\ \frac{1}{N} \sum_{\mathbf{k}} g(\mathbf{k}) g(\mathbf{q} - \mathbf{k}) + 2h^{2} g^{2}(0) g(\mathbf{q}) & \text{for } \mathbf{q} \neq \mathbf{0} \end{cases} \]  

(\text{A-8a})

\[ \phi_{\mathbf{k}}(\mathbf{q}) = \begin{cases} \frac{1}{N} \sum_{\mathbf{k}} g(\mathbf{k}) g(\mathbf{k}) & \text{for } \mathbf{q} = \mathbf{0} \\ \frac{1}{N} \sum_{\mathbf{k}} g(\mathbf{k}) g(\mathbf{q} - \mathbf{k}) + 2h^{2} g^{2}(0) g(\mathbf{q}) & \text{for } \mathbf{q} \neq \mathbf{0} \end{cases} \]  

(\text{A-8b})
and
\[ \phi_{h \alpha}(q) = 1 - \left(\frac{a^2}{N}\right)[g^2(q) + 2\delta q\theta N h^2 g^3(0)]. \quad (A \cdot 9) \]

By differentiating (A · 5) and (A · 6) with respect to \( h \), the relations
\[ 2 \frac{\partial T(q)}{\partial h} = [1 - \phi_{h \alpha}^{-1}(q)] K \left( \frac{\partial s}{\partial h} \right) + 2a^2\delta q \theta h g^2(0) \]
and
\[ K \left( \frac{\partial s}{\partial h} \right) = 2h g^2(0)/\phi_{h \alpha}(0) \nu_{h \alpha}(0) \]
are obtained.

If we define magnetization formally as
\[ M = \frac{\partial F}{\partial h} = h g(0) \left| \frac{1}{h - 2 T(0) s} \right|_{s = r}, \]
it turns out to be expanded in \( 1/n \):
\[ M = h g(0) - \frac{1}{n} \left[ \frac{\partial s}{\partial h} \right] \left[ \frac{1}{N} \sum_q J_{h \alpha}(q) - a^2 J^*_{h \alpha}(0) \right] + O(1/n^2), \quad (A \cdot 11b) \]
where we make use of (3 · 9), (3 · 10) and (A · 10a). Introduced functions are denoted as
\[ J_{h \alpha}(q) = \left[ \begin{array}{c} g(q) \\
\frac{\partial g(q)}{\theta h} \\
- \frac{h^2 g^4(q)}{\phi_{h \alpha}(q) g^2(q)} \end{array} \right] \quad \text{for } q = 0 \quad (A \cdot 12a) \]
and
\[ J^*_{h \alpha}(0) = \left[ \begin{array}{c} g(0) \\
\frac{\partial g(0)}{\theta h} \\
- \frac{h^2 g^4(0)}{\phi_{h \alpha}(0) g^2(0)} \end{array} \right] \quad \text{for } q \neq 0 \quad (A \cdot 12b) \]
and
\[ J^*_{h \alpha}(0) = \left[ \begin{array}{c} g(0) \\
\frac{\partial g(0)}{\theta h} \\
- \frac{h^2 g^4(0)}{\phi_{h \alpha}(0) g^2(0)} \end{array} \right] \quad (A \cdot 12c) \]

* In our paper we must follow various limiting procedures in such an order that \( n \to \infty \) (only in case of the spherical limit), \( N \to \infty \) and \( h \to 0 \). A new parameter \( r \) means the value of \( s \) at the critical point.
By the use of (A·10b)~(A·11b) and also with the help of (A·5), the relation between $s$ and $r$ results in

$$Ks - 2T(q) = \left[ Kr - 2T(q) - \frac{1}{n} \frac{\phi_{ha}(0)}{\phi_{ha}(q)} Q \right]_{s=r} + O(1/n^2) \quad \text{(A·13)}$$

with $Q$ defined as

$$Q = \frac{2}{\phi_{ha}(0) \nu_{ha}(0)} \left[ \frac{1}{N} \sum_q \frac{J_{ha}(q)}{\nu_{ha}(q)} - a^2 J_{ha}^*(0) \right]. \quad \text{(A·14)}$$

Insertion of (A·13) into (A·4) together with scale transformations such as $K M^2 \rightarrow M^2$, $a^2 \rightarrow K^2 a^2$, $K g(q) \rightarrow g(q)$ and $T(q) \rightarrow K T(q)$ gives rise to the equation of state:

$$\tilde{K} = \frac{1}{N} \sum_q g(q) + M^2 + \frac{2}{n} \left[ \frac{1}{N} \sum_q \frac{J_{Ma}(q)}{\nu_{Ma}(q)} - a^2 J_{Ma}^*(0) \right] + O(1/n^2). \quad \text{(A·15)}$$

Subscripts $M$ of $J_{Ma}(q)$, $J_{Ma}^*(0)$ and $\nu_{Ma}(q)$ show that $h g(0)$ in $J_{ha}(q)$, $J_{ha}^*(0)$ and $\nu_{ha}(q)$ should be replaced by $M$. Setting $a=0$ in (A·15), it agrees with the equation of state of the pure system.\(^5\)

References

6) K. Husimi, Proceedings of the International Conference of Theoretical Physics, Kyoto and Tokyo, 1953, p. 531.  