results will be confirmed. It is interesting to note that equation (16) for the higher frequency (ω₁) is identical with Hahn's equation, equation (1), for the chatter frequency. This is true because λ, the cutting proportionality factor, and kₚ, Hahn's work constant, are equivalent. An explanation for the modulation observed by Hahn and Salje derives from the fact that although one frequency was dominant in its tests giving rise to the variation between the chatter and resonant frequencies, the other must also have existed to some extent. Its existence produced some beating which assured at least a partial modulation of the chatter. This situation also occurred in the General Electric test illustrated in Fig. 1 where the vibrations are only partially modulated. However, as shown in Fig. 5, the General Electric tests run on 8-in. steel pipe displayed a nearly 100 percent modulation. This effect was caused by oscillations of nearly equal strength at the two frequencies ω₁ and ω₂, and for this situation three interesting observations were made which, with some manipulation, can also be explained by equations (16) and (17).

Briefly the observations were:

1. The chatter vibrations consisted of a splitting of the pipe's resonant frequency into two bands equally spaced from the resonant frequency.
2. The two bands followed the decreasing resonant frequency throughout the cut.
3. The modulation rate increased during the cut. In order to explain these results we first assume

\[ \frac{\lambda}{k} \ll 1 \quad (18) \]

Equation (18) allows us to approximate equations (16) and (17) with the two equations

\[ \omega_1 = \omega_0 \left( 1 + \frac{\lambda}{2k} \right) \quad (19) \]
\[ \omega_2 = \omega_0 \left( 1 - \frac{\lambda}{2k} \right) \quad (20) \]

and from equations (19) and (20) we see that ω₁ is greater than ω₀, the resonant frequency, by the value \( \omega_0 \lambda / 2k \) while ω₂ is less than ω₀ by the same amount. Thus, the chatter vibrations (ω₁ and ω₂) did indeed consist of a splitting of the pipe's resonant frequency into two bands which remained equally spaced from and followed the resonant frequency at all times.

The modulation effect can be demonstrated by adding the two solutions for x of equation (3). Therefore, employing the fact that the two solutions were of nearly equal strength and with equations (3), (13), (19), and (20), we can write:

\[ x = e^{i\omega t} \left[ e^{i\omega_1 t} + e^{i\omega_2 t} \right] \]

or

\[ x = 2e^{i\omega t} \cos \left( \frac{\omega_0 \lambda}{2k} \right) t \quad (21) \]

Equation (22) consists of the product of a carrier wave having the frequency ω₀, the pipe's resonant frequency, and a modulation wave of frequency

\[ \omega_m = \frac{\omega_0 \lambda}{2k} \quad (23) \]

Now, from Fig. 5 the carrier frequency was around \( \omega_0 = 2\pi \times 630 \text{ rad/sec} \), while the modulation rate was \( 2\omega_m = 2\pi \times 26 \text{ rad/sec} \), so we can write

\[ 2\omega_m = 2\pi \times 26 = \frac{\omega_0 \lambda}{k} \]

This then justifies equation (18) as a valid assumption, making the results derived from it correct.

Finally, if we replace \( \omega_0 \) in equation (23) by \( \sqrt{k/m} \) we have as the modulation frequency

\[ \omega_m = \frac{\lambda}{2\sqrt{k/m}} \quad (24) \]

and this result clearly accounts for the final observation that the modulation rate increased during the run; this is true because both k, the workpiece stiffness, and m, its effective mass, decreased monotonically throughout the cut while λ, which was a function of the constant cutting parameters, remained unchanged.

In conclusion then, the results of several experimental investigations concerning the nature of machining chatter have been discussed with major emphasis placed on the amplitude modulated behavior of the vibrations. Of the several results considered, each found explanation in the lag differential equation postulated by Doi and Kato for the machining process. The fact that the frequency related phenomena of several independent tests are confirmed by the lag equation, which was not derived from frequency considerations, seems to justify it as a valid mathematical model for the machining process. Although the lag equation was shown to account for the main features of the tests run by the General Electric Company, it gives no explicit explanation for the side bands which were mentioned in regard to the discussion of Fig. 7 and which also occurred to some extent in all of the frequency analysis (Figs. 6 through 9). This inadequacy suggests further work, the results of which would undoubtedly indicate that the side bands are the result of coupling between the workpiece vibrations and the mechanically very complicated machine-foundation system.

**References**


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**DISCUSSION**

Theodore R. Comstock

Based on the experimental observations presented in this paper, the lag equation of Doi and Kato is not justified as a mathematical model. The stability analysis for single point turning processes, as developed by Tobias, and refined by Merritt, will also yield two possible chatter frequencies. Furthermore, the frequency
equation is identical in form to Thompson's equation (15). These statements are verified as follows.

For a structure that can be represented as a single degree of freedom, the equations which form the basis of Merritt's graphical stability analysis are

\[ F_c(t) = -\lambda x(t) + \alpha x(t - T) \]

\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F_c(t) \]

It is assumed, as in the paper under discussion, that the tool-workpiece displacement in the direction of the uncut chip thickness is due only to movement of the workpiece. \( F_c \) is the magnitude of the cutting force and \( T \) is the time required to complete one revolution of the workpiece. The remaining symbols are as defined by Thompson.

These equations can be combined as

\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + (k + \lambda)x - \lambda x(t - T) = 0 \]

Assuming

\[ x = e^{i\omega t} \]

yields the expression;

\[ -\omega^2 + \frac{c}{m} \omega + \frac{(k + \lambda)}{m} - \frac{\lambda}{m} e^{-i\omega t} = 0 \]

Equating the real and imaginary part of the last equation to zero gives

\[ -\omega^2 + \frac{(k + \lambda)}{m} = \frac{\lambda}{m} \cos\omega T \]

\[ \frac{c}{m} \omega = -\frac{\lambda}{m} \sin\omega T \]

The sum of the squares of the last two equations yields a frequency equation of the same form as obtained in the paper under discussion. This is true in spite of the fact that the sign and nature of the lag term in the governing equation is different from Thompson's. Therefore, double valued frequency equations of similar form can be obtained from at least two different mathematical models.

Finally, Thompson has not mentioned the constraint that is placed on the value of the lag, \( h \), before his predicted chatter frequencies can occur. The values of frequency, as determined by Thompson's equation (15), can be substituted into his equation (5) and the corresponding values of \( h \) can be found for \( \alpha = 0 \).

The author agrees with those who have found that the most significant transport delay is associated with regeneration of the chip. Furthermore, the author believes that the analysis developed by Tobias, and extended by Merritt, can not only predict the vibration behavior observed by Thompson, but it provides a more complete understanding of the chatter phenomenon.

R. S. Hahn\(^4\)

The author has presented a very interesting analysis and has concluded that two chatter frequencies should occur, one higher than the natural frequency and one lower. Many other investigators have developed chatter theories which predict only the higher chatter frequency. For example, Merritt,\(^5\) Peters,\(^6\)

\(^6\) Un critère de stabilité dynamique pour machines outils Peters & Vanherck CIRP, Univ. of Louvain Belgium, 1962.

Gurney and Tobias\(^7\) all predict that the threshold of regenerative chatter will occur at a frequency somewhat greater than the uncoupled natural frequency, yet the author, starting from essentially the same differential equation arrives at two frequencies. It would be enlightening if the author could discuss this point. In particular, is something lost when the Laplace transform is taken and feedback control theory applied?

Another point to be made concerns the statement after equation (12), wherein the author states that after a steady state of vibration has been attained \( \alpha = 0 \). This is contrary to linear vibration theory which usually treats only the threshold conditions for which the vibration amplitudes are very small and sets \( \alpha = 0 \) as the condition that a small incipient vibration would neither grow nor die out. As is well known, the steady state amplitude to which an unstable system will approach is determined by nonlinearities of the system. Setting \( \alpha = 0 \) has no relation to determining steady state amplitude.

R. Snoeys\(^8\)

I have read the interesting contribution made by Mr. Thompson related to the modulation phenomenon of chatter vibrations and I would like to suggest some additional points:

1 Self excited chatter vibrations are (for almost 90 percent) a matter of regeneration of undulations on the workpiece surface. Therefore, I would rather suggest that the corresponding equation of the machining process based upon this regenerative effect would be, at least, an equally qualified starting point compared to Dol and Kato's lag equation for discussing the dynamic behavior of the cutting process.

2 If the chatter phenomenon observed during these tests is due to some kind of regenerative effect, the rotational speed may have a great influence. In such a case, a periodic rotational speed variation may be an alternative explanation for the modulation, and therefore, it would also be of interest to know if this rotational motion is not affected by the amplitude of vibrations.

3 Although the assumption of a single degree of freedom system could possibly be proved for an experimental setup similar to the lathe test of Fig. 2, it would be more difficult to justify such an assumption for the experimental setup of Fig. 3.

I am somewhat confused by the decrease of the resonant frequency of the workpiece-machine system as a function of the progression of the cut.

If one assumes that a bending mode of the workpiece has been involved, it seems that the workpiece shape, after the cutting operation, came closer to a beamtype with constant bending resistance and thus yielding a higher value for the natural frequency.

I would suggest trying a relative excitation method for determining more accurately the dynamic response of the workpiece-tool system in order to check how many vibrational modes are really involved in the considered frequency range.

4 It has often been observed that forced vibrations can also modulate the self-excited vibration signals.

This can easily be demonstrated by an eccentrically mounted workpiece. The discussor suggests checking for any other external vibration sources which can contribute to chip thickness variations.

Of course, in a brief contribution similar to this ASME paper, not all these aspects could be discussed profoundly.

However, I hope that this discussion will stimulate further discussion on this topic.

\(^8\) Professor, Katholieke Universiteit te Leuven, Instituut voor Werkstukkunde, Herstel-Leuven, Belgium.
Author's Closure

The author wishes to thank each of the discussers for their interesting reviews and would like to begin his closure by commenting on the very significant discussion of Dr. Hahn.

By pointing out that many theories have been developed which predict only the higher chatter frequency, Dr. Hahn has gone to the root of what appears to be a limitation of current models for machine tool stability. That is, Dr. Hahn questions, "is something lost when the Laplace transform is taken and feedback control theory applied?" With this in mind the author reviewed the chatter theories of Merritt and Tobias and would like to make the following comments. Merrit [5] has made gain phase plots of critical loci. They are based on equation (18) of his paper and are critical loci by definition since he has replaced $a$ in the Laplace transformed equations by $jw$. Fig. 7 of Merritt's paper illustrates one such plot. From this graph it appears that the chatter frequency will always be higher than the uncoupled natural frequency. However, in Fig. 7, Merritt has shown only one branch of the phase plot. For example, the phase angle is given as

$$\angle y/y = \tan^{-1} \frac{\Im (y/y)}{\Re (y/y)},$$

and for each frequency, $f$, there are two angles, namely the one plotted by Merritt and one removed from it by 180 deg. Therefore, to complete the picture, there should be loops to the left of the 180 deg point on the abscissa of Fig. 7 as well as to the right. If these loops were plotted, it would be immediately clear that there would always be two frequencies equally split from the uncoupled resonant frequency at which chatter could occur. This is true even though Merritt's Fig. 7 represents a one-degree-of-freedom system and it accounts for the modulation effect, while also indicating that feedback control theory provides a suitable model for machine tool stability. Consideration of both frequencies is also important to Merritt's theory since the critical spindle speeds defined by him for a lathe are doubled by the inclusion of the lower chatter frequencies.

The comments made here for Merritt's theory also apply to the graphical analysis of Gurney and Tobias [7]. For example, Gurney and Tobias have not plotted the harmonic response loci which should appear in the upper half plane of their Fig. 4. With regard to another point raised by Dr. Hahn, the author agrees that the peak amplitudes occurring during chatter are governed by nonlinearities in the mechanical system. However, the choice $\alpha = 0$ is consistent with Merritt's substitution, $s = jw$ during his search for critical loci and although nothing can be inferred as to their peak amplitudes, the chatter vibrations will nevertheless be amplitude modulated by the beating of the two predicted frequencies. An understanding of the nonlinearities and their effects is an area for useful further work. Work in this direction might also help to explain the side bands which were so pronounced during some of the General Electric tests (Fig. 7).

Turning attention now to the comments of Dr. Snoeys, the author agrees that regenerative chatter would serve as an equally suitable starting point for a frequency analysis and would like to point out that since both regenerative chatter and the primary chatter of Doi and Kato are governed by lag equations, the treatment of both is identical. Dr. Comstock has shown this in his discussion. Furthermore, it is interesting to note that neither the time lag itself nor the algebraic sign of the term in which it appears enter the expressions for frequency. This then implies that the modulation effect under consideration is not directly related to the spindle speed as Dr. Snoeys suggests. However, it was found that in the General Electric tests an integer number of modulation envelopes generally occurred per spindle revolution (I for the Lodge and Shipley tests and 6 for the Motch and Merryweather tests). This would seem to indicate that the quantities $\lambda$, $k$, and $m$ of the author's equation (24) are in some way related to the spindle speed. Understanding this phenomenon represents still another area for further research.

Dr. Snoeys has pointed out, quite appropriately, that forced vibrations such as those resulting from an out-of-round workpiece may also cause an amplitude modulation of the cutting vibrations. The author agrees that this can indeed happen and would emphasize that this modulating effect will split the chatter frequencies in the same manner as the lag effect. That is, it will split the uncoupled resonant frequencies into two bands equally spaced from the resonant frequency. The author strongly feels that determining the relation between these two phenomena would greatly improve the understanding of metal cutting chatter. Again, this represents an area for further work.

Finally, a question which was raised at the Winter Annual Meeting and was also expressed by Dr. Comstock relates to constraints on the quantity, $\omega h$. From equation (6) of the author's paper, if $c$ and $\alpha$ are zero, the condition

$$\sin \omega h = 0$$

or

$$\omega h = n \pi$$

must be satisfied. Since the cutting conditions, i.e., spindle speed, etc., were held constant during the General Electric tests, it seems reasonable that the lag, $h$, remained constant. Therefore, the chatter frequency must have changed through discreet steps. Fig. 10 shows that this did indeed occur. It is a real time spectral analysis of a test similar to the one described in the author's paper. The relative amplitude of a vibration at a given frequency and time is given as darkness of the plot. The relative amplitude jumps as the two bands equally spaced from the resonant frequency. The steps occur at about $7 \text{ sec}$ intervals during the 168 sec cut.

The two darkest bands in the illustration, appearing along with their side bands, represent the two modulating chatter frequencies. Further, in answer to Dr. Snoeys' point, the figure shows the decreasing resonant frequency of the steel pipe workpiece corresponding to the removal of material and its loss of hoop mode stiffness.

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Fig. 10 Real time spectral analysis of a chatter test (note that amplitude is represented as darkness of the plot)