Rapidly Rotating Polytropes
and Concave Hamburger Equilibrium

Izumi HACHISU, Yoshiharu ERIGUCHI* and Daiichiro SUGIMOTO*

Department of Aeronautical Engineering
Kyoto University, Kyoto 606
*Department of Earth Science and Astronomy
College of General Education, University of Tokyo
Komaba, Tokyo 153

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Rotating polytropes have equilibrium figures of concave hamburger shape, which bifurcates from Maclaurin-spheroid-like figures and continues into toroids. However, two existing numerical computations of the concave hamburgers are quantitatively in contradiction to each other. Reasons for this contradiction are found to lie in the wrong treatments: One of their methods was applied for deformations too strong to be treated within its limit of applicability so that their boundary condition failed in its convergence of the series and in its analytic continuation into the complex plane.

A modified method of numerical computation is developed which can not only avoid such problems but is still reasonably efficient. With this method we have recomputed sequences of rotating polytropes. We have found the following. When the polytropic index \( N \) is greater than 0.02, the sequence of the Maclaurin-spheroid-like figures terminates by mass shedding from the equator. When \( N < 0.02 \), on the other hand, it continues into a sequence of the concave hamburgers. Contrary to the earlier computation, the Maclaurin spheroids are shown to be the limiting configuration to \( N = 0 \). Some details are also discussed concerning the bifurcation to the concave hamburgers.

§ 1. Introduction

Ninety years ago, Dyson\(^{(1,2)}\) obtained a sequence of toroidal figures for a rigidly rotating, self-gravitating, incompressible fluid by means of perturbation method. Recently Wong\(^{(3)}\) computed numerically a sequence of toroid-like figures, which is an extension of Dyson's sequence to a region with relatively small angular momentum. Hereafter, we will call them the Dyson-Wong toroids (or the Dyson-Wong sequence).

In astrophysical problems containing axisymmetric rotating celestial bodies, many authors quoted Maclaurin spheroids to compare their results with. Therefore, it is important to understand the relation between the Maclaurin sequence and the Dyson-Wong sequence. It is also important to understand the relation between the cases of incompressible and of compressible fluids. For these purposes Fukushima et al.\(^{(4)}\) made numerical computations of hydrostatic equilibrium solutions for rapidly and rigidly rotating polytropes with very small compressibilities and they suggested the following. F1) The Maclaurin spheroid is a special solution and does not represent the limiting case of the rotating poly-
trope to the incompressible fluid. F2) The figure changes from a spheroid-like to a concave hamburger and may connect to or come close to the Dyson-Wong toroid as the angular momentum is increased.

Recently, Eriguchi and Sugimoto computed the case of the incompressible fluid somewhat more analytically. Their results are summarized as follows. ES1) A good parameter that describes the sequence of equilibria is the ratio $f$ of the radius of the equator $a$ to that of the pole $b$ ($f = a/b$). ES2) As $f$ increases, the configuration becomes flatter, and then it becomes a concave hamburger as illustrated in Fig. 1. ES3) When the non-dimensional angular momentum $j$ and velocity $\omega$, which are defined just below, reach the values of $j = 0.1475$ and $\omega^2 = 0.08150$, the radius to the pole vanishes, i.e., $f$ becomes infinity; afterwards the concave hamburger continues to a sequence of ring-like structures which is identical with the Dyson-Wong sequence. Here, $j$ and $\omega$ are defined as

$$j = J / (4\pi G M^{10/3} \rho_c^{-1/3})^{1/2},$$

$$\omega = \Omega / (4\pi G \rho_c)^{1/2},$$

where $J$, $M$, $\rho_c$, $\Omega$ and $G$ are the total angular momentum, the total mass, the mass-density at the center, the angular velocity and the gravitational constant, respectively. Hereafter we shall call these concave hamburgers Eriguchi and Sugimoto (ES) sequence.

Though the results by Fukushima et al. and by Eriguchi and Sugimoto

![Fig. 1](https://example.com/fig1.png)

![Fig. 2](https://example.com/fig2.png)

**Fig. 1.** A typical concave hamburger for the incompressible case ($N = 0$) with $j = 0.14904$, $\omega^2 = 0.08276$ and $f = 10.2$. Views from the direction of $\theta = 70^\circ$ (a) and of $\theta = 90^\circ$ (b).

**Fig. 2.** Squares of the dimensionless angular velocities plotted against the non-dimensional angular momentum for the Maclaurin spheroid (MS; dashed), for the Dyson-Wong toroid (DW; dotted) and for the Eriguchi-Sugimoto sequence (ES; dotted). Also plotted are sequences similar to those computed by Fukushima et al. for $N = 0.1$ (thin solid) and for $N = 0.01$ (thick solid), though they will be proven to be wrong in the present paper.
appear qualitatively consistent with each other, the values of \( j \) and \( \omega^2 \) of the former are very different from those of the latter (see Figs. 2 and 3). In the present paper, we will investigate the reason for these differences and will show that they are ascribed to a wrong treatment of the outer boundary conditions in Fukushima et al.'s work. Then we will make recomputation for Fukushima et al.'s problem by using a revised computational code.

Our new computational method is based on the older one which was originally proposed by Eriguchi and later advanced by Fukushima et al. In this method an elliptical differential equation is transformed into a hyperbolic one after Garabedian by analytically continuing it into the complex plane. Thus, we will refer to it as EFG method.

In § 2, we shall reformulate the EFG method for later discussion. In § 3, we shall criticize the work of Fukushima et al. and the EFG method. In § 4, our new computational method (EFGH method) will be formulated by correcting the treatment for the boundary conditions. Finally in § 5, results of our new numerical computation will be given to show that, contrary to Fukushima et al.'s conclusion F1) above, the Maclaurin sequence is, in reality, the limiting case of the polytropes toward the incompressibility.

§ 2. EFG method

For later discussion we summarize the EFG method after Eriguchi and Fukushima et al. We shall treat the axi- and equatorially symmetric poly-
tropes. For independent variables we take the polar coordinate $(r, \theta)$ in a plane containing the axis of rotation.

If all physical variables are analytic functions of $(r, \theta)$, we can continue the variable $\theta$ into a complex plane as

$$\theta \rightarrow \theta + i\eta,$$

$$\partial / \partial \theta \rightarrow -i \partial / \partial \eta.$$  

Then we treat $r$ and $\eta$ as two new independent variables and $\theta$ as a parameter. Therefore, $\partial^2 / \partial \theta^2$ is replaced with $-\partial^2 / \partial \eta^2$, and equations of hydrostatic equilibrium of the rotating self-gravitating polytropic star become hyperbolic partial differential equations. We can integrate them in the direction of increasing $r$, when the Cauchy data are given in the central region $(r \sim 0)$ of the polytropic star. We shall reach a point where the density of the real axis ($\eta = 0$) vanishes in the first place. We shall refer to the regions inside/outside this point as the interior/exterior. Their boundary will be referred to as surface of the polytropic star, and will be denoted by

$$r = r_e(\theta).$$

2.1. Interior solution

Before the analytic continuation, the basic equations of the hydrostatic equilibrium are given by

$$\frac{\partial y_1}{\partial t} = -2y_1 - \frac{\partial y_2}{\partial \theta} - y_2 \cot \theta - y_3^N e^t,$$

$$\frac{\partial y_2}{\partial t} = -y_2 + \frac{\partial y_1}{\partial \theta},$$

$$\frac{\partial y_3}{\partial t} = (N+1)^{-1} \left( y_1 + \omega^2 e^t \sin^2 \theta \right) e^t,$$

where we use the transformations and normalizations as

$$t = \ln(r/R_0),$$

$$y_1 = -(R_0/\phi_0) \frac{\partial \phi}{\partial r},$$

$$y_2 = -(R_0/\phi_0)(1/r) \frac{\partial \phi}{\partial \theta},$$

$$y_3 = (P/P_c)^{(N+1)},$$

$$\phi_0 = P_c/\rho_c,$$

$$R_0 = [P_c/(4\pi G \rho_c^2)]^{1/2},$$

$$t_e(\theta) = \ln[r_e(\theta)/R_0].$$

Here, $\phi$, $\rho$ and $P$ are the gravitational potential, the mass density and the pressure, respectively. The subscript $c$ denotes the values at the center ($r = 0$;
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Near the center the interior solutions are expanded

\[ \phi = \frac{1}{\phi_0} \sum_{n=1}^{\infty} a_{2n}(\theta) \left( \frac{r}{R_0} \right)^{2n}, \tag{16} \]

\[ \varphi_3 = 1 + \frac{1}{N+1} \left[ -\sum_{n=1}^{\infty} a_{2n}(\theta) \left( \frac{r}{R_0} \right)^{2n} + \frac{1}{2} \left( \frac{r}{R_0} \right)^2 \omega^2 \sin^2 \theta \right]. \tag{17} \]

Such expansion is possible, because the gravitational potential has no singular point near the center. Eriguchi\(^6\) and Fukushima et al.\(^4\) took only the first two terms in the expressions above, i.e.,

\[ a_2 = D_2 P_2(\cos \theta) + \frac{1}{6}, \tag{18} \]

\[ a_4 = D_4 P_4(\cos \theta) - \frac{1}{14} \frac{N}{N+1} \frac{\omega^2}{3} + D_2 P_2(\cos \theta) + \frac{1}{20} \frac{N}{N+1} \frac{\omega^2}{3} \frac{1}{6}, \tag{19} \]

and they neglected the higher order terms. (We may include further terms \( D_6, D_8 \) and so on. See the Appendix for corresponding equations.)

In practice, all relevant equations are analytically continued by Eq. (3). For fixed values of \( \theta \) and \( \omega^2 \) we integrate the analytically continued equations corresponding to Eqs. (6)~(8) in the direction of increasing \( t \) to obtain \( y_k(t, \eta; \theta) \) for \( k=1,2 \) and 3. The initial values of integration at \( r<R_0 \) [i.e., \( t\rightarrow -\infty \), see Eq. (9)] are calculated by means of Eqs. (16)~(19) where the values of \( D_2 \) and \( D_4 \) are assumed in advance. However, the integration thus obtained does not necessarily fit to the exterior solution because the assumed values of \( D_2 \) and \( D_4 \) are, in general, different from their true values. We seek for such true values as eigenvalues by means of appropriate scheme of iteration.

2.2. Exterior solution

The exterior solution must contain information that there is not any mass distributed outside the surface. Eriguchi\(^6\) and Fukushima et al.\(^4\) took the following expressions,

\[ \phi = -\sum_{n=0}^{\infty} I_{2n} P_{2n}(\cos \theta)/r^{2n-1}, \tag{20} \]

where \( I_{2n} \) have to be determined by solving the eigenvalue problem as well as \( D_{2n} \). The method of fitting between the interior and the exterior solution of Fukushima et al.\(^4\) is different from that of Eriguchi.\(^6\)

2.3. Eriguchi's method of fitting

The method of Eriguchi\(^6\) is as follows. For different values of \( \theta \) ranging
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From \( \theta_i = 0 \) to \( \theta_i = \pi/2 \), we obtain the interior solutions \( y_k(t, \eta; \theta_i) \) for \( l = 1, 2, \ldots, L \). Then we fit the interior to the exterior solution on the real axis for each point of the surface, i.e.,

\[
y_k^{(\text{int})}(t_{e, l}, 0; \theta_i) = y_k^{(\text{ext})}(t_{e, l}, 0; \theta_i)
\]

for \( k = 1, 2 \) and \( l = 1, 2, \ldots, L \),

where \( y_k^{(\text{int})} \) and \( y_k^{(\text{ext})} \) denote the interior and the exterior solution, respectively, and where

\[
t_{e, l} = \ln[r_0(\theta_i)/R_0].
\]

The exterior solution \( y_k^{(\text{ext})} \) is calculated from Eqs. (20), (10) and (11). We seek for the values of \( D_2n \) and \( D_4n \) so that Eq. (21) is satisfied. It should be noticed that the boundary conditions are applied here only in the real space (\( \eta = 0 \)). Therefore, the analyticity of the solution is not necessarily required in the whole complex space. The necessary and sufficient condition is that the gravitational potential is continuous to the first order derivative (\( C^1 \)-class) through the real surface.

In this method, however, we need to perform integrations for different values not only of \( D_2 \) and \( D_4 \) but also of \( \theta_i \). It requires much computation.

2.4. Fukushima et al.'s method of fitting

To avoid so much computation, Fukushima et al. fitted the interior solution to the exterior by using the analytic continuation of the boundary condition into the complex plane. Then the fitting is required to be done only at one point with any value of \( \theta = \theta_0 \), i.e.,

\[
y_k^{(\text{int})}(t_{e}(\theta_0), \eta_m; \theta_0) = y_k^{(\text{ext})}(t_{e}(\theta_0), \eta_m; \theta_0)
\]

for \( m = 1, 2, \ldots; k = 1, 2 \) (23)

with which the eigenvalues of \( D_2 \) and \( D_4 \) are determined.

For different values of \( \theta \), the integration with the same values of \( D_2 \) and \( D_4 \) automatically satisfies the condition corresponding to Eq. (23). Therefore, the full solutions are obtained with much smaller amount of computation than those in Eriguchi's method. However, this method is valid only when the \( y_k \)'s are analytic functions throughout the whole space.

\section*{§ 3. Criticism}

\textbf{Criticism 1.} The expansion of Eq. (20) to the infinite series is not convergent on the surface when the equilibrium configuration deforms much from a sphere. We shall show it for the incompressible case, i.e., for the Maclaurin spheroids. The coefficients in Eq. (20) are expressed as
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\[ I_{2n} = \frac{4\pi G\alpha a^2 b}{(2n+1)(2n+3)}(b^2 - a^2)^n. \]  

(24)

The series expansion is convergent only when

\[ \left| \frac{P_{2n+2}(\cos \theta)}{P_2(\cos \theta)} \frac{b^2 - a^2}{r^2} \right| < 1 \]

(25)

for infinitely large \( n \). For the pole this becomes \(|a^2 - b^2|/b^2 < 1\), and the expansion is not convergent when the ratio \( f > 2^{1/2} \).

Nevertheless, Eriguchi's \(^6\) numerical results for the equilibrium models are almost correct, because he computed only the models without such a great deformation. He treated only the cases with appreciable compressibility for which the mass shedding from the equator took place before \( f \) reaching \( 2^{1/2} \).

However, numerical results by Fukushima et al. \(^4\) are incorrect for models with \( f > 2^{1/2} \).

One might imagine that the situation would be better if we used the expansion of \( \phi \) in terms of spheroidal wave functions instead of Eq. (20). Indeed, the Maclaurin spheroid can be described analytically with the first two terms of such expansion. For equilibrium models with further deformations such as the concave hamburgers, however, our numerical experience suggests that the convergence is not appreciably better than the case of Eq. (20).

**Criticism 2.** In Fukushima et al.'s \(^4\) method we cannot use the analytic continuation through the surface. The analytic continuation is possible when and only when there exist derivatives by \( \theta \) of any order. At the surface, however, there exist no higher order derivatives of the gravitational potential \( \phi \) than second order.

This situation is intuitively understood as follows. At the surface of a spherically symmetric non-rotating polytrope, the gradient of the density and thus the third derivative of \( \phi \) are discontinuous to the direction of \( r \). For the rotating deformed polytrope, the derivatives with respect to \( \theta \) cross the corresponding surface; the discontinuity in the direction of \( r \) for the spherical case is transferred into the discontinuity in the direction of \( \theta \) for the deformed case. Therefore, the numerical results obtained by Fukushima et al. \(^4\) are incorrect also in this sense. [As discussed in § 2.3, Criticism 2 does not apply to Eriguchi's method.]

**Criticism 3.** There is a problem also in the inner boundary condition; they did not check the convergence of Eq. (16) either. This series expansion is surely convergent on the real axis (\( \eta = 0 \)), since \( r \) can be chosen to be small enough in the central region. However, the convergence is not guaranteed in the region far from the real axis. For large values of \( \eta \) and small value of \( \exp(t_1) \), Eq. (16) tends to the asymptotic form as
where the subscript 1 denotes the point where the inner boundary condition is imposed. A necessary condition for the convergence is given by

\[ |D_{2n+2}/D_{2n}| < \exp[-2(\eta_1 + t_1)], \tag{27} \]

where \( \eta_1 \) is the maximum value of \( \eta \) in the domain of dependence of our hyperbolic differential equation, i.e., wave equation. Here, \( \eta_1 + t_1 \) is related to the equatorial radius of the surface, i.e.,

\[ \exp(\eta_1 + t_1) = \exp[\ell_e(\theta)]. \tag{28} \]

Therefore, Eq. (27) is reduced to

\[ |D_{2n+2}/D_{2n}| < \left[ \frac{r_e(\theta)}{R_0} \right]^2. \tag{29} \]

Eriguchi\(^6\) computed only the cases with relatively weak deformation where Eq. (29) is satisfied. His results are correct in this sense. However, Fukushima et al.\(^4\) computed even the cases with stronger deformation where Eq. (29) seems to be marginally violated and thus the convergence of Eq. (16) is subject to some doubt.

\section*{§ 4. EFGH method}

If we claim the analyticity only for the interior solution and give it up across the surface as was done by Eriguchi\(^6\), we can avoid Criticism 2 even for Fukushima et al.'s method.\(^9\) On the other hand, Eriguchi's\(^6\) method cannot be applied to highly deformed figures as discussed in Criticism 1. So we need different treatments of the exterior solution and the boundary condition. We shall show in this section that such scheme of computation is possible. The resultant method will be called modified EFG or EFGH method.

\subsection*{4.1. Integral representation of the gravitational potential}

We can automatically include the boundary conditions, if we choose the integral representation for the gravitational potential. Then we need no boundary fittings unlike Eriguchi\(^9\) and Fukushima et al.\(^4\). Now we can be free from Criticism 2. It is given by
\[ \phi = -4\pi G \int d^3 r \frac{\rho(r')}{|r - r'|} \]
\[ = -4\pi G \sum_{n=0}^{\infty} P_{2n}(\cos \theta) \int_0^{\pi/2} \sin \theta' d\theta' P_{2n}(\cos \theta') \]
\[ \times \left[ \int_0^r dr' r'^{2n+2} \rho(r', \theta') + \int_r^\infty dr' r'^{2n-2} \rho(r', \theta') \right]. \quad (30) \]

In this representation the convergence of the series expansion is guaranteed. Therefore we can also be free from Criticism 1, the most important criticism.

4.2. Method of obtaining the interior solution

Hereafter we will refer to the interior solution only for the region \( r < r_e(\theta) \) \([t < t_e(\theta)]\). For the computation of the gravitational potential using Eq. (30), we have to know the density distribution throughout the star in advance. The density distribution can be obtained by assuming the values of \( D_2, D_4 \) and \( \theta_1 \) \((l = 1, 2, \ldots, L)\), and, in case of highly deformed star, \( D_6 \) and \( D_8 \), and then by solving Eqs. (6) \~ (8) as described in § 2. However, this requires a great amount of computation, as mentioned in § 2.3.

In order to reduce the amount of computation we shall make use of the analyticity of the interior solution for the region \( r < r_e(\theta) \). Once choosing some value of \( \theta = \theta_0 \) we can perform numerical integration to obtain the interior solution \( \gamma_k(t, \eta; \theta_0) \) \((k = 1, 2, 3)\). The numerical integration is performed only for one value of \( \theta_0 \). For other values of \( \theta \) we can obtain \( \gamma_k(t, \eta; \theta) \) by utilizing the analyticity of the solution.

It goes as follows. The interior solution is expressed by using Legendre polynomials as

\[ \gamma_1(t, \eta; \theta_0) = \sum_{n=0}^{\infty} F_1^{2n}(t) P_{2n}(\cos(\theta_0 + i\eta)), \]
\[ \gamma_2(t, \eta; \theta_0) = \sum_{n=0}^{\infty} F_2^{2n}(t) \frac{\partial}{\partial \theta} P_{2n}(\cos(\theta_0 + i\eta)), \]
\[ \gamma_3(t, \eta; \theta_0) = \sum_{n=0}^{\infty} F_3^{2n}(t) P_{2n}(\cos(\theta_0 + i\eta)). \quad (31) \]

Here the \( \gamma_k(t, \eta; \theta_0) \)'s are known from the integration for different values of \( \eta \). Thus the linear equation (31) is solved for the coefficients \( F_k^{2n}(t) \). Because of the analyticity of the interior solution, Eq. (31) with the same value of \( F_k^{2n}(t) \) holds for any value of \( \theta_0 \) so far as \( t < t_e(\theta_0) \). If we choose in the direction of the largest radius to the surface, we can obtain the whole interior solution without performing integration for other values of \( \theta \).

The surface values of \( r_e(\theta) \) for other values of \( \theta \) correspond to the first zero point of the density thus obtained, i.e., of \( \gamma_3 \). When we compute the potential by
means of Eq. (30), the integration is performed only in the region \( r < r_e(\theta) \). Therefore, we need not worry about mass densities which may appear outside the surface.

4.3. Equilibrium condition and numerical scheme

The potential thus obtained is, in general, not consistent with \( Y_1 \) and \( Y_2 \) or, in other words, with Eqs. (10) and (11). Therefore we have to seek for correct values of the pre-assumed parameters \( D_2, D_4 \) and \( \omega^2 \). (In practice, we have treated \( \omega^2 \) as an eigenvalue, because we have specified the radius-ratio \( f \) as a model parameter.) The criterion for the consistency is conveniently expressed by the equilibrium condition that the total potential, i.e., the sum of the gravitational and rotational potential, should have the same value everywhere on the surface. It is expressed as

\[
\phi[r_e(\theta), \theta] + \frac{1}{2} \left( \frac{r_e(\theta)}{R_0} \right)^2 \omega^2 \sin^2 \theta = C.
\] (32)

Here \( C \) is a constant. Its value can be known only after the consistent solution is obtained.

The consistency of the potential is searched for by Gauss-Newton method with the criterion that a deviation from the real equilibrium \( Q \) should be minimum. The choice of \( Q \) is somewhat arbitrary except that it should be minimum for the real equilibrium. In our computation we choose

\[
Q = L^{-1} \sum_{t=1}^{L} (C_t - \bar{C})^2 + \left[ r_e(\pi/2)/r_e(0) - f \right]^2,
\] (33)

where the \( C_t \)'s are the values of \( C \) in Eq. (32) for \( \theta = \theta_t \) and \( \bar{C} \) is their mean value. [If we specify a model parameter other than \( f \), we need to replace the last term of Eq. (33) correspondingly.]

Numerical details of our results will be given in the next section. Here we shall discuss them only in relation with a check for the validity of the analyticity of the interior solution. For a set of parameters of the real equilibrium we have computed \( F_l^{2n}(t) \) independently for different values of \( \theta_0 \). The resultant values of \( F_l^{2n}(t) \) have been found to agree with each other within the accuracy of a few percent.

So far in this section we have not referred to Criticism 3. If one wished to be completely free from Criticism 3, one would have to develop much more sophisticated method than our EFGH method. However, even our EFGH method has a wide range of applicability for which Criticism 3 is irrelevant. [Some of Fukushima et al.'s results came to lie somewhat out of such range of applicability, because they were wrong in the sense of Criticisms 1 and 2. When they were corrected, the corresponding solution came into the range of applicability.]
§ 5. Results and discussion

5.1. Limit to incompressibility

We have calculated the models with relatively small values of the polytropic index, i.e., \( N = 0.0, 0.01, 0.02, 0.05, 0.1, 0.2, 0.3, 0.4 \) and \( 0.5 \). For each value of \( N \) we have obtained a sequence by increasing the value of \( f \). As for series expansion, we have taken account of the terms up to \( a_8 \) for Eq. (16) and up to \( P_{16} \) for Eqs. (30) and (31). Results are summarized in Figs. 3~5 and in Table I. Here, \( T \) and \( W \) are the rotational energy and the gravitational energy, respectively.

For the polytropes with \( N > 0.02 \), mass begins to shed from the equator at the points indicated with circles in Fig. 3. Therefore the concave hamburger configuration does not appear. For nearly incompressible cases \((N < 0.02)\), on the contrary, the equilibrium figures become concave hamburgers, which are essentially the same as Eriguchi-Sugimoto sequence. Their details are seen in Fig. 4. The values of the stability indicator \( T / |W| \) in Fig. 5 imply that the concave hamburger is stabler than the Maclaurin spheroid with the same mass.

Table I. Rotating polytropes for different values of the polytropic index.

| \( N \) | \( f \) | \( \omega^2 \) | \( j \) | \( \bar{\rho}/\rho_0 \) | \( T/|W| \) | Shape* |
|---|---|---|---|---|---|---|
| 5.0 | 0.09635 | 0.13725 | 1.0 | 0.33663 | \( S \) |
| 5.8 | 0.09104 | 0.14360 | 1.0 | 0.35126 | \( S \) |
| 6.6 | 0.08763 | 0.14702 | 1.0 | 0.35850 | \( S \) |
| 7.0 | 0.08646 | 0.14799 | 1.0 | 0.36031 | \( C \) |
| 0.0 | 7.8 | 0.08488 | 0.14896 | 1.0 | 0.36168 | \( C \) |
| 8.7 | 0.08379 | 0.14927 | 1.0 | 0.36143 | \( C \) |
| 10.2 | 0.08276 | 0.14904 | 1.0 | 0.35950 | \( C \) |
| 12.2 | 0.08207 | 0.14822 | 1.0 | 0.35612 | \( C \) |
| 14.2 | 0.08172 | 0.14714 | 1.0 | 0.35260 | \( C \) |
| 0.1 | 1.5 | 0.07396 | 0.05732 | 0.88567 | 0.10454 | \( S \) |
| 2.8 | 0.10509 | 0.10385 | 0.88706 | 0.24181 | \( S \) |
| 3.6 | 0.09967 | 0.12113 | 0.88310 | 0.28813 | \( S \) |
| 4.4 | 0.09084 | 0.13519 | 0.86998 | 0.32022 | \( S^* \) |
| 0.2 | 3.5 | 0.09225 | 0.11941 | 0.75546 | 0.27872 | \( S^* \) |
| 0.3 | 1.5 | 0.06505 | 0.05819 | 0.69954 | 0.10447 | \( S \) |
| 2.6 | 0.09093 | 0.10020 | 0.67887 | 0.22620 | \( S \) |
| 2.95 | 0.08929 | 0.10722 | 0.66322 | 0.24418 | \( S^* \) |
| 0.4 | 2.55 | 0.08355 | 0.09730 | 0.58092 | 0.21431 | \( S^* \) |
| 0.5 | 1.5 | 0.05689 | 0.05865 | 0.54684 | 0.10317 | \( S \) |
| 2.0 | 0.07406 | 0.08106 | 0.52604 | 0.16671 | \( S \) |
| 2.3 | 0.07631 | 0.08943 | 0.50845 | 0.18956 | \( S^* \) |

*\( S, C \) and * denote spheroidal-like figure, concave hamburger figure and the point just preceding the mass-shedding.
and the same total angular momentum.

Comparing our result in Fig. 3 with that of Fukushima et al. in Fig. 2, we find that they are greatly different from each other for \( j \geq 0.05 \). From our results we can conclude that the Maclaurin spheroid is the limiting shape of the rapidly rotating polytropes to the incompressibilities in a wide range of the angular momentum so far as the configuration is axisymmetric. This conclusion is essentially different from that by Fukushima et al. \( ^4 \).

5.2. Discussion

In order to see the nature of the concave hamburger more closely, the eigenvalues specifying the interior solutions are plotted in Fig. 6 against the radius ratio \( f \) for the incompressible case (\( N = 0 \)). The concave hamburger bifurcates at the point with \( f = 5.84 \) where the Maclaurin spheroid is neutrally stable against perturbation of the type of \( P_4 \) of the spheroidal coordinate \( \eta \). \( ^8\),\(^{9,10,11} \)
Near \( f = 6.0 \), \( P_4(\cos \theta) \)-type deformation suddenly turns from a gradual to a rapid grow. Then, \( P_6 \)-type and \( P_8 \)-type deformations follow it near \( f = 6.6 \) and \( 7.0 \), respectively. (Their absolute values have no direct significance in Fig. 6, because \( \frac{r_e(\theta = \pi/2) / R_0^{p_n-2}}{D_2} \) is multiplied to \( D_{2n} \). The values of \( D_{2n} \) is of course decreasing, as the value of \( n \) increases.)

In the framework of the linearized stability theory, this \( P_4(\cos \theta) \)-type perturbation may lead to both configurations i.e., one, the concave hamburger and the other, a “Sombrero”-like figure. [“Sombrero” is the pet name given to the galaxy NGC 4594 (M104) in Virgo cluster, which looks like two sombreros joined one another with their bottoms.] Our computational code did not find any solution of “Sombrero”-like figure even when such a type of perturbation was assumed in the initial model of iteration. However, further study is necessary to exclude or realize a possibility for the existence of such figures with finite degree of deformation.

As for differentially rotating polytropes with \( N = 0 \), Bodenheimer and Ostriker\(^{10} \) discussed the following. For a given value of \( N \) we can consider a sequence of equilibrium figures with different values of angular momentum. Along this sequence there is a point with the maximum angular velocity where the figure is spheroid-like. In this sense the behavior of such sequence is represented by the Maclaurin sequence.\(^{11} \)

Many authors have quoted such discussion. In view of our present results, however, such discussion will not always be the case. For \( N < 0.02 \) there exist not only the Maclaurin sequence but also other sequences such as concave hamburgers. One might say that it is so only for a narrow range of \( N \). If we assume a differential rotation with outwardly decreasing angular velocity, however, a core of the star will become a concave hamburger before the mass shedding takes place from its equator of the envelope. Therefore, even for the polytropes with appreciably or much larger values of \( N \), we would have to consider not only the Maclaurin sequence but also the concave hamburger and other sequences as those which represent their equilibrium figures. In this sense more studies will be needed as for the equilibrium figures with differential

Fig. 6. Indicators of deformation from a sphere, \( D_{2n} \frac{r_e(\theta = \pi/2) / R_0^{p_n-2}}{D_2} \), are plotted for \( n = 2, 3 \) and 4 against the axis ratio \( f \). Points where they suddenly turn from gradual to rapid grow are indicated with arrows.
rotation.

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Appendix

—— Further Terms of the Interior Solution ——

Coefficients for the further terms $a_6(\theta)$ and $a_8(\theta)$ are expressed as

\[
a_6(\theta) = D_8 P_6(\cos \theta)
\]

\[
+ \frac{1}{22} \left\{ -\frac{N}{N+1} D_4 + \frac{9}{35} \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} + D_2 \right)^2 \right\} P_4(\cos \theta)
\]

\[
+ \frac{1}{36} \left\{ \frac{N}{14(N+1)} \left( \frac{\omega^2}{3} + D_2 \right) - \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} - \frac{1}{6} \right) \right\} P_2(\cos \theta)
\]

\[
+ \frac{1}{7} \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} + D_2 \right)^2 P_2(\cos \theta)
\]

\[
+ \frac{1}{42} \left\{ \frac{N}{20(N+1)} \left( \frac{\omega^2}{3} - \frac{1}{6} \right) + \frac{1}{2} \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} - \frac{1}{6} \right)^2 \right\}
\]

\[
+ \frac{1}{10} \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} + D_2 \right)^2 \right\}
\]

\[
(A \cdot 1)
\]

\[
a_8(\theta) = D_8 P_8(\cos \theta)
\]

\[
+ \frac{1}{30} \left\{ -\frac{N}{N+1} D_6 + \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} + D_2 \right) D_4 \frac{5}{11} \right\}
\]

\[
- \frac{1}{6} \frac{N(N-1)(N-2)}{(N+1)^3} \left( \frac{\omega^2}{3} + D_2 \right)^2 \frac{18}{77} P_6(\cos \theta)
\]

\[
+ \frac{1}{52} \left\{ \frac{1}{22} \frac{N}{N+1} \left\{ \frac{N}{N+1} D_4 - \frac{9}{35} \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} + D_2 \right)^2 \right\} \right\}
\]

\[
+ \frac{N(N-1)}{(N+1)^2} \left\{ -\left( \frac{\omega^2}{3} - \frac{1}{6} \right) D_4 + \left( \frac{\omega^2}{3} + D_2 \right) D_4 \frac{20}{77} \right\}
\]

\[
- \frac{1}{14} \frac{N}{N+1} \left( \frac{\omega^2}{3} + D_2 \right)^2 \frac{18}{35} \right\}
\]
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\[
+ \frac{1}{6} N(N-1)(N-2)(\frac{\omega^2}{3} + D_2)\left\{ 3\left(\frac{\omega^2}{3} + \frac{1}{6}\right)\frac{\omega^2}{3} + D_2\right\}^{\frac{18}{35}}
\]

\[
- \frac{108}{385}\left(\frac{\omega^2}{3} + D_2\right)^3 \right\} P_1(\cos \theta)
\]

\[
+ \frac{1}{66} \left[ -\frac{1}{36} \frac{N}{N+1} \left( \frac{1}{14} \left( \frac{N}{N+1} \right)^2 \left( \frac{\omega^2}{3} + D_2 \right) \right) \right.
\]

\[
- \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} - \frac{1}{6} \right) \left( \frac{\omega^2}{3} + D_2 \right) + \frac{1}{7} N(N-1) \left( \frac{\omega^2}{3} + D_2 \right)^2
\]

\[
+ \frac{N(N-1)}{(N+1)^2} \left\{ \left( \frac{\omega^2}{3} + D_2 \right) \left( \frac{\omega^2}{6} - \frac{1}{6} \right) + \left( \frac{\omega^2}{3} + D_2 \right) D_1 \frac{2}{7}
\]

\[
- \frac{1}{14} \frac{N}{N+1} \left( \frac{\omega^2}{3} + D_2 \right)^2 \frac{2}{7} + \frac{1}{20} N \left( \frac{\omega^2}{3} - \frac{1}{6} \right) \left( \frac{\omega^2}{3} + D_2 \right)
\]

\[
+ \frac{1}{6} \frac{N(N-1)(N-2)}{(N+1)^3} \left\{ -3 \left( \frac{\omega^2}{3} - \frac{1}{6} \right) \left( \frac{\omega^2}{3} + D_2 \right)
\]

\[
+ 3\left( \frac{\omega^2}{3} - \frac{1}{6} \right) \left( \frac{\omega^2}{3} + D_2 \right) \frac{22}{7} + \left( \frac{\omega^2}{3} + D_2 \right)^3 \frac{3}{7} \right\} P_3(\cos \theta)
\]

\[
+ \frac{1}{72} \left[ -\frac{1}{42} \frac{N}{N+1} \left( \frac{1}{20} \left( \frac{N}{N+1} \right)^2 \left( \frac{\omega^2}{3} - \frac{1}{6} \right) \right) \right.
\]

\[
+ \frac{1}{2} \frac{N(N-1)}{(N+1)^2} \left( \frac{\omega^2}{3} - \frac{1}{6} \right)^2 + \frac{1}{10} N(N-1) \left( \frac{\omega^2}{3} + D_2 \right)^2
\]

\[
+ \frac{N(N-1)}{(N+1)^2} \left\{ \left( \frac{\omega^2}{3} - \frac{1}{6} \right) \left( \frac{\omega^2}{6} - \frac{1}{6} \right) - \frac{1}{14} N \left( \frac{\omega^2}{3} + D_2 \right)^2 \frac{1}{5}
\]

\[
+ \frac{1}{6} \frac{N(N-1)(N-2)}{(N+1)^3} \left\{ \left( \frac{\omega^2}{3} - \frac{1}{6} \right)^3
\]

\[
+ 3\left( \frac{\omega^2}{3} - \frac{1}{6} \right) \left( \frac{\omega^2}{3} + D_2 \right) \frac{2}{5} - \left( \frac{\omega^2}{3} + D_2 \right)^3 \frac{2}{35} \right\} \right].
\]

(A \cdot 2)

References