tion in an Enclosed Rectangular Cavity," *Mechanical and Chemical Engineering Transactions*, The Institution of Engineers, Australia, vol. MC1, no. 1, 1965, p. 43.

3 C. F. Kettleborough, "Turbulent and Inertia in Slider Bearing," *ASLE Transactions*, September, 1965.

4 E. J. Hahn and C. F. Kettleborough, "Solution for the Pressure and Temperature in an Infinite Slider Bearing of Arbitrary Profile," to be published in the *Journal of Lubrication Technology*.

## APPENDIX

The following list identifies cases referred to in the text.

| Identifying letter | Reynolds no. Re | | | Pr | Eckert no. Rt | | Variable | | |
|---|---|---|---|---|---|---|---|---|---|
| | Inertia terms | Convective terms | Pressure density equation (18) | Prandtl number | Viscous dissipation | Expansion work | $\mu$ | $k$ | $c_p$ |
| Z | 0 | 0 | 1.0 | 0 | 0 | 0 | No | No | No |
| A | 1.0 | 0 | 1.0 | 0 | 0 | 0 | No | No | No |
| B | 1.0 | 1.0 | 1.0 | 0.717 | 0 | 0 | No | No | No |
| C | 1.0 | 1.0 | 1.0 | 0.717 | 0.3934 | 0 | No | No | No |
| D | 1.0 | 1.0 | 1.0 | 0.717 | 0.3934 | 0.3934 | No | No | No |
| E | 1.0 | 1.0 | 1.0 | 0.717 | 0.3934 | 0.3934 | Yes | No | No |
| F | 1.0 | 1.0 | 1.0 | 0.717 | 0.3934 | 0.3934 | Yes | Yes | No |
| G | 1.0 | 1.0 | 1.0 | 0.717 | 0.3934 | 0.3934 | Yes | Yes | Yes |
| H | 1.0 | 0 | 1.0 | 0.717 | 0.3934 | 0.3934 | Yes | Yes | Yes |

Discrimination error 0.2 percent in all the foregoing cases.

G1 As for G in the foregoing but with discrimination error 0.1 percent.

For Case G and G1 $c_p = a + bT$ (equation (7*a*)).

G2 As for G1 but $c = A + BT - CT^2$ (equation (7*b*)).

## DISCUSSION

### W. K. Mueller[2]

The considerable body of computations presented in this paper appears to be of dubious value because two of the four governing equations are incorrect for the specified problem.

As may be verified from a number of sources, including the reference below,[3] equation (3) of the paper, for the compressible fluid considered, is missing a factor of 4/3 in the viscous term. Equation (3) should read

$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \frac{4}{3}\frac{\partial}{\partial y}\left[\mu\frac{\partial v}{\partial y}\right]$$

It can also be verified that the specific heat at constant pressure, $c_p$, is incorrectly placed in equation (4) of the paper. The space derivatives in the convective term of the energy equation do not operate on $c_p$. Additionally, until the perfect gas specification is made, the pressure term in the energy equation has the additional factor shown. Equation (4) should read

$$\rho c_p\left[u\frac{\partial T}{\partial K} + v\frac{\partial T}{\partial y}\right] = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \beta T u\frac{\partial p}{\partial x} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

where $\beta$ is defined by

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\Big)_p$$

It is impossible to estimate the error introduced in the results as a consequence of these incorrect equations without repeating many of the calculations of the author. Considering the avowed purpose of the paper, however, it would seem that such calculations are necessary.

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[3] L. Howarth, *Modern Developments in Fluid Dynamics, High Speed Flow*, vol. 1, Oxford University Press, London, England, 1953, pp. 379, 56.

### T. B. Swearingen[4] and D. M. McEligot[5]

It was a delight to see the author's attack on the laminar forced convective problem *with the inclusion of fluid property variation* for a change. He is to be commended for presenting a powerful technique for the solution of the simultaneous partial differential equations of convective heat transfer. It is hoped that more applications of this method will be available soon. However, on studying the paper a few questions and comments arise which, it is hoped, will help make the paper more useful.

Physically, the problem described seems to be the flow of air through a passage so thin, compared to the length, that the pressure drop causes a much more dominant effect on the density than that caused by rather large temperature variation. Outlet conditions are specified. It is presumed that the application would be to bearing technology, but the discussers do not have sufficient experience in that field to recognize under which conditions the relative movement of the surfaces can be neglected. It probably would be worthwhile to point out the physical analog of each of the cases examined.

Concerning the governing equations: (1) The author's technique of selecting the dimensionless variables to reduce these variables to a common order of magnitude does not necessarily yield first and second derivatives of the same order of magnitude. This assumption may not be reasonable in the hydrodynamic entrance region (or in the thermal entrance region in the case of compressible fluids) since the velocity and temperature gradients can become large and, for example, the terms in the transverse momentum equation, in spite of their small coefficients, can become significant [5].[6] (2) The transverse pressure gradient has been shown to be significant in the entrance regions in a paper by Wang and Longwell [6]. (3) Omission of the viscous momentum transfer in the axial direction prohibits treatment of the phenomenon of choking at the exit of the tube [7]. This is not a serious restriction but is one which should be recognized as the Mach number approaches unity. (4) No boundary condition is listed for the temperature, $\theta$, with respect to $x$.

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[5] Associate Professor, Aerospace and Mechanical Engineering Department, Tucson, Ariz. Mem. ASME.

[6] Numbers in brackets designate Additional References at end of discussion.

While the coarse mesh size used may have inherently large discretization errors, the method presented seems to be one of the few available for economically solving second-order partial differential equations of the parabolic type. However, the large length-to-width ratio of this application in conjunction with the coarse mesh probably tends to mask some interesting behavior. The thermal and hydrodynamic entry regions occur almost entirely within the first increment. Since the exit Mach number is only about 0.4, the viscous dissipation and expansion work terms (with both included) only begin to become important in the last increment (with the possible exception of case III($c$)).

The velocity distribution shown in Fig. 3 do not seem reasonable for the sharp entrance of Fig. 1($a$). Perhaps a preferable interpretation would be that these are the profiles necessary in a smooth passage at a distance of $1000H$ from the exit for the conditions given.

Since much of the paper seems to be devoted to proving the validity of the method, it might be appropriate to compare some of the intermediate results to other methods available in the literature. Case II($b$), constant properties without dissipation or expansion work, might be compared to the work of Han [8] or the annular analyses of Heaton, Reynolds, and Kays [9] (radius ratio of unity gives parallel plates). Case D, constant properties with dissipation, might be compared to the paper of Hwang, Kneiper, and Fan [10].

The following considerations of the conclusions seem warranted:

1 Fluid inertia is probably negligible due to low outlet Mach number rather than low temperature.

2 It has not been shown that dissipation and expansion work terms are significant in comparison to convective terms for the important case of property variation—this would require case G with the Eckert number equal to zero. The comparable check for constant properties, B versus D, shows rather close agreement for heat transfer and no large difference for pressure drop.

3 The results show one should not consider the viscosity variation without the thermal conductivity variation. But they do not show that the converse is true. In fact, in case III($c$) (and perhaps in III($b$), as well) the axial variations of the boundary conditions are so slight, in the context of the long channel, that the temperature variation could almost be considered invariant. In that case the heat flux across the cross section can be considered constant (without the exception of the dissipation effects where the Mach number is high) and the problem reduces to conduction across an "effectively solid" medium with variable thermal conductivity—G and H are approximately the same since $\partial t/\partial x \approx 0$.

After choosing suitable generalized parameters, it would be desirable to use the conditions which seem most meaningful physically, case G, to provide solutions for a range of boundary conditions which might be encountered in practical problems. Perhaps a variety of linear axial temperature gradients and outlet Mach numbers and Reynolds numbers would be a good starting point. Also the use of this method to search for acceptable simplified methods would help the design engineer. For example, Table 2 shows that for long channels with exit Mach number of 0.4, and presumably less, the most complete analysis only gives a 2.3 percent difference from the prediction one would make using the incompressible profile evaluated at the local density.

During the interim since the author submitted this paper, a number of other papers considering the effect of variable properties on laminar flow have appeared. Readers interested in this paper may find references [5] and [11–13] useful. These papers consider "entrance" problems.

#### Additional References


5 P. M. Worsoe-Schmidt and G. Leppert, "Heat Transfer and Friction for Laminar Flow of Gas in a Circular Tube at High Heating Rate. Solutions for Hydrodynamically Developed Flow by a Finite-Difference Method," *International Journal of Heat and Mass Transfer*, vol. 8, Oct. 1965, pp. 1281–1301.

6 Y. L. Wang and P. A. Longwell, "Laminar Flow in the Inlet Section of Parallel Plates," *AIChE Journ.*, vol. 10, 1964, pp. 323–329.

7 P. M. Worsoe-Schmidt, PhD thesis, Stanford University, 1964, p. 11.

8 L. S. Han, "Simultaneous Developments of Temperature and Velocity Profiles in Flat Ducts," *Proceedings, Second International Heat Transfer Conference*, 1961, pp. 591–597.

9 H. S. Heaton, W. C. Reynolds, and W. M. Kays, "Heat Transfer in Annular Passages. Simultaneous Development of Velocity and Temperature Fields in Laminar Flow," *International Journal of Heat and Mass Transfer*, vol. 7, 1964, pp. 763–781.

10 C. L. Hwang, P. J. Kneiper, and L. T. Fan, "Effect of Viscous Dissipation on Heat Transfer Parameters for Flow Between Parallel Plates," *Z. Angew. Math. Phys.*, vol. 16, 1965, pp. 599–610.

11 G. Poots and M. H. Rogers, "Laminar Flow Between Parallel Plates, With Heat Transfer, of Water With Variable Physical Properties," *International Journal of Heat and Mass Transfer*, vol. 8, Dec., 1965, pp. 1515–1535.

12 R. G. Deissler and A. F Presler, "Analysis of Developing Laminar Flow and Heat Transfer in a Tube for a Gas With Variable Properties," *Proceedings, Third International Heat Transfer Conference*, vol. I, 1966, p. 250.

13 P. M. Worsoe-Schmidt, "Heat Transfer and Friction for Laminar Flow of Helium and Carbon Dioxide in a Circular Tube at High Heating Rate," *International Journal of Heat and Mass Transfer*, vol. 9, Nov. 1966, pp. 1291–1295.


## Author's Closure

In reply to Professor Mueller, the author agrees that equation (3) should have a factor 4/3 in front of the last term, but this does not effect the conclusion that $\partial\bar{p}/\partial\bar{y}$ is much smaller than $\partial\bar{p}/\partial\bar{x}$ leading to the usual boundary-layer approximation given in equation (15). The energy equation (4) is identical to equation (14), p. 380, of his reference (footnote 3) with the time-dependent terms removed. With the density relation given in equation (8), $\beta T$ is equal to one. However, the author agrees that the space derivatives in the connective term do not operate on $c_p$. The nondimensional values $C$ should more correctly be placed outside the brackets in equations (14) and (22). The existing results indicate that the inclusion of $c_p$ variation has very little effect (compare F and G). Case G in Table 5—being the case with the largest temperatures—has been recalculated as shown in Table 6. Taking into account that there are small differences between operation with CDC 3200 and IBM 7094 Fortran, line 3 of Table 6 indicates that the effect of $c_p$ variation is even smaller than originally considered. This can be seen also from the fact that the correct term $\left(c_p \dfrac{\partial T}{\partial x_i} = (a + bT)\dfrac{\partial T}{\partial x_i}\right)$ whereas, previously, $\dfrac{\partial}{\partial x_i}(c_pT) = (a + 2bT)\dfrac{\partial T}{\partial x_i}$ had been used.

In reply to Professors Swearinger and McEligol, the author would like to stress that the object of the work was to obtain some idea of the magnitude of the effect of property variations and hence it would not be possible to point out the physical analog of each of the cases examined. Flow of air in thin passages is the basis of hydrostatic bearings which now operate under a large range of temperature conditions.

The governing equations used are those associated with the normal boundary-layer assumptions. Entrance regions effects are not considered, i.e., no restrictions are placed on the temperature and velocity distributions across the section; the converged valves are those consistent with the boundary conditions in the $x$-direction ($u = 0 = v$ on the $x$ boundaries, boundary temperatures are specified at the top of each table). Further work is in progress to consider the effects of fluid property variations on entrance region flow using the boundary-layer equations and the more complete solution of reference [6]. A recent publication[7] shows that there is good correlation between experimental results and theoretical results using the boundary-layer equations. The author agrees with the comment about the coarse mesh size (being a limitation of the computer originally used). However,

[7] Atkinson, B., et al., "Measurement of Velocity Profile in Developing Liquid Flows," *AIChE Journal*, Vol. 13, No. 1, 1967, p. 17.

## Table 6

PRESSURES (NON-DIMENSIONAL)

| | | | | | | | | | | | Nu |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 16.992 | 15.967 | 14.896 | 13.769 | 12.574 | 11.294 | 9.908 | 8.395 | 6.690 | 4.502 | 0 | 1.822 |
| 16.967 | 15.943 | 14.872 | 13.746 | 12.552 | 11.272 | 9.888 | 8.375 | 6.673 | 4.490 | 0 | 1.820 |
| 16.949 | 15.927 | 14.857 | 13.731 | 12.538 | 11.259 | 9.875 | 8.363 | 6.657 | 4.467 | 0 | 1.820 |
| 16.954 | 15.930 | 14.860 | 13.734 | 12.541 | 11.261 | 9.877 | 8.365 | 6.660 | 4.475 | 0 | 1.819 |

COMPUTED VALUES CASE G
TABLE 5

Line 1 - using CDC 3200 Fortran (Commonwealth Scientific and Industrial Research Organization, Sydney, Australia)

Line 2 - using IBM 7094 Fortran (Texas A&M University, College Station, Texas)

Line 3 - using IBM 7094 Fortran with Cp outside the spatial derivatives - i.e. Cp = a + bT

Line 4 - using IBM 7094 Fortran with Cp not a function of temperature - i.e. T = 0 in equation 7a

Line 1 and 2 include Cp inside the spatial derivatives

## Table 7

PRESSURES

| | | | | | | | | | | | Nu |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 17.204 | 16.191 | 15.133 | 14.022 | 12.845 | 11.587 | 10.229 | 8.752 | 7.105 | 4.991 | 0.0 | 1.901 |
| 17.203 | 16.190 | 15.132 | 14.021 | 12.844 | 11.586 | 10.229 | 8.753 | 7.107 | 4.995 | 0.0 | 1.886 |
| 12.420 | 11.674 | 10.892 | 10.068 | 9.194 | 8.256 | 7.241 | 6.126 | 4.866 | 3.256 | 0.0 | 1.817 |

Line 1 Case G, Table 5 with Compression work eliminated

Line 2 Case G, Table 5 with Viscous Dissipation and Compression work eliminated

Line 3 Case F, Table 5 with Constant Viscosity

some recent calculations with a finer mesh indicate that the Nusselt number should be increased by about 5 percent.

With regard to Fig. 3 these are the "fully developed" velocity distributions at entry. For simple Poiseuille flow the velocity distribution given by equation (28) is independent of $x$. Some further cases have been calculated as suggested and are given in Table 7. This table indicates that for the conditions given, dissipation and expansion work terms are not as important as property variation. Table 7 also shows the effect of keeping the viscosity constant but retaining the thermal conductivity variation. It would seem that the pressure distribution is dominated by the viscosity, whereas the Nusselt number is dominated by the thermal conductivity.

The suggestion that solutions be obtained for a range of boundary conditions suitable for design engineers is a good one. It is hoped to complete this after an investigation of other effects already mentioned.

The author would like to thank the discussers for their comments and suggestions and also Prof. H. J. Sneck of Rensselaer Polytechnic Institute who, in a private communication, first pointed out that $c_p$ should be removed from the spatial derivatives.