Global Aspects of the Dissipative Dynamical Systems. I

— Statistical Identification and Fractal Properties of the Lorenz Chaos —

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Several empirical concepts which become practically necessary to characterize the ergodic motion in the dissipative dynamical systems are explained concretely by carrying out with the Lorenz model. Chaotic motions are characterized not only by the geometrical features in the phase space but also by the statistical aspects in the time series of the dynamical variables. These global aspects are understood in terms of the fractal properties appearing in the dimension spectrum of the strange attractor and the Hausdorff dimension of the time series. Furthermore, the internal stiffness and the markovian order of the chaotic orbit are discussed in relation to the time series analysis based on the auto-regressive model. The numerical methods proposed here will be applied to the problem of the chaotic response under the periodic perturbation (in Part II).

§ 1. Introduction

The chaotic motions in the deterministic systems have the sensitive dependence on the initial condition, and that is generally far from completely integrable and unpredictable in the global sense. For the sake of the global understanding of the chaotic orbits, therefore, the statistical identification becomes unavoidable in place of the prediction of the individual orbit irrespectively whether the system is dissipative or conservative. However, it is not possible at all to understand the qualitative difference of the ergodic behavior between the dissipative and conservative dynamical systems only by the statistical prediction of the time series of the dynamical variables. In order to characterize the difference between them, the fractal properties must be taken into account. The concept of 'fractal' was discussed extensively concerning the hydrodynamical turbulence, especially intermittency. In the framework of the dynamical system theory, the fractal property comes to appear in the spatial structure in phase space (the so-called strange attractor) as well as in the temporal behavior of the dynamical variables. There still remain many problems unsolved about the relation of the real turbulence in the distributed systems and the lumped dynamical systems, if the concept of 'fractal' is expected to play an important role in understanding their ergodic behavior on the unified basis. Indeed, the strange attractor is considered to be a premonitory pattern of the stochastic behavior in the continuous system with infinite many degrees of freedom.
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One of the purposes of the present paper is to elucidate the global features of the chaotic flow in the dissipative systems with main concerns to the fractal properties. In Part I, with the Lorenz model we will explain several computational methods that become necessary to characterize the fractal properties and the nonperiodicity of the chaos, and in Part II these methods will be applied in the analysis of the forced Lorenz system.\(^{12}\)

The discussion of the present paper is limited to the Lorenz chaos,

\[
\begin{align*}
\dot{X} &= -\sigma (X - Y), \\
\dot{Y} &= rX - Y - XZ, \\
\dot{Z} &= -bZ + XY.
\end{align*}
\]

(\(\sigma = 10, \ r = 28, \ b = 8/3\)) \hfill (1)

The fractal properties of the Lorenz system come to appear in two manners; one is in the cantor-like structure of the attractor, and another is in the time series of the flip-flop jump process between two kinds of the spiral motions around the right or the left foci (see Fig. 1(a)). The former is characterized by the fractal dimension spectrum of the attractor, and the latter by the Hausdorff dimension of the jump process. The characteristic time scale of the spiral motion is much less than that of the flip-flop process. Taking account of the qualitative difference in these two time scale, the time course of \(x(t)\) is separated into two parts as shown in Fig. 1(b),

\[
x(t) = P(t) + O(t),
\]

(2)

where \(P(t)\) and \(O(t)\) describe the flip-flop jump process and the spirally rotating motion respectively. (Fig. 1(b))

The Lorenz model was studied extensively under the various kinds of interest,\(^{11-37}\) e.g., the estimation of the limit cycle orbit, the bifurcation structure,
orbital instability, strange attractor, intermittency, spectral analysis, invariant measure, statistical mechanical discussion and noise effect. Furthermore, one dimensional realization was discussed from the mathematical viewpoints.\textsuperscript{29)–32)} Among these many works, especially, the following must be referred in the context of the present paper; the diffusion properties of the time series\textsuperscript{35)} and the self-similarity dimension of the Lorenz attractor.\textsuperscript{36)} In § 3 we show that the diffusion property observed by Farmer et al.\textsuperscript{35)} is well explained by the markovian identification of the flip-flop jump process mentioned above, and in § 4 the dimension of the strange attractor is discussed in relation to the covering property in the tangent space, where we are led to a formula that differs from that obtained by Mori.\textsuperscript{36)}

Another purpose of the present paper is to propose the characteristics which describe the flexibility (or stiffness) of the chaotic orbit. The characteristics are closely connected to the markovian order of the chaotic time series. Consider the following auto-regressive model of the chaotic time course $a(t)$,

$$\beta(t) = \int_{\tau_0}^{t} A(t-\tau)\beta(\tau)d\tau + \epsilon(t), \quad (3)$$

where $\beta(t)$ is the statistical predictor of the original time series $a(t)$, $A(t)$ is the memory kernel, and $\epsilon(t)$ is the random noise. $\tau_0$ is called the markovian time of $a(t)$. Here the time averages of $a(t)$ and $\beta(t)$ are assumed to be zero. Two parameters $\tau_0$ and $\epsilon(t)$ must satisfy a certain relation in order that the statistical properties of $a(t)$ are well reproduced by the predictor $\beta(t)$. By the theory of the statistical identification,\textsuperscript{38)} the relation is derived from the minimum criterion of the final prediction error (F.P.E). The relative intensity of the random part $\langle \epsilon^2(t) \rangle / \langle \beta^2(t) \rangle$ is considered to characterize the internal flexibility of the chaotic orbit, that is, the random part compensates the regressive feature of the first term on the r.h.s. of Eq. (3) and keeps the orbit chaotic.

In § 2 the markovian order and the flexibility parameter are discussed in relation to the induction period of the non-linear systems. The spectral density function obtained by the Fast Fourier Transformation is compared with the result of the auto-regressive analysis.\textsuperscript{38),39)} The theoretical reconstruction of the time correlation function is discussed in Appendix B. The numerical scheme for the calculation of the dimension spectrum is given in § 4.

\section{Time series analysis of the Lorenz chaos}

\subsection{Power spectral density and time correlation function}

In order to get the stable estimation of the power spectral density (P.S.D.) from a long but finite record, we need to perform a few important operations to the record beforehand, e.g., averaging and/or windowing. Among the various
methods for the spectral estimation, here, we compare the following two methods; 

**Method 1**: the averaged P.S.D.

A long record of the time series \( a(t) \) (\( 0 \leq t \leq nT \)) is divided into \( n \) subdata; \( a(t) = \sum_{i=1}^{n} a_i(t) \), where \( a_m(T_m - t) = a(T_m - t) \) (\( 0 \leq t \leq T \)), and \( a_m(T_m - t) = 0 \) (otherwise), and \( T_m = mT \). The fourier components \( A_\rho \) of \( a(t) \) is calculated in each subregion (\( t_{m-1} < t < t_m \)) by the F.F.T.,

\[
A_\rho^j = \frac{1}{N} \sum_{k=1}^{N} a_j \left( T_{j-1} + \frac{kT}{N} \right) e^{-i2\pi(\rho k/N)}
\]

and the averaged spectral distribution \( \langle |A_\rho|^2 \rangle \) is given by

\[
\langle |A_\rho|^2 \rangle = \frac{1}{n} \sum_{j=1}^{n} |A_\rho^j|^2
\]

which gives the mean power of each component with the frequency \( f = \rho / T \).

**Method 2**: auto-regressive (A-R) analysis based on the F.P.E. criterion\(^{38}\)

Let us consider the discrete point data \( \{ a(n\Delta t), n=1,2,\cdots,N \} \) of the continuous data, where \( \Delta t \) is the interval of the data sampling, and the average value of \( a(t) \) to be zero; \( \frac{1}{N} \sum_{n=1}^{N} a(n\Delta t) = 0 \). The time correlation function of \( a(t) \) is reproduced by

\[
R(m\Delta t) = \frac{1}{N-m} \sum_{s=1}^{N-m} a((s+m)\Delta t)a(s\Delta t),
\]

\( (m=0,1,2,\cdots) \)

when \( \Delta t \) is taken to be small enough. The auto-regressive representation is given by

\[
a(n\Delta t) = \sum_{m=1}^{J} A(m)a((n-m)\Delta t) + \eta_n.
\]

The first term of the r.h.s. is the systematic part and the second is the non-systematic irregular part. \( J \) is the adjustable parameter which characterizes the markovian order of the A-R model for \( a(t) \). If the value of \( J \) is taken to be large enough and adequately, the non-systematic part \( \eta_n \) is assumed to be a random perturbation. As known in the theory of the system identification, the statistical properties of \( a(n\Delta t) \) are well approximated by the predictor \( \beta(n\Delta t) \) for \( a(n\Delta t) \), which satisfies the following relations,

\[
\beta(n\Delta t) = \sum_{m=1}^{J} A'(m)\beta((n-m)\Delta t) + \varepsilon_n,
\]

if the markovian order \( J \) and the coefficients \( A'(m) \) satisfy the following equations,
Here \( \varepsilon_n \)'s are the white gaussian process with the same variance as \( \eta_n; \langle \varepsilon_n^2 \rangle = \langle \eta_n^2 \rangle = \sigma_n^2 \). The l.h.s. of Eq. (9) is called the final prediction error, which describes the likelihood measure of the predictor for the original process \( a(t) \). The spectral density function \( S(f) \) is also estimated as

\[
S(f) = \Delta t \left| 1 - \sum_{m=1}^{M} A^M(m) \exp[-i2\pi fm\Delta t] \right|^2.
\]

The time correlation function \( K(S\Delta t) \) is given by

\[
K(S\Delta t) = \int_{-\infty}^{\infty} \langle A_\rho \rangle e^{i2\pi sf\Delta t} df
\]

\[
\cong R(S\Delta t).
\]

Figure 2 shows the comparison of the P.S.D. of the Lorenz chaos obtained by both the methods. The averaged P.S.D. has a number of small irregular fine peaks, which are the artifacts originated not only from the discretization of the data sampling but also from the finiteness of the data length. The artifacts will be repaired in some extents if we prolong the record time \( nT \), and then the averaged P.S.D. will converge to a smooth function. On the other hand, we get the smooth P.S.D. function from the latter estimation, which is the most probable asymptotic estimation of the P.S.D. under the conditions that the sampling time \( \Delta t \) and the data length \( N \) are fixed.

The long time behavior of the correlation function is transformed into the low
frequency part of the P.S.D (Figs. 2 and 3). The averaged P.S.D. is less reliable than that of the latter method in the low frequency part. Around the zero frequency \( f < 0.05 \), the P.S.D. is convex just like the Lorenzian spectrum for the brownian motion, but in a little higher frequency regime \( 0.06 < f < 1.0 \), the curvature of the P.S.D. becomes negative, where the P.S.D. is approximately as

\[
P.S.D. \propto f^{-1/2}. \quad \text{(see Fig. 4)}
\]

\[
(13)
\]

(B) **Induction period and stiffness of the Lorenz chaos**

The minimum criterion of the F.P.E. provides some information about the non-periodic behavior under consideration. The most important one of them is the markovian order \( M \) in Eq. (9), which describes the speed of the information loss. Figure 5 shows the variation of the F.P.E. for the various values of \( J \). The F.P.E. is minimum at \( J = M = 5 \sim 8 \), and the values of \( A^M(m) \) are given in Table I. The contribution of \( A^M(m) \) reduces sharply for \( m = 8 \). This suggests that there are two different characteristic time scale in the Lorenz chaos as pointed in § 1. The time constant of the statistical decay process of the A-R model is determined by the distribution of the eigenvalues \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_M) \),

\[
A^M(0) = \sum_{m=1}^{M} A^M(m) \lambda^m \equiv 1.
\]

\[
(14)
\]

Here \( \lambda_i^{-1} \) stands for the dissipation rate of the information by the \( i \)-th internal decay mode immersed in the nonlinear process. However, it is not practical to characterize the statistical aspects of the chaotic motion only by the relaxation
Table I. Auto-regressive model \( \langle x^2(t) \rangle \approx 20.30, \langle y^2(t) \rangle \approx 80.53, \Delta t = 0.1 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A^m(m) )</td>
<td>1.378</td>
<td>-1.060</td>
<td>0.694</td>
<td>-0.403</td>
<td>0.265</td>
<td>-0.156</td>
<td>0.100</td>
<td>-0.001</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A^{10}(m) )</td>
<td>1.377</td>
<td>-1.062</td>
<td>0.695</td>
<td>-0.406</td>
<td>0.268</td>
<td>-0.164</td>
<td>0.110</td>
<td>-0.008</td>
<td>0.025</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A^{12}(m) )</td>
<td>1.377</td>
<td>-1.062</td>
<td>0.695</td>
<td>-0.406</td>
<td>0.267</td>
<td>-0.162</td>
<td>0.107</td>
<td>-0.007</td>
<td>0.014</td>
<td>0.034</td>
<td>-0.016</td>
<td>0.015</td>
</tr>
</tbody>
</table>

time of the internal mode. For the sake of the better understanding of the nonlinear relaxation, the markovian time \( M \cdot \Delta t \) may take the place of the linear relaxation time. In general, the qualitative behavior of the memory kernel \( A^m(m) \) changes sharply around the markovian time as shown in Table I. Such qualitative jump is considered to be a kind of the induction phenomena in statistical sense, therefore the markovian time may be called 'induction period'.

The stiffness parameter discussed in § 1 is closely connected to the markovian order, to say more precisely, the stiffness of the non-periodic time course \( y(t) \) is defined by

\[ \langle y^2(t) \rangle / \langle \sigma y^2 \rangle \approx 3.97 \text{. (for the Lorenz chaos)} \]  

In the present paper, the discussion is limited in the one-dimensional A-R model, but the stiffness parameter of the chaotic orbit is defined in the same manner for the high-dimensional A-R model which will be discussed elsewhere.

§ 3. Recurrence time distribution at the level crossing point

(A) Realization of the symbolic dynamics for the Lorenz chaos

Let us denote the zero-crossing time of the \( x \)-variable by \( t_i \) \((i = 1, 2, \cdots; t_1 < t_2 < \cdots)\). Upward and downward crossing occurs at \( t = t_{2n-1} \) and \( t = t_{2n} \) alternatively. The recurrence time or the first passage time \( T_i \) is defined by

\[ T_i = t_{i+1} - t_i \text{. } (i = 1, 2, \cdots) \]  

The return map \( g_T \) for the recurrence time is defined as follows:

\[ g_T : T_i \rightarrow T_{i+1} \text{.} \]  

*) The significance of the induction period in the nonlinear systems is discussed in relation to the ergodic problems of the lattice vibration.
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Figure 6 is the numerical result of $g_T$ for the Lorenz chaos. The mapping has a two dimensional lattice structure on the whole. Therefore, the recurrence time can be discretized by the appropriate quantization of time. Indeed, the recurrence time $T_i$ is uniquely transformed into the integer $n_i$, where $n_i$ denotes the rotation number of $O(t)$ during the time interval $T_i$ (see Fig. 1 and Eq. (2)). The transformation $h: T_i \rightarrow n_i$ is a kind of coarse-graining which enables us to rewrite Eq. (17) into

$$g_n : n_i \rightarrow n_{i+1} .$$  \hfill(18)

If we use the discrete time $n$ measured by the rotation number, and the state is symbolized by $+P$ (or $-P$) for the right (or left) spiral motion, the flip-flop process $P(t)$ in Eq. (2) is described by the following symbolic dynamics $\bar{g}$.

$$\bar{g} : P_{n-1} \rightarrow P_n ,$$  \hfill(19)

where

$$P_n = \begin{cases} P, & \text{(for } x(t) > 0) \\ -P, & \text{(for } x(t) < 0) \end{cases}$$

Both dynamics $g_n$ and $\bar{g}$ in Eqs. (18) and (19) are the symbolic realization of the Lorenz chaos. The former expression is useful for the analysis of the intermittent aspect,\textsuperscript{12} but in the present paper the latter realization is mainly used for the convenient sake.

(B) Markovian property of the Lorenz chaos

Here the discussion is concentrated to the flip-flop process described by $P_n$. Let us divide a long data record of $P_n$ ($0 < n \leq m \cdot N$) into $N$ subsets $I_i$ ($i=1, 2, \cdots, N$) with each interval $m$. When the number of the jump event is $j_i^m$ in the subset $I_i$, the probability density of the jump number $j$ in the interval $m$ is defined by

$$P(j, m) = \frac{1}{N} \sum_{i=1}^{N} \delta_{j,j_i^m} ,$$  \hfill(20)

where $\delta_{i,j}$ is Kronecker's delta. The distribution of the recurrence time $P(m)$ is calculated by
Fig. 7. Distribution of the recurrence time. Fig. 8. Distribution of the flip-flop jump.

\[ P(m) = \frac{1}{N_0 - 1} \sum_{i=1}^{N_0-1} \delta_{m,n_i}, \]  

(21)

where \( N_0 \) is the total number of the zero-crossing points during \( 0 < n < m \cdot N \), and \( n_i \) is the \( i \)-th recurrence time as given in Eq. (18). If we consider \( s \) independent processes with the different initial condition, these expressions are refined by their averages.

Figures 7 and 8 show these distribution function for the Lorenz chaos defined by Eqs. (20) and (21). \( (s=3, N \cong 3 \times 10^2, m=10) \). Each numerical plot is well approximated by the exponential or the poissonian distribution respectively,

\[ P(m) = \bar{m}(1 - \bar{m})^{m-1}, \]  

(22)

\[ P(m, N) = \frac{N!}{m!(N-m)!} (\bar{m})^m (1 - \bar{m})^{N-m}, \]  

(23)

where the value of the mean recurrence coefficient \( \bar{m} \) is well adjustable by \( \bar{m} \approx 0.44 \) in both cases, which is shown by the solid line in each figure. In other words, the Lorenz chaos is well identified by the poissonian process with the transition probability \( p_{++} \) and \( p_{+-} \),

\[ p_{+-} = p_{-+} = \bar{m} = 0.44, \]

\[ p_{++} = p_{--} = 1 - \bar{m} = 0.56. \]  

(24)

Here we have taken account of the symmetry of the Lorenz chaos, i.e., the stationary distribution \( p_+ = p_- = 1/2 \). The correlation function of the present symbolic dynamics is given by \( (1 - 2\bar{m})^N (N=\text{integer}) \). This is well in line with the estimation of the markovian time based on the A-R model in § 2.

(C) Entropy, fractal dimension and diffusion property

The randomness of the chaotic time series is characterized by the Kolmogorov-Sinai entropy \( H_{KS} \) and the Hausdorff dimension \( D_h \), which are ex-
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pressed as follows for the markovian case,$^{42}$

\[
H_{KS} = -(p_{++} \ln p_{++} + p_{+-} \ln p_{+-}), \quad (p_+ = 1/2)
\]

By use of Eq. (24) for the Lorenz chaos they become

\[
H_{KS} = 0.685 \quad \text{and} \quad D_H = 0.989 \ .
\]

The Hausdorff dimension \(D_H\) is the relative estimate of the entropy \(H_{KS}\), and that has the following special information about the self-similarity of the ergodic motion; when we consider the infinite dyadic sequence \(\omega_n (P_n = \omega_n; \; \omega_n = 0 \text{ or } 1, \; n = 1, 2, \ldots)\), \(\omega_1 \omega_2 \omega_3 \cdots\) defines a certain real number \(\beta_1\) \((0 < \beta_1 < 1)\) in dyadic expansion. The time shift operator \(\epsilon^{\beta_n}\) \(: P_n \rightarrow P_{n+1}\), defines the following transformation from \(\beta_1\) to \(\beta_2\), namely a mapping of unit interval,

\[
\epsilon^{\beta_n} : \beta_i \rightarrow \beta_{i+1}. \quad (28)
\]

In general, the invariant set of this transformation has the self-similar structure in the unit interval, and that its measure is singular to the Lebesgue measure. Such ergodic limit set \((Q)\) is characterized by the Hausdorff dimension \(D_H\) of the corresponding markovian process of \(P_n (n = 1, 2, 3, \ldots)\).$^{42}$

Figure 9 shows the Hausdorff dimension for the simple markovian case where the process is symmetric; \(p_+ = p_- = 1/2, \; (p_{+-} = X)\). At \(X = 1/2\), the measure of the invariant set \(Q\) is absolutely continuous and uniform (same as for the Bernoulli shift), and the dimension is unity. By reducing the value of \(X\) from the middle \((X = 1/2)\), the dimension decreases monotonically to zero, and near \(X = 0\) the periodic aspect comes to appear continuously and the similarity level (or the uniformity) of the set becomes quite low. The same situation appears at \(X = 1\), where the apparent period is 2. These kinds of phenomena originated from the reduction of the self-similarity level are closely correlated to the intermittency. For the present case \(r = 28\), however, the uniformity of the distribution is relatively high and the deviation from the Bernoulli shift is very small.$^{54}$

From the poissonian property of the Lorenz chaos, the following diffusive relations are easily derived,

\[
\sqrt{\langle \Delta n^2 \rangle_{n}} \sim \sqrt{N}, \quad (29)
\]
where \( \langle \Delta n^2 \rangle_N \) and \( \langle \Delta N^2 \rangle_n \) are the variances of the jumping number \( n \) during the time \( N \) and of the time interval \( N \) that is necessary for the \( n \) jumps. Denoting the number of the maximum point in \( x(t) \) \( (t < N) \) by \( m \), the variance \( \langle \Delta m^2 \rangle \) becomes

\[
\sqrt{\langle \Delta m^2 \rangle} \sim N^{\nu} \quad (\mu^* = 0.5)
\]

by use of the trivial relation, \( m = N - n/2 \) (Fig. 1). The recent experiment is supporting the fact that this relation is realized in the Lorenz chaos, i.e., \( \mu^* = 0.497 \).35

§ 4. Dimension spectrum of the strange attractor

(A) Orbital instability43-46)

The invariant set \( A \) under the contracting map is called attractor in this paper, if almost all orbits in the neighborhood of \( A \) approach \( A \) itself asymptotically. When the orbit on such invariant set is non-periodic, i.e., neither fixed point nor (quasi-) periodic point, the set \( A \) is called strange attractor. More precisely speaking, the strange attractor must have the following property: almost all nearby orbits on the strange attractor unfolds exponentially in their time course.

\[
\frac{1}{T} \ln \frac{|x_2^T - x_1^T|}{|x_0^T - x_1^T|} \sim \lambda > 0 . \tag{32}
\]

Here \( x_i^t \) \( (i = 1, 2) \) is the phase point at time \( t \), \( x_i^0 \) is the initial point, and \( |r| \) is the Euclidian distance of the vector \( r \). The l.h.s. of Eq. (32) is called the Lyapounov characteristic exponent of the dynamical system, which stands for the separation rate of the orbital distance. When the geometrical dimension of the dynamical system is \( D \) (\( D = 3 \) for the Lorenz system), in general, the expanding rate of the \( D' \)-dimensional volume \( v_0 \) in the tangent space of the corresponding dynamical system is defined by

\[
\lambda_{D'} = \lim_{T \to \infty} \frac{1}{T} \ln \frac{|df^T v_0|}{|v_0|} . \quad (D' = 1, 2, 3) \tag{33}
\]

Here, \( v_0 \) and \( df^T \) are the initial \( D' \)-dimensional volume and the time shift operator in the tangent vector space.21,50-53) For the special case \( D' = 1 \), Eq. (33) is reduced to the expression of the characteristic exponent defined by Eq. (32).

(B) Averaged Lyapounov spectrum

As proved by Oseledec,50) the tangent vector space is decomposed into the
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The direct sum space $\xi^1 \oplus \xi^2 \oplus \cdots \oplus \xi^D$, where the index $x$ denotes the phase point where the tangent space is defined. When the maximum value of the orbital separation rate is denoted by $K_n$ in each space $\xi^n_x$, the volume expanding rate $\lambda_n$ is rewritten as

$$\lambda_n = \sum_{n=1}^{D'} K_n$$

or

$$K_j = \lambda_j - \lambda_{j-1} \quad (j = 1, 2, \cdots, D, \text{ and } \lambda_0 = 0)$$  \hspace{1cm} (34)

for almost all initial volume $u_{D'}$. Here and in what follows $K_1 \geq K_2 \geq \cdots \geq K_D$ is assumed. $\{K_i; i = 1, 2, \cdots, D\}$ is simply called the Lyapounov spectrum in this paper. In general, $\lambda_{D'}$ and $K_i$ are the function of the phase point $x$. Let $\mu$ be the invariant measure $^44)\sim^46)$ satisfying $\mu(f^* A) = \mu(A)$, then the averaged characteristic spectrum is defined as $\bar{\lambda}_D' = \int_{D'} \lambda_{D'} d\mu$ and $\bar{K}_i = \int K_i d\mu$.

In spite of the simplicity, Eq. (33) is not so useful for the numerical estimation of the Lyapounov spectrum by computer, since the convergence of the value $\ln|df^T v_0|$ is quite unstable when $df^T v_0$ goes to zero for large $T$. In order to get the stable estimation of $\{K_i\}$ by computer, we have to perform several preliminary operations, i.e., averaging and orthonormalization; we consider $N$-sample processes $x_{i}^t (i = 1, 2, \cdots, N)$ of the constant period $0 < t < T$ with the different initial condition $x_0^n$. The characteristic exponent $\lambda_{j,x_0}^n$ is calculated by

$$\lambda_{j,x_0}^n = \frac{1}{T} \ln \left| \frac{df^T v_0^n}{v_0^n} \right|, \quad (n = 1, 2, \cdots, N)$$  \hspace{1cm} (35)

where $v_0^n$ and $x_0^n$ are the $j$-dimensional initial volume element ($v_0^n = 1$) and the initial phase point of the $n$-th process. If $x_0^n (n = 1, 2, \cdots, N)$ are distributed in the attractor with the invariant measure $\mu$ of the dynamical system, Eq. (34) is well approximated by

$$\bar{\lambda}_{D'} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_{D',x_0}^n,$$  \hspace{1cm} (34-2)

so long as $T$ is large enough but is not too large to violate the stable estimation of Eq. (35) by computer. In order to pick up the invariant ensemble of $x_0^n$, the following orthonormalization is used,

$$x_0^n = f^{nT} \cdot x_0$$

and

$$v^n = e_1^n \wedge e_2^n \wedge \cdots \wedge e_3^n.$$  \hspace{1cm} (36)

Here, $f^{nT}$ denotes stroboscopic mapping and $e_3^n$ is the unit vector orthogonal to
which is defined by

\[ e_k^n = \frac{d^f e_k^{n-1} - \sum_{j=1}^{k-1} (e_j^n \cdot d^f e_k^{n-1}) e_j^n}{|d^f e_k^{n-1} - \sum_{j=1}^{k-1} (e_j^n \cdot d^f e_k^{n-1}) e_j^n|}, \quad (k = 1, 2, \ldots, D) \]  

where \( e_0^n = 0 \) is assumed. By use of Eqs. (36) and (37), \( e_0^n \) is uniquely determined successively if the initial orthogonal set \( e_i^0 \ (j = 1, 2, \ldots, D) \) is fixed. The numerical scheme defined by Eqs. (36) and (37) is first proposed by Shimada and Nagashima as the exactly possible equality.\(^{19}\) But here we have to say that the scheme is an approximate formula complemented by the several restrictions about the stroboscopic time scale \( T \) and the additional transformation of Eqs. (36) and (37). From the theoretical viewpoint, each mean Lyapounov exponent defined by Eq. (34-2) does not depend on the stroboscopic time scale \( T \), so long as the number of ensemble \( N \) and \( T \) is large enough. On the other hand, however, the variance of the Lyapounov exponent defined by

\[ \overline{\Delta \lambda_D} = \frac{1}{N} \sum_{\pi=1}^{N} (\lambda_D^n)^2 - (\overline{\lambda_D})^2 \]  

may depend on the stroboscopic time scale \( T \). Table II shows each exponent and its fluctuation for the Lorenz chaos.

<table>
<thead>
<tr>
<th>( T )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( K_3 )</th>
<th>( (\Delta K_1)^2 )</th>
<th>( (\Delta K_2)^2 )</th>
<th>( (\Delta K_3)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1/2) T_0 )</td>
<td>0.908 ± 0.003</td>
<td>-14.45</td>
<td>6.414</td>
<td>1.792</td>
<td>3.070</td>
<td></td>
</tr>
<tr>
<td>( T_0 )</td>
<td>0.907 ± 0.005</td>
<td>-14.45</td>
<td>2.151</td>
<td>0.573</td>
<td>0.979</td>
<td></td>
</tr>
<tr>
<td>2( T_0 )</td>
<td>0.907 ± 0.006</td>
<td>-14.45</td>
<td>0.589</td>
<td>0.149</td>
<td>0.303</td>
<td></td>
</tr>
</tbody>
</table>

When the stroboscopic time \( T \) becomes large, the fluctuations \( \overline{\Delta K_i}^2 \) approaches zero and the numerical estimation for each exponent becomes more reliable. However, when the value of \( T \) is too large (\( T \simeq 3T_0 \) for the Lorenz chaos), the numerical estimation of Eq. (35) overflows (or underflows) from the capacity of computer and as the result the fluctuation becomes large again. Namely, we have to adopt an appropriate value of \( T \) to get the most reliable estimation of each Lyapounov exponent.

(C) **Averaged Dimension Spectrum and Topological Entropy**

The geometrical property of strange attractor \( A \) is characterized by the Hausdorff dimension, which has been briefly discussed in the previous section in relation to the markovian property of the chaotic time series. In this section, it will be shown that the same idea can be used for the characterization of the strange attractor.\(^{42}\)

First let us consider the dimension of the set \( A_{i_1,i_2,\cdots,i_{l-1}} \), which is the projection of the attractor \( A \) onto the restricted tangent space spanned by \( \{e_{i_1}, e_{i_2}, \cdots, e_{i_{l-1}}\} \).
The outer measure $l_{d,j}^{i_1,i_2,...,i_{j+1}}$ is defined by

$$ l_{d,j}^{i_1,i_2,...,i_{j+1}} = \inf_{(S)} (\text{Diam: } S)^* N_{i_1,i_2,...,i_{j+1}}, \quad (39) $$

where Diam: $S$ is the diameter of the covering sets $\{S_i; i=1, 2, \cdots\}$, and $N_{i_1,i_2,...,i_{j+1}}$ is the minimum number of the set $S_i$ necessary for covering the whole attracting set $A_{i_1,i_2,...,i_{j+1}}$. Here the diameter is assumed to be uniform for each set $S_i$ for convenience' sake of the computer calculation. Under the same assumption, by use of the same notation as used in Eqs. (36) and (37), $N_{i_1,i_2,...,i_{j+1}}$ and Diam: $S$ are taken as follows,

$$ \text{Diam: } S = \frac{|df^T(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{j+1}})|}{|df^T(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_1})|} \equiv \varepsilon \quad (40) $$

and

$$ N_{i_1,i_2,...,i_{j+1}} = \varepsilon^{-j} |df^T(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_1})|. \quad (41) $$

$\varepsilon$ stands for the smallest diameter. Here we have assumed $K_{i_1} < 0$, and $K_{i_1} < K_{i_k}$ ($k=1, 2, \cdots, j$). $N_{i_1,i_2,...,i_{j+1}}$ and $\varepsilon$ are the functions of the phase point on the attractor where the tangent space is defined, i.e., Eq. (39) is the local expression. Then Eq. (39) is rewritten into the following,

$$ l_{d,j}^{i_1,i_2,...,i_{j+1}} = \exp[T((\alpha - j)K_{i_{j+1}} + \sum_{k=1}^{j} K_{i_k})], \quad (42) $$

where use has been made of Eqs. (34), (40) and (41). The dimension of the set $A_{i_1,i_2,...,i_{j+1}}$ is determined by the change-over point $a_c(i_1, i_2, \cdots, i_{j+1})$, where the outer measure jumps from zero to infinity.\(^{42}\)

$$ D_{x_1,x_2,...,x_{j+1}} = a_c(i_1, i_2, \cdots, i_{j+1}) $$

$$ = j + 1 - (\sum_{k=1}^{j+1} K_{i_k})/K_{i_{j+1}}. \quad (43) $$

The l.h.s. of Eq. (43) is simply called dimension spectrum in this paper. For $\alpha < a_c(i_1, i_2, \cdots, i_{j+1})$, the outer measure defined in Eq. (42) is divergent to infinity, but for $\alpha > a_c(i_1, i_2, \cdots, i_{j+1})$ it converges to zero when the stroboscopic time $T$ is large enough. The number of the spectrum is $\Sigma_{j=0}^{D-j-1}(D!(j+1)!(D-j-1)!$. By the ensemble average used in Eq. (34-2) the averaged dimension spectrum is given by

$$ D(i_1, i_2, \cdots, i_{j+1}) = \frac{1}{N_{x_1,...,x_{j+1}}} \sum_{N_{x_1,...,x_{j+1}}} D_{x_1,x_2,...,x_{j+1}}. \quad (44) $$

Each spectrum characterizes the covering property of the attractor in the corresponding projected space. When we consider the Poincaré map on the special section surface, the Cantor-like structure might come to appear with the cor-
responding spectrum. The dimension \( D_A \) of the attracting limit set \( A \) is defined by the maximum spectrum in the whole space without any restriction,

\[
D_A = D(1, 2, \cdots, D)
= D - \left( \sum_{i=1}^{D} \frac{K_i}{K_D} \right).
\]

In the case of the Lorenz attractor, \( D_A \) and \( D(i, j) \) become as follows,

\[
D_A \cong 2.06,
D(2, 3) = 1, \quad D(1, 3) \cong 1.06,
D(1, 2) = \text{undefined}.
\]

Generally, \( D(i_1, i_2, \cdots, i_{j+1}) \) is undefined in the case \( K_{i_{j+1}} = 0 \), but so long as the condition \( K_{i_{j+1}} < 0 \) is satisfied, the following relations are led

\[
0 < D(i_1, i_2, \cdots, i_{j+1}) < j + 1 \quad \text{for} \quad \sum_{k=1}^{j+1} K_{i_k} < 0,
\]
\[
D(i_1, i_2, \cdots, i_{j+1}) > j + 1 \quad \text{for} \quad \sum_{k=1}^{j+1} K_{i_k} > 0.
\]

Therefore, the dimension of the attractor \( D_A \) is always smaller than that of the geometrical dimension of the phase space \( D \), since \( \sum_{k=1}^{D} K_{i_k} \) is negative in the dissipative system. On the other hand, in the case \( K_{i_{j+1}} > 0 \), \( D(i_1, i_2, \cdots, i_{j+1}) \) may become negative,

\[
D(i_1, i_2, \cdots, i_{j+1}) < 0 \quad \text{for} \quad K_{i_{j+1}} > 0,
\]

namely, the corresponding transformation \( df^T \) in the restricted tangent space span by \( (e_{i_1}, e_{i_2}, \cdots, e_{i_{j+1}}) \) is no more than contracting but stretching for all directions (see Appendix A). Generally, such transformation is not one-to-one correspondence in the projected space. The significance of Eqs. (47)~(49) is explained in Appendix A by using a simple dynamical model.

For the case \( K_{i_{j+1}} > 0 \), the lapping spectrum \( \text{lap}(i_1, i_2, \cdots, i_{j+1}) \) is taken as follows,

\[
\text{lap}(i_1, i_2, \cdots, i_{j+1}) \equiv \exp(\sum_{k=1}^{j+1} K_{i_k})
\]

and the topological entropy \( H_{\text{top}}(i_1, i_2, \cdots, i_{j+1}) \) is correlated to the dimension.

---

*) Recently, another expression of the fractal dimension was derived by Mori under the self-similarity assumption. The interrelation between his expression and the present one has not yet been clarified theoretically, but as will be shown in Part II of the present work, the definition by Eq. (45) is more acceptable for the non-autonomous system. Equation (43) is consistent with the definition given in Refs. 55) and 56).
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The maximum spectrum of the topological entropy $H_{\text{top}}$ is given by $H_{\text{top}} = \sum_{i=1}^{P} K_i$. This is the same relation as surmised to hold in more general classes of hyperbolic systems. The purpose of the present section is not to give the mathematical proof for this conjecture but to emphasize that the computational scheme based on Eq. (42) is consistent with this insight at least from the practical viewpoints.

The mean spectrum of dimension does not depend on the stroboscopic time $T$, so long as $T$ is large enough. However, the reliability in the numerical estimation of the spectrum remarkably depends on $T$ in the same sense as discussed in Eq. (38). Here let us consider the fluctuation of the dimension $D_A$ in order to characterize their reliability,

$$\overline{\Delta D_A^2} = \frac{1}{N} \sum_{\pi=1}^{N} (D_{x_{\pi}}^{1,2,3})^2 - \left( \frac{1}{N} \sum_{\pi=1}^{N} D_{x_{\pi}}^{1,2,3} \right)^2.$$  (52)

Table III shows the mean dimension and its fluctuation for the several stroboscopic time.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$D_A$</th>
<th>$(\Delta D_A)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2} T_0$</td>
<td>2.0490</td>
<td>0.0114</td>
</tr>
<tr>
<td>$T_0$</td>
<td>2.0579</td>
<td>0.0041</td>
</tr>
<tr>
<td>$2T_0$</td>
<td>2.0614</td>
<td>0.0011</td>
</tr>
</tbody>
</table>

§ 5. Discussion

The numerical scheme proposed in the present paper can be applied to the global analysis of the high dimensional chaos. Especially, discussion in §§ 2 and 4 is directly applied without any restrictions. On the other hand, in the realization of the symbolic time series as in § 3, we have to adopt an adequate coarse-graining method to get the best identification. In many cases, such coarse-graining scheme must depend on case by case. It is expected, however, that the $n$-adic expression of the time series will be successfully applied for many cases where the dynamical system under the consideration is endowed with the special
symmetry. Further, the $n$-adic expression becomes significant in the statistical identification of the intermittent chaos. In the last part of the present paper, we will show the theoretical reconstruction of the time correlation function and the fluctuation of the local dimension of the Lorenz chaos.

(A) Theoretical reconstruction of the time correlation function

As mentioned in § 1, the time course of $x$-variable of the Lorenz system is separated into two parts; $x(t) = P(t) + O(t)$. The correlation function $\Gamma(t)$ is given by

$$\Gamma(t) = \langle P(t)P(0) \rangle + \langle P(t)O(0) \rangle + \langle P(0)O(t) \rangle + \langle O(t)O(0) \rangle,$$

where $\langle \rangle$ stands for the ensemble average. If $P(t)$ and $O(t)$ are approximated by the continuous poisson process and the unstable sinusoidal respectively,

$$P(t) = \begin{cases} P_+(x(t) > 0) \\ P_-(x(t) < 0) \end{cases} \quad \text{and} \quad P(m, T) = \frac{(\gamma T)^m}{m!} e^{-\gamma T},$$

$$O(t) \approx e^{\alpha t} \cos(\omega t), \quad \text{and threshold value of } O(t) = O_c,$$

the approximate form of $\Gamma(t)$ is determined analytically as shown in Appendix B. Here $\gamma, \alpha, \omega$ and $P_\pm$ are the adjustable parameters of the Lorenz chaos. ($\gamma \approx 0.44, \alpha \approx 0.21, \omega \approx 6.28, |P_\pm| = O_c \approx 8.485$). As illustrated in Fig. 1, the value of $P(t)$ jumps from $P_\pm$ to $P_\mp$ when $O(t)$ touches at the threshold level $O_c$. If we denote by $O_0$ the initial value of $O(t)$ at the time just after the flip-flop jump, the residence time $\tau$ is estimated as follows,

$$\tau = \frac{1}{\alpha} \ln \frac{O_c}{O_0}.$$

Then, the distribution of the residence (or recurrence) time $P(\tau)$ is easily derived from Eq. (54),

$$P(\tau) = \gamma e^{-\gamma \tau},$$

and the distribution of the initial value $O_0$ becomes as follows,

$$P(O_0) = \frac{\gamma}{\alpha O_0} \left( \frac{O_c}{O_0} \right)^{-\gamma/\alpha}.$$

Denoting the peak point of $O(t)$ in Eq. (55) by $O_n = O(2n\pi/\omega)$, the Lorenz plot from $O_{n-1}/O_c$...
to \( \frac{O_s}{O_c} \) is approximated by an almost linear map whose inclination \( \beta = e^{2\pi \omega} \). This situation is well in line with the numerical calculation of the Lorenz system (see Fig. 10).

(B) **Fluctuation of the local dimension**

Here, we consider the non-uniform stroboscopic map \( f^{T_i} \), whose time interval \( T_i \) is not constant. For example, let us consider the Poincaré section surface \( \Sigma \) in phase space, and denote the successive recurrence time on \( \Sigma \) by \( T_i (i = 1, 2, \cdots) \). For the discrete map \( f^{T_i} \), characteristic exponents \( K_i^* \) (and fluctuations \( (\Delta K_i^*)^2 \)), the local dimension \( D_{AL}^L \) and its fluctuation \( (\Delta D_{AL}^L)^2 \) are defined in the same manner as in Eqs. (44) and (52). Table IV shows each exponent, dimension and their fluctuations for two cases of Poincaré section.

<table>
<thead>
<tr>
<th>Section surface</th>
<th>( K_1^* )</th>
<th>( K_2^* )</th>
<th>( K_i^* )</th>
<th>( (\Delta K_1^*)^2 )</th>
<th>( (\Delta K_2^*)^2 )</th>
<th>( (\Delta K_i^*)^2 )</th>
<th>( D_{AL}^L )</th>
<th>( (\Delta D_{AL}^L)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = 27, \ z &lt; 0 )</td>
<td>0.736 ± 0.08</td>
<td>-14.5</td>
<td>2.218</td>
<td>1.007</td>
<td>0.846</td>
<td>2.0523</td>
<td>0.0077</td>
<td></td>
</tr>
<tr>
<td>( z = 27, \ z &gt; 0 )</td>
<td>0.666 ± 0.11</td>
<td>-14.45</td>
<td>4.787</td>
<td>1.855</td>
<td>2.041</td>
<td>2.0523</td>
<td>0.0120</td>
<td></td>
</tr>
</tbody>
</table>

**Appendix A**

--- **Generalized Baker's Transformation** ---

Let us consider the following transformation \( F \) of the unit rectangular space; twice stretching towards \( x \)-direction and \( 1/r \) time contraction to the \( y \)-direction \((r > 1)\). Excess part is replaced into the upper side of the original unit rectangle. After many times iteration of this mapping \( F^n (n \rightarrow \infty) \), the rectangle is transformed into very thin fibrous set \( A \) for \( r > 2 \). On the other hand, for \( r < 2 \) the mapping is not one-to-one, and that may not be contracting. Baker's transformation is the case of \( r = 2 \). When \( r > 2 \), the Cantor structure comes to appear for the \( y \)-direction as \( n \) goes to infinity. The dimension \( D_A \) of such self-similar set \( (A) \) is given by

\[
D_A = 2 - \frac{K_1 + K_2}{K_2} = 1 - \frac{K_1}{K_2}.
\]

(A1)

where \( (K_1, K_2) = (\ln 2, -\ln r) \) are the Lyapounov spectra defined by Eq. (34) in the text. When the lapping number defined in § 4 is less than unity, the dimension \( D_A \) is always less than 2 and the singular measure is obtained on the fibrous set \( (A) \). For \( 1 < r < 2 \), however, the dimension \( D_A \) is larger than 2, and the invariant measure \( g(x, y) \) becomes a pathological function which is derived from the following functional equation,
Figure 11 shows an example of the numerical solution of Eq. (A-2) for \( r = 3/2 \), where the initial trial function \( g_0 \) is taken as a symmetric piecewise linear function.

For \( r < 2 \), the definition of Eq. (A.1) loses the intuitive meanings as the dimension of the support on which the invariant measure is defined, and it expresses only the covering properties of the transformation under consideration. As the relation, \( \sum_i k_i < 0 \), is satisfied in the dissipative dynamical system, it is reasonable to consider that Eq. (A.1) defines the self-similarity dimension of the attracting limit set (A). The negative dimension occurs only in the projected space as discussed in § 4.

**Appendix B**

--- Theoretical Reconstruction of the Correlation Function ---

Each contribution of Eq. (53) is rewritten as follows,

\[
\Gamma^{pp}(t) \equiv \langle P(0)P(t) \rangle = |P_x|^2 e^{-2\tau t},
\]

\[
\Gamma^{oo}(t) \equiv \langle O(0)O(t) \rangle = \sum_{k=0}^{\infty} (-1)^k \int_0^t dt_1 \gamma e^{-\eta t_1} \int_0^{t_1} \int_{t_1}^t dt_2 \gamma e^{-\eta (t_2-t_1)}\ldots
\]

\[
\times \int_{t_2-t_1}^t dt_k \gamma e^{-\eta (t_k-t_{k-1})} \int_0^{\eta (t_{k-1}-t_{k-2})} [P(O_k) ]_o e^{\omega t_k} \gamma [O_k e^{\omega t_k}] dO_k,
\]

\[
\Gamma^{po}(t) \equiv \langle P(0)O(t) \rangle = \sum_{k=0}^{\infty} (-1)^{k-1} \int_0^t dt_1 \gamma e^{-\eta t_1} \int_{t_1}^t dt_2 \gamma e^{-\eta (t_2-t_1)}\ldots
\]

\[
\times \int_{t_2-t_1}^t dt_k \gamma e^{-\eta (t_k-t_{k-1})} \int_0^{\eta (t_{k-1}-t_{k-2})} [P(O_k) ]_o e^{\omega t_k} \gamma [O_k e^{\omega t_k}] dO_k,
\]

\[
\Gamma^{op}(t) \equiv \langle P(t)O(0) \rangle = \sum_{k=0}^{\infty} (-1)^{k-1} \int_0^t dt_1 \gamma e^{-\eta t_1} \int_{t_1}^t dt_2 \gamma e^{-\eta (t_2-t_1)}\ldots
\]

\[
\times \int_{t_2-t_1}^t dt_k \gamma e^{-\eta (t_k-t_{k-1})} \int_0^{\eta (t_{k-1}-t_{k-2})} [P(O_k) ]_o e^{\omega t_k} \gamma [O_k e^{\omega t_k}] dO_k,
\]

where \( P(O_n) \) is given by Eq. (58) and the residence time was estimated by Eq. (56) in the text. After a little bit tedious calculation, the time correlation function \( \Gamma^{xy}(t) \) is given as follows,
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\[ \Gamma^{\infty}(t) = \frac{\gamma^2(\gamma - \alpha)}{(\gamma + a)(\gamma^2 + \omega^2)((\gamma - \alpha)^2 + \omega^2)} \cdot O e^{2e^{-2\omega t}} + \frac{-\gamma^2(\gamma - \alpha)e^{-at}}{(\gamma + a)(\gamma^2 + \omega^2)((\gamma - \alpha)^2 + \omega^2)} \cdot \frac{\gamma^2\omega^2(1 - e^{-at})}{2(\gamma + a)(\gamma^2 + \omega^2)a} \]

\[ + \frac{\gamma^2\omega^2(2\gamma + a)(e^{-at} - 1)}{(\gamma + a)(\alpha^2 - 4\omega^2)} \cdot \frac{\gamma((2\alpha + \gamma)^2 + 2\omega^2)e^{-at}}{(2\alpha + \gamma)((2\alpha + \gamma)^2 + 4\omega^2)} \right] \times O e^{2e^{-\omega t}} \cos(\omega t), \]  

\[ \Gamma^{\rho 0}(t) = \left[ \frac{-\gamma^3 e^{-\omega t}}{(\gamma^2 + \omega^2)(\gamma + a)} \cdot \frac{\gamma(\alpha + \gamma)}{(\alpha + \gamma)^2 + \omega^2} \right] O e^{2e^{-\omega t}} \cos(\omega t) + \left[ \frac{\gamma^2\omega}{(\gamma + a)(\gamma^2 + \omega^2)} \right] \times O e^{2e^{-\omega t}} \sin(\omega t), \]  

\[ \Gamma^{\rho p}(t) = \frac{\gamma(\gamma - \alpha)}{((\gamma - \alpha)^2 + \omega^2)} O e^{2e^{-\omega t}} \]

\[ + \left[ \frac{\gamma(\gamma - \alpha)e^{-at}}{((\gamma - \alpha)^2 + \omega^2)} - \frac{\gamma(\gamma + a)e^{-at}}{(\gamma + a)^2 + \omega^2} \right] O e^{2e^{-\omega t}} \cos(\omega t) \]

\[ + \left[ \frac{\gamma\omega}{((\gamma - a)^2 + \omega^2)} + \frac{\gamma\omega}{(\gamma + a)^2 + \omega^2} \right] O e^{2e^{-\omega t}} \sin(\omega t). \]  

\[ \text{References} \]

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Note added in proof: The recent letter from Dr. I. Shimada shows $D_4=2.047$ for the stroboscopic time $T=1$. 