Restoration of the Local Gauge Symmetry
and Color Confinement in Non-Abelian Gauge Theories

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Restoration of the local gauge symmetry and its connection to color confinement is investigated in non-Abelian gauge theories with covariant gauge fixing. We consider the Noether current $J^a_{\mu}$ of the local gauge transformation with transformation functions $A^a(x)$ linear in $x^\mu$; $A^a(x)=\delta^{a\mu}x_\mu$. This current is conserved only in the physical subspace of the state vector space and in perturbation theory contains a massless pole communicating to the gauge field. We define the local gauge symmetry restoration as the disappearance of this massless "Goldstone" pole from $J^a_{\mu}$. The restoration condition is obtained and it coincides exactly with the color confinement criterion proposed earlier by Kugo and Ojima. Quarks and other colored particles are shown to be confined in the local gauge symmetry restored phase by using the Ward identities of $J^a_{\mu}$.

§ 1. Introduction

It is now widely believed that the phenomenon of quark confinement (or more generally color confinement) in QCD is a manifestation of the fact that the system is in its disordered (symmetry restored) phase.\textsuperscript{1} What kind of symmetry is broken in the perturbative vacuum and restored in the true confining vacuum? Often suggested is the local gauge symmetry.\textsuperscript{2} However, the whole local gauge symmetry cannot be restored since, in the continuum theory, we have to break it explicitly (i.e., fix the gauge) in quantizing the gauge system. So it must be a local gauge symmetry of a special kind that matters.

Such a symmetry is well known in QED where confinement does not occur.\textsuperscript{3} If we adopt in QED the covariant gauge fixing of the type $-(1/2\alpha)(\partial_\mu A^\mu)^2$, the gauge-fixed Lagrangian still has an residual local gauge invariance with the transformation function $A(x)$ linear in $x_\mu$; $A(x)=\varepsilon_\mu x^\mu$, $\varepsilon_\mu=$ constants. [The transformation property of the photon field $A_\mu(x)$ is defined by $A_\mu(x)=\partial_\mu A(x)$.] If we denote by $Q_\mu$ the conserved Noether charge of this local gauge symmetry, it effects the following transformation on the photon field $A_\mu$:

$$[Q_\mu, A_\nu(x)]=ig_{\mu\nu},$$

expressing that the $Q_\mu$-symmetry is spontaneously broken and the photon is the corresponding Goldstone vector boson.

One naive expectation is that in the true confining vacuum of QCD the corresponding symmetry is restored and the perturbatively massless gluon disappears from the physical spectrum. However, if we try to treat the gauge-fixed continuum theory, the matter is not so simple because i) there is no such a simple
residual local gauge symmetry in the non-Abelian case, and ii) the connection between the symmetry restoration and color confinement is not obvious at all. So the precise formulation seems to be still lacking.

In this paper we investigate this problem of local gauge symmetry restoration in non-Abelian gauge theories with covariant gauge fixing and its connection to color confinement. As the characterization of restoration of the local gauge symmetry, we have two alternative ways which are essentially equivalent to each other. One is to examine the *exact* residual local gauge symmetry present in the gauge-fixed system, whose transformation functions $A^a_\mu(x)$ depend (non-locally) on the gauge field $A_\mu^a$. It is interesting owing to its close similarity to the symmetry restoration in the 2-dimensional non-linear $\sigma$-models. However, we do not discuss it here leaving the details to a separate publication. The other way, which we take in this paper and is more suitable for the discussion on color confinement, is based on the manifestly covariant operator formalism of non-Abelian gauge theories of Kugo and Ojima, and treats the Noether current of the local gauge transformation with transformation functions $A^a_\mu(x)$ linear in $x_\mu$, the same as for the residual local gauge symmetry in QED. It is not exactly conserved, but is conserved in the physical subspace only. In perturbation theory, this current contains a massless pole communicating to the gauge field $A_\mu^a$, and we search for the possibility that this massless "Goldstone" pole disappears from the current in the same way as the restoration of the usual global symmetries. We call this phenomenon restoration of the "local gauge symmetry". [We have attached the quotation marks because the current is not exactly conserved.]

The condition for the "local gauge symmetry" restoration is obtained and it is found to just coincide with the color confinement criterion proposed earlier by Kugo and Ojima by taking a rather different point of view. This condition also coincides with the restoration condition for the *exact* residual local gauge symmetry mentioned above.

The rest of this paper is organized as follows. In § 2, we construct the Noether current of the local gauge transformation and obtain the restoration condition of the "local gauge symmetry". In § 3, we classify the phases of a gauge theory by the help of the global gauge symmetry and the "local one". Quark confinement in the "local gauge symmetry" restored phase is shown to be realized by using the Ward identity (WI) of the Noether current constructed in § 2. Section 4 is devoted to the discussion, where the analogy with the symmetry restoration in the 2-dimensional non-linear $\sigma$-models is briefly described.

§ 2. Restoration of the "local gauge symmetry"

We consider a (non-)Abelian gauge theory with covariant gauge fixing, the
Lagrangian density of which is given as follows:  
\[ \mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}, \]  
\[ \mathcal{L}_{\text{inv}} = -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \mathcal{L}_{\text{matter}}(\varphi, D_\mu \varphi), \]  
\[ \mathcal{L}_{\text{GF}} = -\partial_\mu B \cdot A^\mu + \frac{1}{2} a B \cdot B, \]  
\[ \mathcal{L}_{\text{FP}} = -i \partial_\mu \bar{c} \cdot D_\mu c, \]  
where  
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu, \]  
\[ D_\mu c = (\partial_\mu + g A_\mu \times c), \quad D_\mu \varphi = (\partial_\mu - ig T \cdot A_\mu) \varphi. \]  
\[ \mathcal{L}_{\text{matter}} \] in (2.1a) is the gauge-invariant Lagrangian density of matter fields \( \varphi \), \( \mathcal{L}_{\text{GF}} \) (2.1b) is a covariant gauge fixing term with Lagrange multiplier fields \( B^a(x) \) included, and \( \mathcal{L}_{\text{FP}} \) (2.1c) is the corresponding Faddeev-Popov ghost term. The \( T^a \)'s in the covariant derivative \( D_\mu \) (2.2) are the generators of the gauge group \( G \) in the representation to which \( \varphi \) belongs.

The system (2.1) has an important conserved charge, the BRS charge \( Q_B \),  
\[ Q_B = \int d^3x \left\{ B \cdot (D_0 c) - \partial_0 B \cdot c + \frac{1}{2} ig \partial_0 \bar{c} \cdot (c \times c) \right\}, \]  
which generates the BRS transformation:  
\[ [iQ_B, A_\mu] = D_\mu c, \quad [iQ_B, B] = 0, \]  
\[ [iQ_B, c] = -\frac{1}{2} g (c \times c), \quad [iQ_B, \bar{c}] = iB, \]  
\[ [iQ_B, \varphi] = ig (T \cdot c) \varphi. \]  
\( Q_B \) has a particular property of being nilpotent, i.e.,  
\[ Q_B^2 = 0. \]  
Following Ref. 5), we specify the physical subspace \( \mathcal{V}_{\text{phys}} = \{ |\text{phys}\rangle \} \) of the total state vector space by the subsidiary condition  
\[ |\text{phys}\rangle \in \mathcal{V}_{\text{phys}} \iff Q_B |\text{phys}\rangle = 0. \]  
In particular, the vacuum \(|0\rangle\) is assumed to be annihilated by \( Q_B \):  
\[ Q_B |0\rangle = 0. \]  

2.1. Noether current of the local gauge transformation

Before fixing the gauge, the "classical" Lagrangian \( \mathcal{L}_{\text{inv}} \) (2.1a) has an in-
variance under the local gauge transformation
\[ \delta^A A_\mu(x) = D_\mu A(x) = (\partial_\mu + g A_\mu \times) A(x), \]
\[ \delta^A \varphi(x) = ig(T \cdot A(x)) \varphi(x) \] (2.8)
with \( A^\alpha(x) \) arbitrary c-number functions of space-time \( x_\mu \). Of course, this local
gauge symmetry is explicitly broken in the quantum (gauge-fixed) Lagrangian \( \mathcal{L} \)
(2.1) owing to \( \mathcal{L}_{GF} \) and \( \mathcal{L}_{FP} \). However, let us study the Noether current
of the transformation (2.8), which becomes a key object in the following discussion.

As for the transformation rule for the fields \( B, c \) and \( \bar{c} \), which is not known a
priori, we assign a simple extension of the global gauge transformation:
\[ \delta^A B(x) = g B(x) \times A(x), \quad \delta^A c(x) = g c(x) \times A(x), \]
\[ \delta^A \bar{c}(x) = g \bar{c}(x) \times A(x). \] (2.9)
Now, the Noether current \( J^a_\mu \) is obtained from the Lagrangian (2.1) and the
transformation law (2.8) and (2.9):
\[ J^a_\mu(x) = g J^a_\mu(x) \cdot A(x) + F^a_\mu(x) \cdot \partial^\mu A(x), \] (2.10)
where \( J^a_\mu(x) \) is the conserved Noether current of the global gauge symmetry of
the system (2.1):\(^5\)
\[ J^a_\mu = -i(T^a \varphi)_i \frac{\partial}{\partial \varphi_i} \mathcal{L}_{matter} + (A^\nu \times F^a_\nu)^a + (A_\mu \times B)^a \]
\[ - i(\bar{c} \times D_\mu c)^a + i(\partial_\mu \bar{c} \times c)^a. \] (2.11)
Then, taking the divergence of the current \( J^a_\mu \), we get
\[ \partial^a J^a_\mu = (Q_B, (D_\mu \bar{c})^a) \partial^\mu A^a. \] (2.12)
In calculating (2.12), we have used the conservation of the global current \( J^a_\mu \)
(\( \partial^a J^a_\mu = 0 \)) and the equation of motion of the gauge field \( A^a_\mu \):\(^5\)
\[ \partial^a F^a_{\mu \nu} + g J^a_\nu = (Q_B, (D_\nu \bar{c})^a). \] (2.13)

From Eq. (2.12), we see that \( J^a_\mu \) is conserved in the physical subspace \( C_V^{phys} \) (2.6):
\[ \langle \beta | \partial^a J^a_\mu(x) | \beta \rangle = 0 \quad \text{for } |\alpha\rangle, |\beta\rangle \in C_V^{phys}. \] (2.14)

2.2. Abelian case

When the gauge group \( G \) is \( U(1) \) (i.e., in the case of QED), the r.h.s. of (2.12)
\[^5\) This is a general property common to the Noether current of a transformation which is a
symmetry for the local gauge-invariant Lagrangian \( \mathcal{L}_{inv} \) (2.1a) but not so for the gauge-fixed one,
and moreover, is commutable with the BRS transformation.\(^5\) This is easily seen by noting that \( \mathcal{L}_{GF} = \mathcal{L}_{FP} \)
can be expressed as a particular anti-commutator form \( \{Q_B, M\} \) for any choice of \( \mathcal{L}_{GF} \).
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reduces simply to $\partial_\nu B \partial^\nu A$, which becomes a gradient form $\partial_\nu B$ if we take for $A(x)$ a special one

$$A(x) = x_\nu.$$  \hspace{1cm} (2.15)

So, we have a conserved current $j_{\mu,\nu}$ corresponding to the local gauge transformation (2.15):

$$j_{\mu,\nu} = J_{\mu,\nu} = g J_{\mu} x_{\nu} + F_{\mu\nu} - g_{\mu\nu} B,$$

$$\partial^\mu j_{\mu,\nu} = 0.$$  \hspace{1cm} (2.16)

This is the residual local gauge symmetry in QED stated in the previous section.

Let $Q_\mu$ be the conserved charge obtained from the current $j_{\mu,\nu}$ (2.16),

$$Q_\mu = \int d^3 x j_{0,\mu}(x),$$  \hspace{1cm} (2.17)

then it satisfies the following commutation relation with the photon field $A_\mu(x)$

$$[Q_\mu, A_\nu(x)] = ig_{\mu\nu}.$$  \hspace{1cm} (2.18)

So the $Q_\mu$-symmetry is always spontaneously broken and the photon field $A_\mu$ contains the corresponding Goldstone boson. However, we should note that there are two alternative ways satisfying (2.18) corresponding to two possible phases realized in QED. i) Coulomb phase: The Goldstone boson is the massless vector photon itself. ii) Higgs phase: The Goldstone boson associated with the spontaneous breakdown of the global $U(1)$ symmetry plays at the same time the role of the Goldstone boson of $Q_\mu$-symmetry breaking. Although, in the Higgs phase, the physical mode of $A_\mu$ is massive, the massless unphysical Goldstone mode $\chi$ is contained in $A_\mu$ in the form $A_\mu \sim \partial_\mu \chi$ in the manifestly covariant formalism adopted here. Also, the global $U(1)$ current $J_\mu$, which appears in $j_{\mu,\nu}$ in the form $g J_\mu x_\nu$, contains a massless mode of $B$; $g J_\mu \sim \partial_\mu B$, as seen from (2.13). Equation (2.18) is a consequence of the massless pole between $\chi$ and $B$; $[\chi(x), B(y)] = iD(x - y)$. See, e.g., Ref. 5).

2.3. Non-Abelian case

In the non-Abelian case, the transformation (2.8) and (2.9) cannot be a symmetry of the Lagrangian (2.1) for any choice of the c-number functions $A^a(x)$ except for the case $A^a(x) = \text{constant}$ corresponding to the global gauge transformation. However, if we take for $A^a(x)$ functions linear in $x_\mu$ in analogy with the residual symmetry in QED (2.15),

$$A^b(x) = A_{\mu}^b(x) = \delta_{\mu\nu} x_\nu,$$  \hspace{1cm} (2.19)

and denote $J^a_{\mu,\nu}$ in this case by $J^a_{\mu,\nu}$,
then, in perturbation theory, this $J_{\mu,\nu}^a$ contains a massless "Goldstone" pole communicating to $A_{\mu}^a$ in the same way as in QED, and hence the current divergence term in the WIs of $J_{\mu,\nu}^a$

$$\int d^4x e^{ip\cdot x} \partial^a \langle T J_{\mu,\nu}^a(x) X \rangle$$

(2.21)

with $X$ a product of fields, cannot be dropped in the limit $p \to 0$. In this sense, the "local gauge symmetry" of $J_{\mu,\nu}^a$ is "spontaneously broken" in the perturbative vacuum.

Although, in QED, this massless pole in the current $J_{\mu,\nu}$ (or $j_{\mu,\nu}$ (2.16)) is always present as seen from Eq. (2.18), there might be a possibility in the non-Abelian case that it disappears and (2.21) vanishes in the limit $p \to 0$. We call this phenomenon restoration of the "local gauge symmetry" as stated in § 1. To see whether this is possible, consider the following WI:

$$\int d^4x e^{ip\cdot(x-y)} \partial^a \langle T J_{\mu,\nu}^a(x) A_{\lambda}^b(y) \rangle$$

$$= i \langle (D_\mu A_{\nu}(x))^b \rangle + \int d^4x e^{ip\cdot(x-y)} \langle T(Q_{\beta}, (D_\nu \bar{c}(x))^a) A_{\lambda}^b(y) \rangle$$

$$= i(g_{\nu\lambda} - p_{\nu} p_{\lambda}/p^2) \delta^{ab} - ig \int d^4x e^{ip\cdot(x-y)} \langle T(A_\nu \times \bar{c}(x))^a(D_\lambda c(y))^b \rangle,$$

(2.22)

where $\langle T \cdots \rangle$ denotes the covariant $T^*$-product, $A_{\nu}(x)$ is given by (2.19), and we have assumed that the Lorentz invariance is not broken and hence $\langle A_{\mu}^a(x) \rangle = 0$. The divergence of the current $J_{\mu,\nu}^a$ is given by

$$\partial^a J_{\mu,\nu}^a = \{ Q_{\beta}, (D_\nu \bar{c})^a \},$$

(2.23)

as seen from (2.12). Use has also been made of Eqs. (2.4), (2.7) and the 2-point function

$$\int d^4x e^{ip\cdot(x-y)} \langle T B^a(x) A_{\lambda}^b(y) \rangle = \delta^{ab} p_{\lambda}/p^2.$$

(2.24)

As is easily seen from diagrammatical consideration, the last term in (2.22) is transversal, i.e.,

$$\int d^4x e^{ip\cdot(x-y)} \langle T(A_\nu \times \bar{c}(x))^a(D_\lambda c(y))^b \rangle$$

$$= g(g_{\nu\lambda} - p_{\nu} p_{\lambda}/p^2) \int d^4x e^{ip\cdot(x-y)} \langle T(A_\nu \times \bar{c}(x))^a(A_\rho \times c(y))^b \rangle$$

$$= -g(g_{\nu\lambda} - p_{\nu} p_{\lambda}/p^2) u^{ab}(p^2),$$

(2.25)
where $u^{ab}(p^2)$ is defined by
\[ u^{ab}(p^2) = f \int d^4 x e^{ip(x-y)} \left< T(A_a(x) \times \bar{c}(x))^a \right> \]
By using (2.25), Eq. (2.22) is rewritten as
\[ \int d^4 x e^{ip(x-y)} \partial_x U^{ab}(x,y) \]
In order for the massless "Goldstone" pole between $J_{\mu a}$ and $A_\mu^a$ to disappear, the l.h.s. of (2.27) must vanish in the limit $p \to 0$, which implies\)
\[ \lim_{p \to 0} (\delta^{ab} + g^2 u^{ab}(p^2)) = 0. \]
This is the desired condition for the restoration of the "local gauge symmetry". Unexpectedly, the condition (2.28) coincides exactly with the color confinement criterion proposed by Kugo and Ojima in Refs. 6) and 5).** They showed by the asymptotic field argument that the global color charge $Q^a$ vanishes in the physical subspace $\mathcal{C} \mathcal{V}_{\text{phys}}$:
\[ \langle a|Q^a|\beta\rangle = 0 \quad \text{for } |a\rangle, |\beta\rangle \in \mathcal{C} \mathcal{V}_{\text{phys}}, \]
if i) the condition (2.28) is fulfilled, and ii) the global gauge symmetry is unbroken. They also showed that a rather abstract expression of color confinement (2.29) is equivalent to the color confinement by quartet mechanism,\footnote{The parameter $u^{ab}$ in Refs. 6) and 5) is just equal to $g^2 u^{ab}(0)$ in this paper.} according to which all the colored asymptotic states, if any, are necessarily unphysical and cannot be seen in the physical subspace $\mathcal{C} \mathcal{V}_{\text{phys}}$. Our finding here is that the color confinement phenomenon is interpretable as a result of the restoration of the "local gauge symmetry". The additional assumption of the unbroken global gauge symmetry is unnecessary as we shall see in the next section.

§ 3. Color confinement and the "local gauge symmetry" restoration

3.1. Classification of phases

Now, we have two criteria to specify the phases of a gauge theory: One is
spontaneous breakdown or non-breakdown of the global gauge symmetry, and the other is breakdown or non-breakdown of the “local gauge symmetry”. For the latter the l.h.s. of (2·28) serves as the order parameter. Therefore, in principle, there are four possible cases.* However, we infer that the phase with breakdown of the global gauge symmetry but with non-breakdown of the “local one” is impossible. The situation is the same as in the Higgs phase of QED which we have explained in the previous section: When the global gauge symmetry is spontaneously broken, there is a massless Goldstone pole in the global current \(J_{\mu}^{a}\), which necessarily causes the breakdown of the “local gauge symmetry” since the current \(J_{\mu}^{a}\) contains \(J^{a}_{\mu}\) in the form \(gf_{\mu}^{a}x_{\nu}\).

Thus, we are led to the following classification of the phases of a gauge theory.

i) Higgs phase; where both the global gauge symmetry and the “local one” are broken. The gauge bosons are massive due to the Higgs mechanism. Breakdown of the “local gauge symmetry” is caused by that of the global one.

ii) Coulomb phase; where the “local gauge symmetry” is broken but the global one is not. We have massless vector gauge bosons, which cause the breakdown of the “local gauge symmetry”.

iii) Confinement phase; where both the global gauge symmetry and the “local one” are unbroken. This phase is characterized by the condition (2·28), and has a chance to be realized only in the non-Abelian case. The reason why we call this phase confinement phase is clear from the discussion given in Ref. 5). The gauge bosons become totally unphysical.

3.2. Quark confinement in the third phase

In the following, we will present an independent argument for color confinement in the third phase paying special attention to quark confinement. Before doing this, let us remember what we have to show to prove quark confinement in QCD in covariant gauges. [Quark fields are introduced into the system, i.e., the matter Lagrangian \(\mathcal{L}_{\text{matter}}\) in (2·1a) contains the quark part \(\bar{\psi}(\not\!q - M)\psi\), so that the Wilson loop expectation value is of no use.] In the manifestly covariant formulation, quark confinement does not necessarily imply that quarks have no asymptotic fields. They may have asymptotic fields so long as they are unphysical, or more precisely, the BRS-transform of the quark field \([iQ_B, \phi] = ig(c \cdot T)\phi\) also has an associated asymptotic field of the same mass. Then, quarks cannot be seen in the physical subspace \(\mathcal{V}_{\text{phys}}\) by quartet mechanism.\(^{9,5}\)

We now show that the quark asymptotic fields, if they exist at all, become necessarily unphysical in the third phase by the help of the WI of \(J_{\mu}^{a}\) (2·20). For this purpose, consider the following equation:

\[^*\) Of course, there are many possibilities of partial breaking of the gauge group \(G\). Here, we consider only the cases of complete breaking or non-breaking, for simplicity.
\[
\int d^4x e^{ip \cdot x} \partial_\mu \langle TJ^\mu_{\nu,\lambda}(x) \phi(y) \bar{\phi}(z) \rangle
\]
\[
= g \left\{ -e^{ip \cdot y} T^a \langle T \phi(y) \bar{\phi}(z) \rangle + e^{ip \cdot z} \langle T \phi(y) \bar{\phi}(z) \rangle T^a \right\}
\]
\[
+ \int d^4x e^{ip \cdot x} \langle T \{ Q_\nu, (D_\nu \bar{c}(x))^a \} \phi(y) \bar{\phi}(z) \rangle
\]
\[
= -g \left\{ e^{ip \cdot y} \bar{\phi} - e^{ip \cdot z} \phi \right\} T^a \langle T \phi(y) \bar{\phi}(z) \rangle
\]
\[
+ g \int d^4x e^{ip \cdot x} \langle T (D_\nu \bar{c}(x))^a (c \cdot T) \phi(y) \bar{\phi}(z) \rangle
\]
\[
= \langle T (D_\nu \bar{c}(x))^a \phi(y) \bar{\phi}(c \cdot T)(z) \rangle,
\]
(3.1)

where use has been made of Eqs. (2.23), (2.4), (2.7) and the fact that the matrix \(\langle T \phi(y) \bar{\phi}(z) \rangle\) with color indices \(i\) and \(j\) commutes with \(T^a\), which is owing to the unbroken global gauge symmetry in the third phase. By some diagrammatical consideration, we can rewrite the last integral in (3.1) as
\[
-ig(p_\nu/p^2)(e^{ip \cdot y} - e^{ip \cdot z}) T^a \langle T \phi(y) \bar{\phi}(z) \rangle
\]
\[
= g^2 (g_{\lambda\lambda} - p_\nu p_\lambda/p^2) \int d^4x e^{ip \cdot x} \langle T (A_\lambda \times \bar{c}(x))^a (c \cdot T) \phi(y) \bar{\phi}(z) \rangle
\]
\[
= \langle T (A_\lambda \times \bar{c}(x))^a \phi(y) \bar{\phi}(c \cdot T)(z) \rangle,
\]
(3.2)

where the first term is the contribution of the graphs in which \(\partial_\nu \bar{c}\) in \(D_\nu \bar{c}\) is connected directly to \(c\) in \((c \cdot T)\phi\) or \(\bar{\phi}(c \cdot T)\) without interactions, while the rest contributions of \(\partial_\nu \bar{c}\) with interactions constitute the \(p_\nu p_\lambda/p^2\)-part in the second term. In rewriting the latter contribution to \(p_\nu p_\lambda/p^2\)-form, we have used the WI of \(Q_\nu\):
\[
g \langle T (B \times \bar{c}(x))^a (c \cdot T) \phi(y) \bar{\phi}(z) \rangle
\]
\[
- \langle T (B \times \bar{c}(x))^a \phi(y) \bar{\phi}(c \cdot T)(z) \rangle
\]
\[
= -\langle T \{ Q_\nu, (B \times \bar{c}(x))^a \phi(y) \bar{\phi}(z) \rangle = 0.
\]
(3.3)

Fourier transforming with respect to \(y\) by \(q\), we get from (3.1) and (3.2)
\[
\int d^4x e^{ip \cdot x} \partial_\mu \int d^4y e^{iq \cdot y} \langle TJ^\mu_{\nu,\lambda}(x) \phi(y) \bar{\phi}(z) \rangle
\]
\[
= -g \left\{ \frac{\partial}{\partial p^\mu} - \frac{p_\lambda}{p^2} \right\} \left[ e^{ip \cdot q \cdot z} T^a \{ S_f(p + q) - S_f(q) \} \right]
\]
\[
+ g (g_{\lambda\lambda} - p_\nu p_\lambda/p^2) e^{ip \cdot q \cdot z} \{ G_\lambda^a((c \cdot T)\phi, \bar{\phi}; p, q) \}
\]
\[
- G_\lambda^a(\phi, \bar{\phi}(c \cdot T); p, q),
\]
(3.4)

where we have defined
\[ iS_F(p) = \int d^4x e^{ip \cdot x} \langle T \psi(x) \overline{\psi}(0) \rangle, \]

\[ G_F^a(X, Y; p, q) = g \int d^4x \int d^4y e^{ip \cdot x + iq \cdot y} \langle T(A_1 \times \overline{c}(x))^a X(y) Y(0) \rangle. \]  

(3.5)

Now, from the assumption that the "local gauge symmetry" is restored, the l.h.s. of Eq. (3.4) tends to zero in the limit \( p \to 0 \), which implies

\[
\lim_{p \to 0} (g_{\nu \lambda} - p_{\nu} p_{\lambda}/p^2) \left\{ T^a \frac{\partial}{\partial q^\nu} S_F(q) - G_F^a((c \cdot T)\psi, \overline{\psi}; p, q) \right. \\
+ G_F^a(\psi, \overline{\psi}(c \cdot T); p, q) \} = 0.
\]  

(3.6)

Define \( G_F^a \) by separating the quark propagator leg from \( G_F^a \)

\[ G_F^a((c \cdot T)\psi, \overline{\psi}; p, q) = \tilde{G}_F^a((c \cdot T)\psi, \overline{\psi}; p, q) S_F(p + q), \]

\[ G_F^a(\psi, \overline{\psi}(c \cdot T); p, q) = S_F(q) \tilde{G}_F^a(\psi, \overline{\psi}(c \cdot T); 0, q). \]  

(3.7)

Then, (3.6) yields

\[
T^a \frac{\partial}{\partial q^\nu} S_F(q) = \tilde{G}_F^a((c \cdot T)\psi, \overline{\psi}; 0, q) S_F(q) \\
- S_F(q) \tilde{G}_F^a(\psi, \overline{\psi}(c \cdot T); 0, q),
\]  

(3.8)

where \( \tilde{G}_F^a \) is the part of \( G_F^a \) not proportional to \( p_\nu \). We have depicted Eq. (3.8) in Fig. 1.

![Diagrammatic representation of Eq. (3.8)](https://example.com/diagram)

(3.8)

Fig. 1. Diagrammatic representation of Eq. (3.8).

From Eq. (3.8) (or Fig. 1), we can deduce the following: If the quark has an asymptotic field and its propagator \( S_F(q) \) has a pole\(^*) \) at \( q^2 = m^2 \), then the two \( \tilde{G}_F^a(0, q) \) in (3.8) (which are related to each other by charge conjugation) must have a simple pole at the same position \( q^2 = m^2 \). Note that the \( q \)-differentiation in the first term of (3.8) increases the order of the pole by one. This means that there exist bound states of mass \( m \) in the channels \((c \cdot T)\psi \) and \( \overline{\psi}(c \cdot T)\),\(^\text{**} \) and

\(^*) \) This need not be a simple pole but may be a pole of higher order or even one of fractional order such as in the electron propagator in QED.

\(^\text{**}) \) The possibility of the bound state in the channel \( A_\psi \) or \( \overline{\psi}A \) is excluded. This is because the presence of these bound states implies that there is a pole at \( q^2 = m^2 \) in the quark self-energy part, which contradicts the assumption that \( S_F(q) \) has a pole at \( q^2 = m^2 \).
thus the assumed quark asymptotic state (of mass $m$) is confined by quartet mechanism. Similar consideration applies to any other colored particles such as gluons.

§ 4. Discussion

In this paper we have investigated the possibility of the restoration of the “local gauge symmetry” in a gauge theory which is broken in perturbation theory due to the presence of massless gauge bosons even when the global gauge symmetry is unbroken, and found that this restoration is equivalent to the realization of color confinement. Unfortunately, we have not succeeded in showing that the restoration actually occurs in QCD. If we try to calculate the l.h.s. of (2·28) perturbatively, it is logarithmically infrared divergent and does not make sense. So, some dynamical mass generation is expected in the gauge field system.

One may wonder how such a condition as (2·28) can be satisfied in QCD, which has no intuitive meaning as the area law of the Wilson loop. It may seem miraculous that $g^2 u^{ab}(p^2)$ is just equal to $-\delta^{ab}$ in the limit $p \rightarrow 0$. The non-linear $\sigma$-models in 2-dimensions will provide an instructive example. Let us take the $O(N)/O(N - 1)$ model, where in the perturbative vacuum the $O(N)$ symmetry of the action spontaneously breaks down to $O(N - 1)$ and we have $N - 1$ massless Goldstone bosons. We adopt the Lagrangian of Bardeen et al.:

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 \left[ 1 + \frac{1}{4} \lambda \phi^2 \right]^2$$

with $N - 1$ fields $\phi_i$ ($i = 1 \sim N - 1$) transforming under the (apparently broken) $O(N)/O(N - 1)$ symmetry operation $\delta_i$ ($i = 1 \sim N - 1$) as

$$\delta_i \phi_j = \left( 1 - \frac{1}{4} \lambda \phi^2 \right) \delta_{ij} + \frac{1}{2} \lambda \phi_i \phi_j .$$

Then, as the restoration condition for this $O(N)/O(N - 1)$ symmetry, we have

$$\langle \delta_i \phi_j \rangle = \delta_{ij} \left( 1 - \frac{1}{4} \lambda \langle \phi^2 \rangle \right) = 0 ,$$

neglecting the $1/N$ non-leading term $\langle \phi_i \phi_j \rangle$. Note the similarity between the conditions (2·28) and (4·3). The latter is also logarithmically infrared divergent in the perturbative calculation. However, we know by the $1/N$ expansion that (4·3) is actually satisfied* [i.e., equals to the gap equation] and the fields $\phi_i$ become massive in the true vacuum. Similar results are strongly expected for

*) Of course, we know the occurrence of symmetry restoration from Coleman’s theorem11) without explicit calculations.
QCD by replacing the $O(N)/O(N-1)$ symmetry and the fields $\phi_i$ by the "local gauge symmetry" and the gauge fields $A_\mu^a$, respectively. A possible way to prove (2·28) would be to show that it is equal to the "gap equation" of the gauge field system.

These analogies between the non-Abelian gauge theories in 4-dimensions and the non-linear $\sigma$-models in 2-dimensions will become more transparent if we consider the exact residual local gauge transformation in non-Abelian gauge theories mentioned in § 1. This will be explained in a separate paper.\textsuperscript{12}

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**References**

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