Finite Temperature Excitations of the \textit{XYZ} Spin Chain

Masatoshi IMADA

\textit{Institute for Solid State Physics, University of Tokyo, Tokyo 106}

(Received February 20, 1982)

A procedure is presented to obtain finite temperature excitations of the \textit{XYZ} spin chain. Numerical results agree with the analytic expressions derived by Johnson, Krinsky and McCoy at $T=0$. The numerical results also agree with those of the Heisenberg-Ising ring by Johnson at $T=0$. Sample dispersion curves are illustrated for the "scaling region", which is expected to have an equivalence to the massive Thirring model. Temperature dependences are calculated for some types of elementary excitations.

§ 1. Introduction

We start with the one-dimensional anisotropic Heisenberg Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{N} [J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z] \quad (1\cdot1)$$

with a periodic boundary condition. Here $S_i^\mu$ is the $\mu$ component of the spin operator for site $i$. Many authors have applied the Bethe Ansatz formalism to construct the eigenvectors and to determine the energy spectrum of the spin chain. The most comprehensive formulation has been given by Baxter\textsuperscript{2} for the \textit{XYZ} spin chain (1\cdot1). The energy spectrum is obtained from the solutions of a set of transcendental equations, which correspond to the periodic boundary condition in the Bethe Ansatz formalism.

The pioneering work by Yang and Yang\textsuperscript{3} has made it possible to treat finite temperature problems of one-dimensional quantum systems. They have introduced the variational method to obtain the free energy of the one-dimensional Bose gas with repulsive delta function interaction. This method has been applied by Gaudin\textsuperscript{4} to obtain the thermodynamic properties of the Ising-like \textit{XXZ} spin chain (i.e., $J_z=J_\mathcal{Y}$ and $|J_x|<|J_z|$). Takahashi and Suzuki (TS)$^5$ have been successful in treating the thermodynamics of the \textit{XY}-like \textit{XXZ} spin chain (i.e., $J_z=J_\mathcal{Y}$ and $|J_x|>|J_z|$) and further the \textit{XYZ} spin chain. Free energy and low temperature specific heat have been given by solving a set of coupled non-linear integral equations.

On the other hand it is necessary to calculate the excitation energy spectrum, if one wishes to obtain correlation functions. Johnson, Krinsky and McCoy\textsuperscript{6} have calculated the low-lying excitation energies of the \textit{XYZ} Hamiltonian at the
Finite Temperature Excitations of the XYZ Spin Chain

ground state. Yang and Yang\(^9\) have obtained a formalism to obtain elementary excitations at a finite temperature. Finite temperature dispersion curves have been given by Johnson\(^7\) for the Ising-like XXZ spin chain.

The purpose of this paper is to obtain the dispersion curves of the XYZ model as a starting point of the investigation of correlation functions. There are several one-dimensional quantum systems, which are proposed to be equivalent to the XYZ model in some limiting cases. Massive Thirring model\(^6\),\(^8\) and quantum sine-Gordon system\(^10\),\(^11\) are examples. The XYZ spin chain is also useful to investigate these systems.

In § 2 we briefly summarize the analysis of TS for the XYZ spin chain. In § 3 we present a formalism to obtain a finite-temperature dispersion curves of the XYZ model. Numerical results are given in § 4 in several cases. In particular our calculation contains the region \(|J_x| > |J_y| - |J_z|\) in which the equivalence to the massive Thirring model and the quantum sine-Gordon system is expected. Section 5 is devoted to some discussion.

§ 2. Preliminary

When we introduce Baxter’s parametrization for the XYZ model (1·1):

\[
J_z = J_z \cosh(2\xi, l)
\]

and

\[
J_y = J_z \sinh(2\xi, l),
\]

the transcendental equations

\[
\left\{ \frac{H(\xi(x_\alpha + i))}{H(\xi(x_\alpha - i))} \right\}^N = -\exp\left( \frac{2\pi\nu}{Q} \right) \prod_{\beta=1}^{N/2} \left\{ \frac{H(\xi(x_\alpha - x_\beta + 2i))}{H(\xi(x_\alpha - x_\beta - 2i))} \right\}
\]

should be satisfied for \(N/2\) parameters \(x_1, x_2, \ldots, x_{N/2}\), where \(Q = K_{\nu}/\xi', l' = \sqrt{1 - l^2}\) and \(\nu\) is a certain integer. Here \(K_{\nu}\) denotes the complete elliptic integral of the first kind of modulus \(l'\). Jacobi’s theta function \(H(z)\) is expressed by the elliptic theta function as

\[
H(z) = \theta_1(z/(2K_{\nu}); iK_{\nu}/K_{\nu'}).\]

The parameters \(\{x_i\}\) correspond to quasi-momenta in the Bethe Ansatz formalism, and Eq. (2·2) expresses a periodic boundary condition. In the XYZ model, Baxter has shown that the parameters \(\{x_i\}\) should also satisfy the sum rule

\[
\sum_{\alpha=1}^{N/2} \xi x_\alpha = K_{\nu'} \nu' + iK_{\nu},
\]

where \(\nu'\) is a certain integer.
In the thermodynamic limit \( N \to \infty \), a solution \( x_1, \ldots, x_{N/2} \) of Eq. (2.2) are grouped into various lengths of strings. A string is characterized by a common real abscissa, an order \( n \) and a parity \( v \). A string of order \( n \) has \( n \) particles given by

\[
x_{a,n}^{\text{a,k}} = x_{a,n}^n + (n+1-2k)i, \quad k = 1, 2, \ldots, n \quad \text{for } + \text{parity}
\]

and

\[
x_{a,n}^{\text{a,k}} = x_{a,n}^n + (n+1-2k)i + p_0^i, \quad k = 1, 2, \ldots, n \quad \text{for } - \text{parity},
\]

where \( x_{a,n}^n \) is the common real abscissa. If we define the following series of numbers:

\[
p_0 = \frac{K_l}{\zeta}, \quad p_1 = 1, \quad \nu_i = \frac{p_{i-1}/p_i}{p_{i-2} - p_{i-1} \nu_{i-1}},
\]

\[
m_0 = 0, \quad m_i = \sum_{k=1}^{i} \nu_k,
\]

\[
y_{-1} = 0, \quad y_0 = 1, \quad y_1 = \nu_1, \quad y_2 = \nu_1 \nu_2 + 1, \quad y_i = y_{i-2} + \nu_i y_{i-1}
\]

and

\[
n_j = y_{i-1} + (j - m_i)\nu_i \quad \text{for } m_i \leq j \leq m_{i+1}, \quad j = 1, 2, \ldots
\]

TS have assumed that an order \( n \) of a string should be a component of \( n_j \). The parity \( v_j \) of the string of order \( n_j \) is given by

\[
v_1 = 1, \quad v_{m_1} = -1
\]

and

\[
v_j = \exp(\pi i [(n_j - 1)/p_0]) \quad \text{for } j \neq 1, m_1.
\]

This assumption has been rigorously proved in the case of \( XY \)-like \( XXZ \) spin chain. The condition (2.6) turned out to be the normalizability condition for the wave function and further this condition is expected to be reasonable for the \( XYZ \) model. The logarithm of (2.2) has the form

\[
N\ell_j(x\ell_j) = 2\pi I\ell_j + \sum_{k=1}^{M_j} \sum_{\beta=1}^{M_j} \theta_{j,\beta}(x_{\ell,\beta} - x_{\ell,j}), \quad \alpha = 1, \ldots, M_j,
\]

where \( M_j \) is the number of bound states of order \( n_j \). Functions \( \ell_j \) and \( \theta_{j,\beta} \) are defined as follows:

\[
\ell_j(x) = \begin{cases} 0 & \text{for } \frac{n}{p_0} = \text{integer,} \\ i \ln \left[ \frac{H(\frac{\zeta}{2}(x + in_j + \frac{1}{2}(1-v_j)p_0))}{H(\frac{\zeta}{2}(x - in_j + \frac{1}{2}(1-v_j)p_0))} \right] & \text{otherwise,} \end{cases}
\]
Finite Temperature Excitations of the XYZ Spin Chain

\[ \theta_{jk}(x) \equiv f(x; |n_j - n_k|, v_j, v_k) + f(x; n_j + n_k, v_j, v_k) + 2 \sum_{i=1}^{\min(n_j, n_k)-1} f(x; |n_j - n_k|+2i, v_j, v_k) \]

and

\[ f(x; n_j, v_j) = t_j(x). \] (2.9)

In the thermodynamic limit, we introduce distribution functions \( \rho_j(x) \) and \( \rho_j^h(x) \) of particles and holes of bound states of order \( n_j \) at a common real abscissa \( x \). Integral equations for the distribution functions are derived from the derivative of Eq. (2.3):

\[ a_j(x) = (-1)^{r(j)}(\rho_j(x) + \rho_j^h(x)) + \sum_{k=1}^{\infty} T_{jk} \ast \rho_k(x), \] (2.10)

where \( r(j) \) is defined by

\[ m_{r(j)} \leq j < m_{r(j)+1} \]

and

\[ a \ast b = \int_{-\infty}^{\infty} a(x-y)b(y)dy. \]

In Eq. (2.10), \( a_j(x) \) and \( T_{jk} \) denote

\[ a_j(x) = \frac{1}{2\pi} \frac{d}{dx} t_j(x) \]

and

\[ T_{jk}(x) = \frac{1}{2\pi} \frac{d}{dx} \theta_{jk}(x). \]

The energy and the total momentum of the system have the form

\[ E = -\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} A a_j(x) \rho_j(x) dx \]

and

\[ P = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} t_j(x) \rho_j(x) dx, \] (2.11)

where \( A = J_2 \pi \xi^{-1} \sin(2\xi, \ell) \). With the minimization of the free energy, we obtain a set of nonlinear integral equations from (2.10) as

\[ \ln \eta_j = -A a_j/T + \sum_{k=1}^{\infty} (-1)^{r(k)} T_{jk} \ast \ln(1 + \eta_k^{-1}). \] (2.12)

Equation (2.12) is equivalent to the following set of equations:
where
\[ \eta_j = \frac{\rho_j^k}{\rho_j}, \]
\[ s_i(x) = \sum_{j=-\infty}^{\infty} s_i(x - 2jQ), \]
\[ s_i(x) = \frac{1}{4\rho_i} \text{sech} \frac{\pi x}{2\rho_i}, \]
\[ d_i(x) = \sum_{j=-\infty}^{\infty} d_i(x - 2jQ) \]
and
\[ d_i(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{\text{ch}(p_i - p_{i+1} + k)} \frac{\text{ch}(p_{i+1} + k)}{\text{ch}(p_i + k)}. \] 

§ 3. Elementary excitations at a finite temperature

In this section we consider the elementary excitations from the thermal equilibrium state at temperature \( T \). Finite temperature excitations have been investigated by Yang and Yang\(^3\) for the one-dimensional Bose gas with repulsive delta function interaction. Johnson et al.\(^7\),\(^13\) have applied their procedure to the XXZ model. This section is devoted to an application of Johnson’s method to the XYZ model.

An elementary excitation is attained by moving a number of particles from their thermal equilibrium positions to some new ones. When a particle is added to (removed from) the thermal equilibrium state for a given temperature \( T \), quasi-momenta of all other particles should slightly shift so that (2·2) is satisfied. This backflow effect is determined as follows: Let the real part of the quasi-momenta of an added (removed) particles be \( x^k_\alpha \), where \( k \) denotes the particles of the order \( n_k \). The positions \( \{ x^j_\alpha \} \) of particles at equilibrium move to a new set of positions \( \{ x^j'_\alpha \} \) by an addition (removal) of a string with \( x^k_\alpha \). Since both sets

\[ \ln(1 + \eta_0) = -\frac{A\delta(x)}{T}, \]
\[ \ln \eta_j = (1 - 2\delta_{m_{i-j}, j}) s_i \ast \ln(1 + \eta_{j-1}) + s_i \ast \ln(1 + \eta_{j+1}) \]
\[ \ln \eta_j = (1 - 2\delta_{m_{i-j}, j}) s_i \ast \ln(1 + \eta_{j-1}) \]
\[ + d_i \ast \ln(1 + \eta_j) + s_{i+1} \ast \ln(1 + \eta_{j+1}) \]
\[ \text{for } m_{i-1} \leq j \leq m_i - 2, \]
\[ \ln \eta_j = (1 - 2\delta_{m_{i-j}, j}) s_i \ast \ln(1 + \eta_{j-1}) \]
\[ + d_i \ast \ln(1 + \eta_j) + s_{i+1} \ast \ln(1 + \eta_{j+1}) \]
\[ \text{for } j = m_i - 1 \]
and
\[ \lim_{j \to \infty} \frac{\ln \eta_j}{\eta_j} = 0, \] (2·13)
of \(\{x_0^i\}\) and \(\{x_0^f\}\) should satisfy (2.8), we have

\[
N[t_i(x_0^i) - t_i(x_0^f)] = \sum_{m=1}^{M_0} \sum_{\beta=1}^{N_0} [\theta_{jm}(x_0^i - x_0^m) - \theta_{jm}(x_0^i - x_0^m)] + \theta_{jn}(x_0^i - x_0^h) + 2\pi l_i, \tag{3.1}
\]

where \(l_i\) is a certain integer and the lower (upper) sign in the rhs denotes an addition (removal) of a string of order \(k\). Now we define

\[
\chi_j(x_0^i, x_0^h) \equiv N(x_0^i - x_0^j), \tag{3.2}
\]

\[
g_{ij}(x, x_0^h) = (-1)^{r_{ij}} r_j(x) \chi_j(x, x_0^h), \tag{3.3}
\]

and

\[
r_j(x) = \rho_j(x) + \rho_j^h(x), \tag{3.4}
\]

where \(x_0^h\) is the real abscissa of the added (removed) particle. Using (2.9), Eq. (3.1) is transformed into

\[
2\pi g_j(x, x_0^h) = -\sum_{m=1}^{M_0} (-1)^{r_{jm}} \int_{-Q}^{Q} \theta_{jm}(x - x') \left[ 1 + e^{\epsilon_{m}(x')/T} \right]^{-1} g_m(x', x_0^h) dx' + \theta_{jn}(x - x_0^h) + 2\pi l_j, \tag{3.5}
\]

where \(\theta_{jm}(x) = \frac{d}{dx} \theta_{jm}(x)\) and \(e^{\epsilon_{m}(x)/T} = \eta_{jm}(x) = \rho_{jm}(x)/\rho_j(x)\). For simplicity we rewrite Eq. (3.5) in the form

\[
\Gamma^{-1}_g = \tilde{i}, \tag{3.6}
\]

which is an abbreviation of

\[
\sum_{n=1}^{\infty} \int_{-Q}^{Q} \Gamma_{mn}(x, x') g_n(x', x_0^h) dx' = i_m(x, x_0^h), \tag{3.7}
\]

\[
i_m(x, x_0^h) = \frac{1}{2\pi} \theta_{mn}(x - x_0^h) + l_m
\]

and

\[
\Gamma_{mn}(x, x') = \delta_{mn} \delta(x - x') + \frac{1}{2\pi} (-1)^{r_{mn}} \theta_{mn}(x - x')(1 + e^{\epsilon_{m}(x')/T})^{-1}. \tag{3.8}
\]

On the other hand, if we differentiate Eq. (2.12) with respect to \(x\), we obtain

\[
\tilde{n} = \frac{\partial \tilde{e}}{\partial x} = -A \frac{\partial \tilde{a}}{\partial x}. \tag{3.8}
\]

Now we will derive expressions for the energy and the momentum change by
the addition (removal) of a string to (from) the thermal equilibrium state. From (2-11) the energy change has the form

$$\Delta E = -\sum_{j=1}^{\infty} \int_0^1 (1+e^{\epsilon(x)/T})^{-1} g_j(x) \, dx \pm A a_h(x^b) \tag{3-9}$$

Using Eqs. (3-6), (3-8), (2-12) and the identity

$$\Gamma_{ki}(x', x) \equiv (1+e^{\epsilon(x)/T}) \Gamma_{ki}(x, x')(1+e^{\epsilon(x)/T})^{-1}, \tag{3-10}$$

Eq. (3-9) is rewritten in the form

$$\Delta E = \mp T \ln \eta_k(x^b) - \frac{T}{2\pi} (R(Q, x^b) - R(-Q, x^b)), \tag{3-11}$$

On the other hand momentum change is given by

$$\Delta P = \mp t_h(x^b) + \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} dx' \, (\mp \theta_{jk}(x'-x^b) + 2\pi l_j) r_j(x') (1+e^{\epsilon(x)/T})^{-1}. \tag{3-12}$$

With the use of Eq. (2-9), Eq. (3-12) is transformed into

$$\Delta P = \mp t_h(x^b) + \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} dx' \, (\mp \theta_{jk}(x'-x^b) + 2\pi l_j) r_j(x') (1+e^{\epsilon(x)/T})^{-1}. \tag{3-13}$$

In Eq. (3-13), $r_j(x) = \rho_j(x) + \rho_j^h(x)$ is determined from the relation

$$r_j(x) = (-1)^{\alpha(j)} \frac{T^2}{A} \frac{\partial}{\partial T} \ln \eta_j \tag{3-14}.$$

Equation (3-14) is proved in Appendix A.

A real elementary excitation is not given by an addition or a removal of only one particle, because total number of particles should always be $N/2$ in the $XYZ$ model. A real elementary excitation is reached by moving a number of particles to new positions. When $m_1(k)$ strings of the order $k (k=1, 2, \cdots)$ move from the set of the equilibrium positions $\{x^b\}$ to new ones $\{x'^b\}$ of $m_2(l)$ strings of the order $l (l=1, 2, \cdots)$, the energy and the momentum change is given by a superposition of Eqs. (3-11) and (3-13), that is

$$\Delta E_{\text{total}} = \sum_{k=1}^{\infty} m_1(k) \sum_{b=1}^{\infty} \int_0^1 \left[ -T \ln \eta_k(x^b) - \frac{T}{2\pi} (R(Q, x^b) - R(-Q, x^b)) \right]$$

$$+ \sum_{l=1}^{\infty} m_2(l) \sum_{a=1}^{\infty} \int_0^1 \left[ T \ln \eta_l(x'^a) - \frac{T}{2\pi} (R(Q, x'^a) - R(-Q, x'^a)) \right] \tag{3-15}$$

and
Finite Temperature Excitations of the XYZ Spin Chain

\[ \Delta P_{\text{total}} = \sum_{k=1}^{\infty} \sum_{j=1}^{m(k)} \left[ -t_k(x_j^k) + \sum_{j'=1}^{j} \int_{-Q}^{Q} dx' \left( -\theta_{jk}(x' - x_j^k) + 2\pi l_j \right) \right] \times r_j(x')(1 + e^{\epsilon_j(x')/T})^{-1} + \sum_{k=1}^{\infty} \sum_{j=1}^{m(k)} \left[ t_k(x_j^k) + \sum_{j'=1}^{j} \int_{-Q}^{Q} dx' \left( -\theta_{jk}(x' - x_j^k) + 2\pi l_j \right) \right] \times r_j(x')(1 + e^{\epsilon_j(x')/T})^{-1}. \]  (3.16)

Here \( \sum_{k=1}^{\infty} m_1(k)k = \sum_{k=1}^{\infty} m_2(l)l \) should be satisfied since the total number of particles added is the same as those removed.

It is worth while noting that the sum rule (2·3) should be satisfied for each eigenstate. In the thermal equilibrium state, the distribution function \( \rho_j(x) \) and \( \rho_j^k(x) \) obviously have a symmetric property \( \rho_j(x) = \rho_j(-x) \) and \( \rho_j^k(x) = \rho_j^k(-x) \). Therefore Eq. (2·3) is automatically satisfied for the solution of (2·12). We can express the sum rule (2·3) for the elementary excitation as

\[ \sum_{k=1}^{\infty} \sum_{j=1}^{m(k)} \left[ kx_j^k + \sum_{j'=1}^{j} \int_{-Q}^{Q} dx' \chi_j(x', x_j^k)(x) \right] = K \nu + iK \nu, \]  (3.17)

where the notations are the same as in (3·15) and (3·16). In Eq. (3·17), \( \rho_j(x) \) is determined from \( \rho_j(x) = \rho_j(x)/(1 + \eta_j(x)) \) by the use of Eq. (3·14) and the solution of Eq. (2·13). While \( \chi_j(x) \) is obtained from the solution of Eq. (3·5). Equation (3·5) is equivalent to the following set of equations

\[ (-1)^{r(j)}(1 + e^{\epsilon_j(x)/T})\chi_j = s_i \times \left[ ((-1)^{r(j-1)}(1 - 2\delta_{m_i-1,j})(1 + e^{\epsilon_j(x)/T}) - (-1)^{i+1})\chi_{j-1} + \frac{1}{2\pi} \Gamma_{jk}(x - x_j^k) \right] \]

for \( m_{i-1} \leq j \leq m_i - 2 \)

and

\[ (-1)^{r(j)}(1 + e^{\epsilon_j(x)/T})\chi_j = s_i \times \left[ ((-1)^{r(j-1)}(1 - 2\delta_{m_i-1,j})(1 + e^{\epsilon_j(x)/T}) - (-1)^{i+1})\chi_{j-1} + d_i \times \left[ ((-1)^{r(j)}(1 + e^{\epsilon_j(x)/T}) - (-1)^{i+1})\chi_j \right] + s_{i+1} \times \left[ ((-1)^{r(j-1)}(1 + e^{\epsilon_j(x)/T}) - (-1)^{i+1})\chi_{j+1} + \frac{1}{2\pi} \Gamma_{jk}(x - x_j^k) \right] \right] \]

for \( j = m_i - 1, \)  (3.18)
where
\[
\Gamma_j (y) = \theta_j (-Q) - (1 - 2 \delta_{m_i,j}) [s_i \ast \theta_{j-1,\lambda}](-Q)
- [s_i \ast \theta_{j+1,\lambda}](Y) + 2\pi (-1)^{i+1} (\delta_{j-1,\lambda} + \delta_{j+1,\lambda}) \int_Q^y dx \hat{s}_i(x)
\]
for \(m_i-1 \leq j \leq m_i-2\),
\[
\Gamma_j (y) = \theta_j (-Q) - (1 - 2 \delta_{m_i,j}) [s_i \ast \theta_{j-1,\lambda}](Y)
- [d_i \ast \theta_{j,\lambda}](Y) - [s_{i+1} \ast \theta_{j+1,\lambda}](Y)
+ 2\pi (-1)^{i+1} \left[ \delta_{j-1,\lambda} \int_Q^y dx \hat{s}_i(x) + \delta_{j,\lambda} \int_Q^y dx d_i(x) - \delta_{j+1,\lambda} \int_Q^y dx s_{i+1}(x) \right]
\]
for \(j = m_i-1\) \hspace{1cm} (3.19)

and
\[
[a \ast b](x) = \int_Q^x a(y)b(x-y)dy.
\]
Equation (3.18) is derived in Appendix B.

§ 4. Numerical results

A dispersion curve is obtained for an elementary excitation at temperature \(T\) in the following way: i) The thermal equilibrium state is determined from the solution of the coupled nonlinear integral equation (2.13). ii) An elementary excitation is reached by moving a number of particles to new positions. We determine the new positions so that the sum rule (3.17) is satisfied. In Eq. (3.17) \(\chi_j\) is obtained from the solutions of (3.18). iii) Energy and momentum change caused by the excitation is calculated from (3.11) and (3.13). A certain integer \(l_i\) is taken to be zero.

Numerical results are checked with those obtained previously in some limits: In the limit \(T \rightarrow 0\), excitation spectrum is reduced to those obtained by Johnson, Krinsky and McCoy.\(^6\) In the case of Ising-like XXZ spin system, dispersion curves agree well with those obtained by Johnson.\(^7\)

In this paper we will report mainly for the region
\[
|J_y - J_z| \ll |J_x| \quad \text{and} \quad |J_x| < |J_z|.
\]
(4.1)

Luther has pointed out that the massive Thirring model is equivalent to the XYZ model on the limit \(J_y - J_z \rightarrow 0\) in the region (4.1). It is useful to investigate the region (4.1) in order to obtain properties of massive Thirring model since scaling properties are expected\(^8\) in the region (4.1). Coleman has shown that the massive Thirring model is equivalent to quantum sine-Gordon model in a term by
Finite Temperature Excitations of the XYZ Spin Chain

Fig. 1. Dispersion curve for the $n=1$ bound state excitation of a ferromagnetic chain. Parameters are taken at $p_0=5$ and $\ell=0.3$.

term comparison of the perturbation series calculation. Therefore there is also a possibility to obtain a useful information for the quantum sine-Gordon equation.

We investigate two types of excitations as follows.

1) Figure 1 shows finite temperature dispersion curves for the ferromagnetic chain with $p_0=5$ and $\ell=0.3$. These parameters correspond to $J_x/J_z=0.984$, $J_y/J_z=0.80$ and $J_z>0$. In Fig. 1, excitation is created by moving one particle from a string of order $n_0(=1)$ and parity $\nu_b(= -1)$ to a string of order $n_1(=1)$ and parity $\nu_b(=1)$. The string of order $n_b$ has only one particle and locates at $x+p_0i$, where $x$ is real. While the string of order $n_1$ has also one particle located on the real axis. This type of excitation corresponds to an $n=1$ "bound state" excitation.\(^\text{*)}\) The temperature dependence of the gap $\Delta$ is given in Fig. 2. At a low temperature, the gap is almost constant and is proportional to $T$ at a higher temperature.

We also obtain $\ell$ dependence of the gap at $T=0$ in Fig. 3. For $\ell\ll1$, the gap $\Delta$ is given by

$$\Delta \approx 4\pi \frac{\sin \frac{2\nu}{\pi - 2\nu}}{\pi - 2\nu} \left( \frac{\ell}{4} \right)^{\frac{\pi(\pi-2\nu)}{\pi-2\nu}} \sin \left( \frac{\pi \nu}{\pi - 2\nu} \right)$$

at $T=0$.\(^\text{**}\) The massive Thirring model is related to the XYZ model in this scaling region.\(^\text{8)}\) Figure 4 shows that the scaling behavior (4·2) is satisfied even at $\ell=0.3$. Therefore we can expect a behavior similar to Figs. 1 and 2 for the bound state excitation of the massive Thirring model. In quantum sine-Gordon system, this type of excitation corresponds to the quantized soliton-antisoliton bound state.

\(^\text{*)}\) For detailed discussion, see Ref. 6.

\(^\text{**}\) See for example, Eq. (7·12) of Ref. 6.
Fig. 2. Temperature dependence of the gap for the same excitation as in Fig. 1.

Fig. 3. The dependence of the gap on $l$ at $T=0$. The solid curve corresponds to Eq. (4:2).

Fig. 4. Dispersion curve for the free state excitation of an antiferromagnetic chain. Parameters are taken at $p_0=5$ and $l=0.3$.

2) Finite temperature dispersions are illustrated for an antiferromagnetic case in Fig. 4. The parameters are taken also at $p_0=5$ and $l=0.3$ ($J_y/J_x=0.984$ and $J_z/J_x=0.80$), while $J_x$ is negative. Figure 4 is obtained from an excitation, which moves an $n_1$-string to an $n_5$-string. This type of excitation is a component of "free state".$^6$ "Free state" excitation is known to have a two-parameter dispersion,$^6$ and thus give the spin wave continuum. Figure 4 corresponds to the lower boundary of the continuum. The temperature dependence is shown in Fig. 5 for the gap, which turns out to be a monotonically decreasing function of $T$.
In the antiferromagnetic case the gap is approximated by

$$\Delta \approx 2\pi \sin \frac{2\zeta}{\zeta} \left( \frac{1}{4} \right)^{\pi/2\zeta}$$  \hspace{1cm} (4.3)

for the free state excitation at $T=0$. Figure 5 shows that (4.3) is satisfied at $l=0.3$.

§ 5. Summary and discussion

A method is presented to investigate finite temperature excitation of the one-dimensional $XYZ$ spin system. In particular, two cases of excitations have been studied: 1) a bound state excitation of the ferromagnetic case and 2) a free state excitation of the antiferromagnetic case. In both cases parameters are taken to satisfy $|J_y - J_z| \ll |J_z|$ and $|J_x| \ll |J_z|$, in which an equivalence is expected to the massive Thirring model. Temperature dependences of the gaps are shown in Fig. 2 for case 1) and in Fig. 5 for case 2).

It should be noted, however, that these temperature dependences do not simply correspond to the thermal renormalization of an elementary excitation. Correlation functions are not determined only from excitations of type 1) or type 2) at a finite temperature. The dynamic structure factor is expected to have a width composed of a dense distribution of the excitation energy. Nevertheless a starting point for the dynamic correlation functions is presented by the study of finite temperature excitations.

Acknowledgements

The author would like to thank Professor M. Takahashi for instructive discussion and comments.

Appendix A

— Derivation of Eq. (3.14) —

If we differentiate Eq. (2.12) with respect to $T$, we obtain

$$a_j = \frac{T^2}{A} \frac{\partial}{\partial T} \ln \eta_j + \sum_{k=1}^{\infty} (-1)^{\tau(k)} \frac{T^2}{A} T_{j,k} \ast \left( \frac{1}{1 + \eta_k} \frac{\partial}{\partial T} \ln \eta_k \right),$$  \hspace{1cm} (A.1)

while Eq. (2.10) is rewritten as

$$a_j = (-1)^{\tau(j)} r_j + \sum_{k=1}^{\infty} T_{j,k} \ast \left( \frac{1}{1 + \eta_k} r_k \right).$$  \hspace{1cm} (A.2)

Making a comparison between (A.1) and (A.2), we get (3.14).
Appendix B

--- Derivation of Eq. (3.18) ---

Using Eq. (3.7) of TS, Eq. (3.5) is rewritten as

\[
(-1)^{r(jl)}(1 + e^{e^{i/T}})\chi_j - s_i \star \left[ (-1)^{r(j)-1}(1 - 2\delta_{m_i,j})\chi_{j-1} + (-1)^{r(j+1)}(1 + e^{e^{i/T}})\chi_{j+1} \right] \\
= (-1)^{r(j)}s_i \star (\chi_{j-1} + \chi_{j+1}) \\
\mp [\theta_{j,k}(x - x_\theta) - s_i \star ((1 - 2\delta_{m_i,j})\theta_{j-\theta,k}(x - x_\theta) + \theta_{j+\theta,k}(x - x_\theta))].
\]

(B·1)

Substituting the integration of Eq. (3.7) of TS, we obtain Eq. (3.18).

References

1) H. Bethe, Z. Phys. 71 (1931), 205.