A Modified Strain-Energy Density Criterion Applied to Crack Propagation[1]

G. C. Sih[2] and E. E. Gdoutos[3]. First of all, the discussers wish to thank the authors for their candid view on applying the strain-energy density criterion to the angle crack problem. The selection of a suitable failure criterion to examine material damage caused by fracture and/or yielding is problematic, because the process can often be prejudiced by the investigator(s) on the basis of how well his or their experimental data agree with the theory. Generally speaking, it is not difficult to show that several competing failure criteria can be made to agree equally well with the data of a single physical problem but it becomes much more demanding to have a single criterion that can consistently explain a multitude of physical phenomena. Of equal importance is that approximations introduced through stress analysis should not be attributed to limitations of the failure criterion. This point was discussed in detail in 1974 [1] with reference to fracture experiments on beryllium. Needless to say, better accuracies are obtained when the complete strain-energy density expression is used rather than just the singular terms. Although the choice of number of terms affects the end results, it has no bearing on the original failure criterion. The so referred to "thinking ability" must indeed be left to the investigator.[4] The versatility of any criterion can only be judged by its *consistency* and *generality* in application.

More specifically, this discussion is intended to clarify the basic ideas behind the strain-energy density theory which apparently have escaped the attention of the authors. The concocted modifications outlined in the paper are found to be groundless and serve no useful purpose. Ironically, the authors' criticisms apply quite appropriately to their own work. For instance, the mean strain-energy density factor $\bar{S}$ as defined by equation (8) in the paper can hardly have more physical meaning than the strain-energy density factor S itself. For a linear elastic material, S can be written as [2]

$$S = S_v + S_d \tag{1}$$

in which S_v corresponds to the dilatational component and S_d to the distortional component. The former is assumed to govern fracture while the latter to yielding. More details on this will be given subsequently. In general, S is associated with the strain-energy density function dW/dV by the relation

$$\frac{dW}{dV} = \frac{S}{r} \tag{2}$$

with r being the linear distance locating a possible failure site.[5] Only in the case of a linear elastic material can S be computed from the stress-intensity factors. The expression in equation (2) applies to a typical material element at a distance[6] r_0 from the site of failure initiation, say a crack tip. The angular position of the element, denoted by θ, determines the direction of fracture or yielding. It suffices to use the singular term in the expansion of dW/dV if information is required only on failure initiation [3]. Of course the entire dW/dV field must be considered for determining the crack trajectory [4]. There is no sense to investigate one additional term at a time unless the truncation error is evaluated. The direction of the element that initiates fracture is assumed to correspond with $S_{\min}$ or $(dW/dV)_{\min}$ for a fixed r and the direction of the element that initiates yielding with $S_{\max}$ or $(dW/dV)_{\max}$. In this connection, the hypotheses A and B posed by the authors are inconsistent. One refers to the position of a specific element for which S possesses a relative minimum and the other considers the values of S for all the elements averaged from $\theta = 0$–360 deg. It is inconceivable how $\bar{S}$ could be claimed to have more physical meaning than S. What the authors have failed to recognize is that both $S_{\min}$ and $S_{\max}$ attain different critical values: one for the initation of fracture and the other for yielding. In a given problem, there may exist a number of $S_{\min}$. It is the maximum of $S_{\min}$ or $S_{\min}^{\max}$ where fracture will first initiate. Furthermore, the critical value of S or S_c can be related to K_{1_c} as

$$S_c = \frac{(1+\nu)(1-2\nu)}{2\pi \mathrm{E}} K_{1c}^2 \tag{3}$$

where E is Young's modulus and can be determined by the K_{1c} tests recommended by ASTM. While S_c can be interpreted as the fracture toughness of the material, $\bar{S}$ has no such meaning.

To be emphasized is that the strain-energy criterion as used by the authors in a concocted fashion represents only a special case of the more general theory [5] based on

$$\frac{dW}{dV} = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} \tag{4}$$

In equation (1), dW/dV applies to all materials, either linear [3, 6]) (nondissipative) or nonlinear [7, 8] (dissipative). σ_{ij} and ϵ_{ij} are the stress and strain components referred to the rectangular Cartesian coordinates. It is worthwhile to review the following basic assumptions [5]:

(1) Yielding and fracture are assumed to coincide with *locations* of maximum of the local maximum and minimum of the strain-energy density function $(dW/dV)_{\max}$ and $(dW/dV)_{\min}$, respectively.

(2) Yielding and fracture are assumed to occur when the maximum of $(dW/dV)_{\max}$ and $(dW/dV)_{\min}$ reach their respective critical values.

(3) The amount of incremental growth r_1, r_2, $\ldots, r_j, \ldots, r_c$ is governed by

$$\left(\frac{dW}{dV}\right)_c = \frac{S_1}{r_1} = \frac{S_2}{r_2} = \ldots = \frac{S_j}{r_j} = \ldots = \frac{S_c}{r_c} = \text{const.} \tag{5}$$

if the process of yielding and fracture leads to global instability,[7] i.e.,

$$r_1 < r_2 < \ldots < r_j < \ldots < r_c \tag{6}$$

and r_c corresponds to the critical ligament size of the material.

[1] Theocaris, P. S., and Andrianopoulos, N. P., and published in the March 1982 issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 49, pp. 81–86.

[2] Professor of Mechanics and Director of Institute of Fracture and Solid Mechanics, Lehigh University, Bethlehem, Pa., Fellow ASME.

[3] Professor of Applied Mechanics, Democritus University of Thrace, Xanthi, Greece. Presently: Visiting Scientist at the Institute of Fracture and Solid Mechanics, Lehigh University, Bethlehem, Pa.

[4] Even in solving a quadratic equation involving two roots, the analyst must have the capability of recognizing that only one of them may be physically admissible.

[5] Failure does not necessarily initiate from an existing macrocrack. It can occur anywhere in the solid depending on the conditions assumed by the criterion.

[6] This is a limiting distance within which the influence of material microstructure must be accounted for.

[7] For a process that leads to arrest in yielding and/or fracture, the ratio S_c/r_c in equation (3) is replaced by S_0/r_0 such that $r_1 > r_2 > \ldots > r_j > \ldots > r_0$.

In contrast to the introductory remarks in the paper, the strain-energy density criterion is fundamentally different from the von Mises yield condition as it attempts to address material damage due to the simultaneous influence of yielding and fracture. The proportion of the distortional and dilatational energy component is weighed automatically by the stationary values of dW/dV regardless of whether the crack is in the elastic portion of the elastic-plastic material [8] or in the fully plastic material[8] [9]. The critical values of dW/dV for yielding and fracture are obviously different and *they occur at different locations*. This interpretation is perfectly clear and requires no modification. In fact, in the neighborhood of any point in a stressed solid, there exists a *local* $(dW/dV)_{\max}$ and $(dW/dV)_{\min}$. Their maximum values $(dW/dV)^{\max}_{\max}$ and $(dW/dV)^{\max}_{\min}$ corresponds to locations of yielding and fracture initiation. For the case of a crack in uniform tension, the former occurs at $\theta_{\max} = \cos^{-1}(1-2\nu)$ where ν is the Poisson's ratio and the latter at $\theta = 0$ deg. The important point is that for ductile materials, yielding and fracture have to be addressed simultaneously.[9] The critical values of $(dW/dV)^{\max}_{\max}$ and $(dW/dV)^{\max}_{\min}$ denote the initiation of local yielding and fracture. There exists another pair of *global* stationary values of $(dW/dV)_{\max}$ and $(dW/dV)_{\min}$ whose critical values govern the global instability of the solid or specimen due to yielding and/or fracture. This condition corresponds to

$$\left(\frac{dW}{dV}\right)_c = \frac{S_c}{r_c} \tag{7}$$

where $(dW/dV)_c$ can be measured experimentally from the area under the true stress-strain curve [10]. Note that from equations (3) and (7), r_c can be determined. Hence, for any fracture process that involves crack initiation, slow growth, and termination at least two of the parameters in equation (7) will have to be specified for a given material. This procedure has been applied to a number of problems involving ductile fracture [11, 12].

Contrary to one's physical intuition, the authors' claim that the lowest applied stress for initiating fracture corresponds to $\beta = 72$ deg rather than $\beta = 90$ deg when the load and crack plane is normal to one another. This was based on $\bar{S}$ possessing a *weak* maximum at $\beta = 72$ deg. They attempted to explain this effect by the influence of Mode I and II interaction for which the discussers cannot comprehend. Mode II prevails only because $\beta \neq 90$ deg. With reference to the work of Sih and Kipp (reference [13] in the paper), the $\beta \simeq 70$ deg phenomenon was clearly explained and attributed to the two-term approximation in the stress expression. The S-criterion cannot correct numerical inaccuracies. Sih and Kipp showed that the lowest failure stress indeed occurred at $\beta = 90$ deg when the exact stress expansions were used while no change was made on the S-criterion. This serves as an excellent example of the danger of concocting analysis and forcing the results to agree with unexplained experimental data. Indeed, the experimental data of Williams and Ewing (reference [5] in the paper) exhibited the $\beta \simeq 70$ deg phenomenon. This effect was due to the Mode I and III interaction and not that of Mode I and II as claimed by the authors. In tensile specimens, there is the tendency for the crack to deviate from the plane normal to the specimen surfaces resulting in the additional influence of Mode III. In such a case, indeed, an exact three-dimensional analysis of the embedded flat elliptical crack solution confirmed [2] that the lowest failure stress corresponded to a Mode I and III loading situation rather then Mode I. This has been known in the open literature for some time.

Somewhat disconnected from the main body of the paper, the authors further concluded a paradox in the S-criterion that was concerned with predicting crack bifurcation [13] due to the dynamic effect of running cracks. Reference was also made to the $\beta \simeq 72$ deg phenomenon which, as explained earlier, refers to crack initiating under static loading. These two situations are clearly not the same and should not be confused with one another. In fact, it was shown in 1976 [14] that for $\nu = 0.21$–0.24, the $S_{\min}$ condition did predict the range of half bifurcation angle of ± 18.84–± 15.52 deg. The results for $\nu = 0.25$ was given in [13] and agree well with the S-criterion prediction. Again, it serves only a necessary condition but not sufficient to justify the verification of the criterion.

In conclusion, the discussers failed to see the advantage of the $\bar{S}$ approach which, in fact, tends to confuse the issue and leads to false conclusions. The semilobes represent no more than the graphical display of results and yield no additional information other than the location of $S_{\min}$. The three assumptions stated earlier for dW/dV are sufficiently general to describe the complex behavior of the damage process by fracture and/or yielding provided that the appropriate stress and/or strain analysis is performed.

[8] It has been shown that the condition $(dW/dV)_{\min}$ still governs the direction of crack growth when yielding at large takes place [9]. This result is shown numerically from the finite element solution of a crack in a fully plastic material.

[9] The modifications in the paper failed to recognize the simultaneous treatment of yielding and fracture and provided no improvements of any kind.

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Authors' Closure

The authors wish to thank the discussers for the chance offered to clarify some basic ideas which drove to *one essential modification* of the *S*-criterion, introduced by Sih.

Before that, we have to clarify a few topics of secondary importance raised in the discussion.

1. We never attributed any limitations of the *S*-criterion to numerical inaccuracies introduced by the singular stress expressions. We just used the exact expressions, instead of the one, two-term, etc. approximations, usually used, which sometimes drive to a *"personalization"* of each term and to conclusions of the kind: "The first term affects the crack initiation, the second controls the crack direction . . ." [1]. There are no individual terms in nature. There are only individual stress fields represented algebraically by expressions more or less accurate. Thus, the senior of the discussers should not feel disappointed, if the *singular* predictions differ from the *exact* ones.

2. The versatility of a criterion is judged primarily by its *rationality* and then by its consistency and generality, as stated by the discussers. From a number of existing failure criteria, having the property of rationality, the best will be the criterion that also possesses consistency and generality. In our opinion, the original *S*-criterion seems to lack the property of rationality.

3. *Thinking ability* is asked from the crack, according to the original *S*-criterion. It does not answer the question "which minimum?" as misleadingly claim the discussers, but answers the question "why and how a minimum?" To the former question a good answer was given by Swedlow [2], the answer being independent of the thinking ability of the crack. The second discusser used this answer in at least one of his papers [3].

4. K_{II}-mode does not prevail for $\beta \neq 0$ deg. It just exists for $\beta \neq 0$ deg, 90 deg. It prevails for $\beta < 45$ deg, where all the criteria are (incidentally?) more or less problematic.

5. The discussers propose an experimental method for the determination of r_c. We have been waiting for it since 1974 [4]. Experimental results for the value of r_c will be helpful, if available, to compare them with the radius of the initial curve of the caustics, which was rationally proposed by us as the boundary of the core region [5, 6].

6. One's physical intuition is useful and productive but also dangerous. Intuitively, we agree with the discussers that it seems unphysical for a crack to propagate easier when $\beta \approx 72$ deg than when $\beta = 90$ deg; we were compelled by the results of extensive experiments performed in PMMA, PCBA, and 57-S Aluminum alloy. In all cases, this extremum was always present, stronger in the brittle PMMA and weaker in the other two ductile materials. We do not accept the discussers' explanation that this extremum is due exclusively to the presence of K_{III}. The contribution of K_{III} to the total strain-energy density is independent of angle β and, either *S* does not play a role in the fracture process (a fact that we do not believe), or the new explanation of the discussers is groundless.

7. Concerning the explanation given by Sih and Kipp (reference [13] in our paper), they explained the *theoretical* extremum of fracture load and not the *experimental* as the discusser's claim. Their statement that the extremum is due to the influence of the second term of the stress expressions is answered in our first remark. They surely know that this extremum in fracture-stress was also predicted [7] some years before the introduction of σ_ϑ-criterion [8], although a completely different algebraic description (by means of an asymptotic expansion) of the stress-field was used. However, the situation is somewhat confusing. Predictions of σ_ϑ and $\bar{S}$-criteria show an extremum somewhere around 70 deg, contrary to the *S*-criterion predictions. Experimental evidence is in favor of this extremum and the discussers felt obliged to fight this remarkable coincidence.

8. $\bar{S}_c$ is equally well connected with the toughness K_{Ic} of the material, as is S_c. Integration of equation (8) of the paper for $\beta = 90$ deg, immediately gives:

$$\bar{S}_c = \frac{(1+\nu)(3-4\nu)}{4\pi E} K_{Ic}^2$$

Having finished with the stuffing material, we will try to explain again our basic ideas that resulted in the introduction of the modified or $\bar{S}$-criterion. Let us consider a simple example. A specimen with a crack at $\beta = 90$ deg is loaded uniaxially in tension (Fig. 1). It is assumed that the critical value S_c of the strain-energy density is known. It is also known that $\vartheta_0 = 0$ deg. In the plane of energy-density, S_c is represented by a circle of constant radius S_c. As the external load increases, we consider an instant when the level of *S*-distribution around the crack tip is as shown in the figure. It is a possible situation since $S_{min} < S_c$. The elementary volume *A* ahead of the crack tip, where the crack is expected to propagate, can bear higher strain densities, according to *S*-criterion. But, what happens with the elementary volume at *B*? It, exactly, bears the critical density, but, still, denies to fail. Why? Other elementary volumes, corresponding to arcs *DE* and *FG*, are more stubborn. How can one accept such a behavior? There are two answers. Either the elementary volumes are entities that possess a *thinking ability* and they know that they have to fail only when they are in the "right" direction, or *S*, having an angular character, cannot serve as the critical quantity. At present, we cannot accept the first alternative. On the other hand, the second alternative (that *S* is irrelevant) can hardly be believed. Thus, we have modified the *S*-criterion, replacing S_{min} by the mean value $\bar{S}$ in the role of the decisive quantity for *crack initiation* and keeping S_{min} as the decisive quantity for *crack direction*. This modification removes the fundamental irrationalities of the original *S*-criterion, leaving its predictions unaffected.

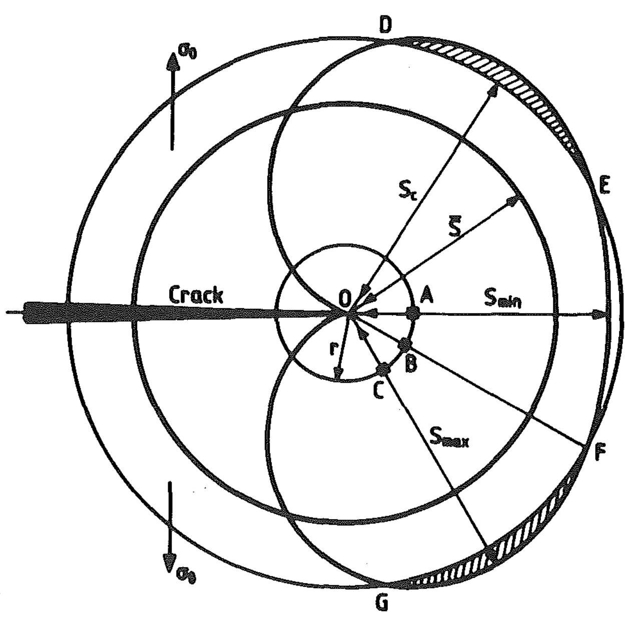

Fig. 1

How does the modified or $\bar{S}$-criterion work? At each load step the mean value of the strain-energy density stored around the crack-tip is geometrically represented by a circle. The crack initiates when the radius of this circle is greater or equal to the radius of the critical S, i.e., $\bar{S} \geqq S_c$. When this relation is fulfilled, the crack propagates to the direction of the minimum energy-density ($S_{\min}$) according to the fundamental laws of mechanics [9].

From a physical point of view, the introduction of $\bar{S}$ implies the necessity of the existence of a low-elastic, strain-energy density level in the neighborhood of the crack tip, which when achieved, permits the initiation of the various fracture and yielding mechanisms.

From the algebraic point of view, S is a positive quantity, increasing with the external load. Thus, S can reach a positive critical value (say S_c) only from below, and this obviously is first reached by the maximum value of S. Therefore, symbols like $S_{\min}^{\max}$ have only a formal value, not interpretable physically.

Let us return, again, to Fig. 1. Concerning the behavior of the elementary volumes corresponding to S-values between DE and FG, the discussers may say that these volumes are at the direction of yielding and thus they do not fracture, being already yielded. But, according to their words ". . . *yielding and fracture have to be addressed simultaneously*" not only for ductile materials, as they say, but for all materials. Perfectly brittle materials do not exist. Simple, the brittle or ductile part of the whole failure character of an individual material predominates more or less in each case. This situation is clearly exemplified in Fig. 1 where, at the given load-level, elementary volume A is still unaffected, B is a little yielded and a little fractured, and C is yielded, according to original S-criterion.

We feel that, exactly, such conclusions are "concocted." The problem asks for a more brave confrontation, where the fundamentally different influence of the two density components S_V and S_D on the failure process must be incorporated. In our opinion, this has already been done by the introduction of a new criterion, the T-criterion [10–12]. According to this criterion the distortional part S_D of the total strain-energy density is responsible for the creation of a yielded zone around the crack tip, as is described by the Mises yield condition, $S_D = \text{const}$. Outside the yielded area, dilational component S_V, being a module of normal stresses, initiates fracture processes like cleavage or hole growth and coalescence, according to modern concepts of fracture mechanics [13–15]. This approximation accurately describes the simultaneous but qualitatively different influence of S_D and S_V to the failure processes, and includes S or $\bar{S}$-criteria as limiting cases for purely brittle materials.

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