Unilateral extension of a two-dimensional shear crack under the influence of cohesive forces

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Summary. We consider the problem of the unilateral extension of a two-dimensional anti-plane crack that initiates spontaneously at a point. The crack extends under the influence of cohesive resistance at the edge and dynamical friction along the crack walls. The stresses in the region beyond the edge of the crack are approximated so that they are exactly equal to the cohesive stresses near the edge of the crack, and are zero on the wavefront. An exact method of solving such problems is also given and can be used to determine the validity of the approximation. We find that the crack will not grow if the cohesion exceeds some critical value; this is consistent with an earlier result obtained by Knopoff, Mouton & Burridge for a similar one-dimensional model of crack propagation.

1 Introduction

The problems of dynamical fracture history are relevant to the construction of models of the earthquake source. From such models, one hopes to find correlations between the physical parameters giving the history of a fracture and the properties of the seismic signal radiated to near and/or distant seismographic stations. In this paper we study problems of crack rupture histories for those cases of two-dimensional cracks in which the rupture is influenced by cohesive forces at the crack tips; we wish to see if the conclusions derived for one-dimensional cases apply in the two-dimensional cases as well.

Problems of the dynamics of crack formation and healing are inherently non-linear in character: they are a form of the Stefan problem, which is that of linear partial differential equations with macroscopically moving boundaries. Other than purely numerical solutions, most progress in attempts to derive analytical solutions to problems of the Stefan type has been from the use of Green's function methods. This paper is no exception in this regard; we carry on in the tradition of the use of Green's functions to derive integral expressions for the relevant quantities.

Our progress on the analysis of dynamical crack histories derives from the work of Burridge & Halliday (1971), which is itself derivative from an earlier contribution by Kostrov (1966). Burridge & Halliday have analysed the problem of the growth and shrinkage of a two-dimensional, anti-plane strain, shear crack under the influence of dynamical
frictional stresses, but without cohesive forces at the crack tips. This case, which the seismologist describes as the extension of a crack that has only relative motions of $SH$ type on the crack walls, has a solution that can be written in terms of double integrals over distance and time involving Green's function kernels; these integrals can be separated if written in terms of characteristic coordinates (Kostrov 1966). In the absence of cohesive forces at the tip, the crack begins to extend sonically; the extension decelerates when the dynamical friction exceeds the stresses driving the crack, i.e. when the stress drop becomes negative. Healing, or shrinkage of the zone of the crack undergoing relative motion, can start at the points of maximum extension of the crack, or it can start in the interior; if the latter condition occurs, then the crack undergoes fission and each segment heals separately.

One of the difficulties with a crack model which does not include effects of cohesion, is that cessation of extension depends on the presence of a negative stress drop, as noted above, and as a consequence, faulting decelerates gradually; in turn, no significant stress concentrations are to be found at the termini of such a crack. Cohesion is a mechanism for guaranteeing that extension of a fault cease abruptly and thus that significantly large stresses be stored at loci of maximum extension. We assume that aftershocks are testimony to the presence of large stress concentrations at the termini of a fault and hence to the importance of cohesion in the fracture process. The residual stress can be invoked to trigger aftershocks by a mechanism of stress tunnelling (Knopoff 1972). Cohesive forces at the edges of cracks were introduced by Barenblatt (1962) to account for a gradual transition in the crack profile between the ruptured part of the fault behind the crack tip and the as yet unruptured segment in advance of the tip (Fig. 1).

One of the problems in invoking cohesion as a significant mechanism in the earthquake process, is that the comparison of cohesive forces with stresses involves a scaling by a characteristic length. The scaling factor is of the order of the linear dimension $L$ of the taper in the crack tip (Fig. 1). For ideal crystalline materials, the dimensions of the taper are probably of the order of a small number of lattice spacings; hence the cohesive forces are probably not too important as influences on the rupture of these materials, in the cases where the lengths of the cracks are of the order of centimetres.

In the case of fracture along pre-existing earthquake faults, we must look to influences other than those of interatomic forces to moderate rupture histories, since we are dealing with objects of much larger dimensions. We propose that the geometrical properties of earthquake faults can be modelled by cohesive forces on an appropriate scale of distances. Realistic earthquake faults do not have the perfectly plane geometries usually attributed to them. Close inspection of faults shows them to display bifurcations, bending, echeloning and other non-planar geometrical features; some branches terminate without continuation, while others reunite. This dendritic geometry is apparently reproduced on all scales on a self-similar basis (Kagan & Knopoff 1980). In order that a fracture on one segment trigger rupture on its non-planar extension, a stress derived from the first segment must be equal to the difference between the strength and the prestress on the second; the problem is complicated because of the tensor nature of the stresses and the roles of normal and shear stresses in comparison with the static friction, but a scalar model suffices for the purpose of pedagogy. For our purpose, we may assume that the termination of the first segment of rupture was

![Figure 1. Graph of relative motion between two faces of a growing shear crack in the vicinity of the crack tip.](https://academic.oup.com/gji/article-abstract/68/1/7/705649)
Extension of a two-dimensional shear crack

accompanied by the freezing of a singular stress concentration into the matter surrounding its edge. It is this stress concentration that we invoke to supply the trigger for the second segment.

Unfortunately, we are unable to solve the mathematics of faulting under conditions of complex dendritic geometries. We take, as a tractable alternative, the notion of projecting the complex structure on to a plane and propose the condition that the new single planar crack continue to extend if the stress function $k_d r^{-1/2}$ at the tip of a crack, arising from dynamic causes, exceeds a quantity $k_c r^{-1/2}$, where $k_c$ is a local characteristic property of material along the fault and $r$ is the distance from the crack tip. In our case, the quantity $k_c$ is the difference between the critical breaking stress and the prestress, scaled by a characteristic distance, such as the distance between echelon segments.

The converse of the above process is also valid. If $k_d < k_c$, then the crack stops its extension abruptly. In this case, we find that the subsequent freezing of the motion of the crack leaves the residual stress $k_d r^{-1/2}$ imbedded in the medium close to the locus of the maximum extension of the crack, as described above. Physically, cohesive forces act as absorbers of stress. This model of extension and cessation of rupture, we describe as one influenced by cohesion in analogy with the more familiar model which is governed by the same mathematical constraints.

We anticipate that cohesion will constrain the velocities of rupture of two-dimensional cracks so that they are subsonic in analogy with corresponding descriptions of one-dimensional cases (Knopoff, Mouton & Burridge 1973; Burridge & Keller 1978). The process whereby freezing is initiated at the point of encounter of a crack with an 'unbreakable obstacle', and a residual stress concentration is left behind as the legacy of that encounter, has already been described in one-dimensional cases (Knopoff et al. 1973).

In many cases of one-dimensional faults the strongest high-frequency seismic signal emanates, not from the point of onset of rupture, but from the point of onset of healing; this phase may dominate the spectral properties of the signal. Knopoff & Mouton (1975) have shown that macroscopic seismic fault parameters such as fault length and stress drop are not well correlated with the spectra of theoretical seismograms for one-dimensional faults that are moderated by the influences of cohesive forces, because of the powerful influence of the healing phase on the seismogram as well as the influence of the subsonic rupture velocity. One important feature of realistic models of faulting that is absent in these one-dimensional model fault studies, is the influence of radiation of cohesive forces. Radiation damping of the fault motions was introduced into one-dimensional models by a parametric scheme (Burridge & Knopoff 1967). Nevertheless we may expect similar inconsistencies between the faulting and spectral parameters for the two-dimensional cases.

Problems of the histories of two-dimensional fracture in the presence of cohesive forces have a significant literature. But in the cases considered up to now in the literature, the fractures are presumed to have developed from an initial, semi-infinite crack (e.g. Kostrov 1966) or an initial finite crack (e.g. Kennedy & Achenbach 1973; Kostrov 1966). These solutions avoid, perforce, the influences of stress fields radiated by one edge of the crack on the extension of the opposite edge. The problem of the history of extensions of fractures initiating at a point are relevant to the problems of earthquake faulting histories; these problems are the focus of this paper and represent the novelty of this contribution. We note that, in order that the rupture initiate at a point, the stress intensity factor $k = 0$ at the point of initiation of fracture since, at this point, the prestress and the strength are equal; spontaneous fracture initiates at points where the cohesion is zero.

As a final introductory comment, we note that the two popular models of cohesion, namely the Griffith critical energy flux model and the Irwin critical shear stress model, give
Incompatible fracture histories (Knopoff et al. 1973; Burridge & Keller 1978; Andrews 1976). In this paper we adopt the critical shear stress model, as has been intimated above.

In this paper we explore analytical solutions to the two-dimensional problems of anti-plane strain rupture initiating at a point in the presence of dynamical friction and study the influence of cohesion on the rupture process. Numerical studies of problems of this type have been given by Das & Aki (1977a, b). Our purpose is to develop analytical solutions, including some in closed form, in order to provide test cases for comparison with the numerical results. More important, we believe analytical methods yield greater computational flexibility and provide greater insights into the significance of the physical processes involved in the onset and cessation of rupture and in the character of the radiated signal, than do numerical methods. Even if these goals are not achieved to the satisfaction of all readers, we believe the analytical solutions represent useful alternatives to the numerical ones. In this paper we focus our attention on the history of extension of two-dimensional, anti-plane cracks that extend unilaterally, for purposes of comparison with the earlier work on one-dimensional unilateral cracks. In subsequent contributions we explore bilateral crack histories and other features of the problem, such as freezing, seismic radiation, and final stress states.

2 Theory

We discuss the problem of the determination of the tearing history of a unilateral anti-plane shear crack under the influence of dynamical friction in the torn region, and the influence of resistance of the material to tearing due to cohesion. The problem has certain difficult aspects associated with the problem of interaction between the propagating and non-propagating ends of the crack. The stationary edge of the crack has an ever-increasing stress concentration due to the accumulation of stresses that would normally be relieved by extension, and can in this case only be relieved by the radiation of SH-waves. Because of the cohesive forces at the crack tip, the propagating edge of the crack moves subsonically. Thus in front of the advancing edge of the crack, there is a stress field which is the sum of the pre-stress, which is known, and the stress field in the wake of the field radiated from the stationary edge. The latter stress field is unknown since the stresses in the neighbourhood of the stationary edge themselves depend on the history of relative motion on the crack faces, which in turn depends on the rate of advance of the moving edge. We calculate the stresses just in advance of the moving crack tip as the difference between the stress accumulated due to the relative motion of the crack faces and that due to the known cohesive forces. Using this result plus the fact that the total shear stress in advance of the elastic wavefronts is just the prestress, we write an approximation to the shear stresses in the region between the elastic wavefront and the edge of the crack. Below we derive the differential equation of motion of the crack tip under this approximation. We test the approximation by solving the problem exactly in one special case; the exact solution is obtained both analytically and by an iterative procedure.

Let \( \tau_0 \) be the x-y component of the shear prestress in the medium before fracture, and let \( \nu \) denote the displacement in the y-direction relative to the prestressed state, where the y-axis is the long direction of the crack. The crack grows in the z-direction; the x-axis is perpendicular to the fault plane (Fig. 2). The crack is located in an infinite, homogeneous, isotropic, elastic medium. We avoid the problem of the imbedding of these faults in a half space. Such problems have been dealt with skillfully by Burridge & Halliday (1971) by a method of images; we can contribute nothing new to this more general feature of the problem. However, for convenience, the y-axis may be considered to be a horizontal direction along a strike-slip fault and the z-axis the direction into the interior of the Earth.
Since we are considering the anti-plane shear crack problem (SH), the only component of the (displacement) motion is that in the $y$-direction

$$u = u(x, z, t)$$

which is independent of $y$. The relative slip between the two faces of the crack is given by

$$[u] = 2u(0^+, z, t) = 2u(z, t), \quad 0 < z < z_1(t)$$

where we assume that the crack starts as a line crack at the origin, which is the pinned edge, and $z_1(t)$ is the location of the moving tip at time $t$. Equation (2) defines the function $u(z, t)$. We let $T(z, t)$ be the difference between the static prestress and the stress during the fracture at $(0, z, t)$, and define

$$u_x(z, t) = v_x(0, z, t).$$

Then

$$-\mu u_x(z, t) = T(z, t), \quad 0 < z < z_1(t)$$

where $\mu$ is the shear wave modulus of the medium; the subscript indicates a partial derivative with respect to the coordinate indicated.

We consider motions $u(z, t)$ in the plane of the fault. Let the characteristic coordinates $(\xi, \eta)$ be:

$$\xi = t + z,$$

$$\eta = t - z.$$  

Figure 2. Coordinate system for unilateral growth of a two-dimensional anti-plane strain shear crack. The crack nucleates at the pinned edge at $0$. The moving edge is at $z_1(t)$. The particle displacements are in the $y$-direction.
Here we assume that the shear wave velocity is 1. Then the relative displacement at any point $P(\xi_P, \eta_P)$ on the fault plane is given by (Burridge & Halliday 1971)

$$u(\xi_P, \eta_P) = -\frac{1}{2\pi} \int_0^{\xi_P} \int_0^{\eta_P} \frac{u_x(\xi, \eta) \, d\xi \, d\eta}{\sqrt{(\xi_P - \xi)(\eta_P - \eta)}}.$$  \hfill (5)

Let $\xi_Q(\eta_P)$ be the locus of $z = z_1(t)$ in the coordinate system $(\xi, \eta)$ (Fig. 3). Since $2u$ is the jump in the displacement across the fault, we may write

$$\int_0^{\eta_P} \frac{d\eta}{\sqrt{(\eta_P - \eta)}} \int_0^{\xi_P} \frac{u_x(\xi, \eta) \, d\xi}{\sqrt{(\xi_P - \xi)}} = 0, \quad \xi_P > \xi_Q(\eta_P).$$  \hfill (6)

Equation (6) implies (Kostrov 1966)

$$\int_0^{\xi_P} u_x(\xi, \eta_P) (\xi_P - \xi)^{-1/2} \, d\xi = 0.$$  \hfill (7)

Letting $\xi_P \rightarrow \xi_Q^-$ (Fig. 3),

$$\int_0^{\eta_P} \frac{u_x(\xi, \eta_P) \, d\xi}{\sqrt{(\xi_P - \xi)}} - \frac{1}{\mu} \int_{\eta_P}^{\xi_Q} T(\xi, \eta) \, d\xi \int_0^{\eta_P} \frac{k(\xi_Q)}{\sqrt{\xi_P - \xi}}$$

$$+ \frac{k(\xi_Q)}{\mu} \int_{\xi_Q}^{\xi_P} \frac{(\xi - \xi_Q)(\eta_P - \xi)}{\sqrt{(\xi_P - \xi)}} \, d\xi = 0$$

where we have assumed that the stress at the edge of the crack is

$$\mu(u_x)_P = k(\xi_Q) (\xi_P - \xi_Q)^{-1/2},$$

i.e. that there is a square root singularity at the advancing edge of the crack. The quantity $k$ is a measure of the strength of the singularity, which we call the dynamical stress intensity factor.

Let $g$ be the coefficient of cohesion at $Q$ (Fig. 3). Then equation (9) may be rewritten as

$$\mu(u_x)_P = k(\xi_Q) (\xi_P - \xi_Q)^{-1/2} = g(z_Q) (z_R - z_P)^{-1/2}.$$  \hfill (10)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image}
\caption{The region of slip of a unilateral crack VOR. The crack nucleates at O. The moving edge of the crack is the curve $z_1(t)$. Coordinates and points in the diagram refer to the mathematical solution of the problem given in the text.}
\end{figure}
in the limit $P \rightarrow Q$. The quantity $g(z)$ is a coefficient that describes the ability of the material to absorb a stress singularity which varies as the reciprocal square root of the distance from the edge of the crack, namely

$$g(z_Q) = \frac{g(z)}{\pi(z_P - z_R)^{1/2}}.$$ 

If the stress is greater than this quantity, further extension of the crack ensues; if the stress is less than this quantity, the crack cannot extend. The coefficient $g(z)$ is specified in advance as one of the parameters of the fault system. In equation (10) we have indicated the self-regulating character of the growth of the crack by equating the dynamical shear stress generated by the growing crack to the strength of the material, which is its ability to absorb the dynamically generated shear stress. Equation (10) is therefore the condition for extension of the crack. By referring to the triangle PQR (Fig. 3), the relationship between the coefficients $k$ and $g$ is given by

$$k = \frac{g(z_1(t_2))}{\pi} \sqrt{\frac{2}{1 - \xi_1(t_2)}}.$$ 

(11)

where

$$Q[\xi_Q(\eta_P), \eta_P] = Q[z_1(t_2), t_2]$$

is the location of $Q$ in the two coordinate systems.

Carrying out the integration of the third term of (8), we have

$$\pi k = \int_{\eta_P}^{\xi_Q(\eta_P)} T(\xi, \eta_P) d\xi - \mu \int_{0}^{\eta_P} u_x(\xi, \eta_P) d\xi.$$ 

The task we now face is to evaluate the second integral which gives the influence on the moving edge of the crack of the dynamical stress field beyond the stationary edge.

We attack the problem of the field beyond the stationary edge as follows. An equation similar to (7) and derived in the same way, is

$$\int_{0}^{\eta} \frac{u_x(\xi, \eta') d\eta'}{\sqrt{\eta - \eta'}} = 0, \quad \eta > \xi$$ 

(13)

where, this time, $\xi$ is constant along the path of integration. For the case $\eta = \eta_P$ (Fig. 3), equation (13) becomes

$$\int_{0}^{\eta_P} \frac{u_x(\xi, \eta) d\eta}{\sqrt{\eta_P - \eta}} = 0, \quad \eta_P > \eta_T, 0 < \xi < \eta_P$$

or

$$\int_{\eta_T}^{\eta_P} \frac{u_x(\xi, \eta) d\eta}{\sqrt{\eta_P - \eta}} = - \int_{0}^{\eta_T} \frac{u_x(\xi, \eta) d\eta}{\sqrt{\eta_P - \eta}}.$$ 

(14)

If (14) is taken to be an Abel integral equation for

$$u_x(\xi, \eta_P), \quad 0 < \xi < \eta_P$$
we have
\[ u_x(\xi, \eta) = \frac{1}{\pi \mu \sqrt{\eta_p - \eta}} \int_0^T T(\xi, \eta) \sqrt{\eta_p - \eta} \, d\eta \]
\[ - \frac{1}{\pi \sqrt{\eta_p - \eta}} \int_0^S u_x(\xi, \eta) \sqrt{\eta_p - \eta} \, d\eta. \] 

(15)

3 Approximate solution

In order to evaluate the second integral of (15) we need to know 
\[ u_x(\xi, \eta), \quad 0 < \eta < \eta_S, 0 < \xi < \eta_p. \]

There are two methods to attack this aspect of the problem, one of them an exact procedure, and the other approximate. We discuss the approximation first. In the above range of \((\xi, \eta)\) we approximate 
\[ u_x(\xi, \eta) \]

as
\[ u_x(\xi, \eta) = \eta k(\eta_S)/\{\mu \eta_S \sqrt{\eta_S - \eta}\}. \] 

(16)

Expression (16) satisfies the condition of zero stress at \(\eta = 0\) and has the correct stress intensity factor at the tip \(S\). The validity of this approximation and its higher-order refinements will be discussed in a later section when we compare the result obtained with (16) with those obtained from exact procedures. Upon substituting (16) in (15) we have
\[ u_x(\xi, \eta_p) = \frac{1}{\pi \mu \sqrt{\eta_p - \eta_T}} \int_0^T T(\xi, \eta) (\eta_T - \eta)^{1/2} \, d\eta \]
\[ - \frac{k(\eta_S)}{\pi \mu \eta_S \sqrt{\eta_p - \eta}} \int_0^S \eta \sqrt{\eta_T - \eta} \, d\eta. \] 

(17)

If we substitute (17) in (12) and use (11), we have
\[ g \left| z_1(t_2) \right| \sqrt{\frac{2}{1 - \dot{z}_1(t_2)}} = \int_{\eta_p}^{\xi} Q(\eta_p) \frac{T(\xi, \eta_p) \, d\xi}{\sqrt{\xi_p - \xi}} \]
\[ - \frac{1}{\pi} \int_0^{\eta_p} \frac{d\xi}{\sqrt{(\xi_p - \xi)(\eta_p - \xi)}} \int_{\eta_S}^{\xi} T(\xi, \eta) \sqrt{\xi - \eta} \, d\eta \]
\[ + \frac{\sqrt{2}}{\pi^2} \int_0^{\eta_p} \frac{g(S) \, d\xi}{\eta_S ((\xi_p - \xi)(\eta_p - \xi)(1 - \dot{z}_1(S)))^{1/2}} \int_0^{\eta_S} \frac{\eta}{\eta_p - \eta \sqrt{\eta_S - \eta}} \, d\eta. \] 

(18)

Equation (18) is the non-linear differential equation of motion of the crack tip where \(\eta_S\) depends on the crack tip locus. It is easily shown that \(\dot{z}_1 = 1\) if \(g = 0\), i.e. the crack extends sonically if cohesion is absent. Equation (18) may be solved numerically for any given stress drop \(T\) and cohesion \(g\) by iteration. We neglect the last term in (18) in order to derive a starting function for the iteration and continue to iterate until the solution converges. Usually only three or four iterations with a Runge-Kutta procedure are sufficient to obtain a solution to this first-order differential equation with reasonable accuracy.
In summary the approximation involves an integration over the path VQ followed by an integration over the triangular region OVC. The latter integration has been broken into two parts: the first region, where the stress drops are known exactly, is bounded by the straight lines OV and VW and the arc OW; the second is the region OCW (with OW an arc) where the stresses in this region are approximated by the function (16).

4 An exact solution

As indicated above, the criterion for the extension of the crack does not depend entirely on the cohesive strength locally, but is also influenced by conditions that prevailed at an earlier stage of the crack history, through a complex non-linear relation. The stresses beyond the cracked region at this earlier time, need to be determined in order to determine the conditions for rupture at any given time. The approximation need not be taken. The integration can be made exact by the following procedure. The integration over the triangular region OVC in the preceding calculation is carried out first by integrating along lines of constant \( \xi \) (such as BT) followed by an integration over \( \xi \) from O to C. We replace the integral from B to T, along a typical line of constant \( \xi \), by the integral from S to T as before and compute the relevant quantities at S by an integral over the line SX plus an integral over the area OAX. The latter integration can be stopped at Y and the process repeated by integrating over a line \( \xi = \) constant through Y followed by the integral over OMY and so on. Each step of the iteration involves an increase in the order of multiple integration. The quality of the ultimate solution depends on the step at which the iteration of the approximate stress equation (16) is truncated. After each iteration we will have moved one step down a staircase path (see Fig. 3) which converges to the origin. Theoretically, if we assume that \( g = 0 \) and is continuous at the origin, a repeated application of (15) to (13) leads us to a fracture criterion in the form of an infinite oscillatory series whose \( n \)th term approaches zero rapidly as \( n \) increases, i.e. the series converges (as it should). Alternatively, as we approach the origin the velocity of rupture approaches 1 if

\[
g(z) = 0(z) \text{ as } z \to 0.
\]

The process then terminates because the rupture velocity is known. In practice, we obtain a series of solutions each of which is obtained by terminating equation (16) at different steps of the staircase. Usually only a few iterations are required to obtain a solution with reasonable accuracy. We illustrate this by an example, in which the condition

\[
g(z) = 0(z)
\]

near \( z = 0 \) is not obeyed.

5 Example: uniform propagation

Let \( T = T_0 \)

and

\[
g = a z^{1/2}
\]

where \( T_0 \) and \( a \) are constants. If we substitute these in (18), we find that the crack velocity \( \dot{z}_1 = V \), where \( V \) is some constant. Hence under a constant dynamic stress drop and cohesion proportional to the square root of distance from the origin, a unilateral crack propagates
with constant velocity; since the crack cannot propagate supersonically, $0 < V < 1$. We compute the relationship between $\alpha$ and $V$. We attack the problem in three ways. In the first place, the problem has an exact analytic solution in closed form by virtue of the self-similarity of cases of uniform extension of one- and two-dimensional cracks initiating at a point. The solution is given in Appendix A. We use this solution as a control to assess the quality of the approximate solution of Section 3 and the iterative solution of Section 4. These comparisons are useful to give guidance in the cases in which closed form solutions are not available.

To develop the numerical procedure, we have reference to Fig. 3. We have, as in (7),

$$\int_T^0 \frac{u_x(\xi_U, \eta)}{\sqrt{\eta_U - \eta}} \, d\eta + \int_T^0 \frac{u_x(\xi_U, \eta)}{\sqrt{\eta_Y - \eta}} = 0. \quad (20)$$

We solve for $u_x(\xi_U, \eta_Q)$ at the point $U$ as a formal solution of Abel's equation as before and get

$$u_x(\xi, \eta_Q) = \frac{T_0}{\pi \mu (\eta_Q - \eta_1)^{1/2}} \int_{\eta_Q}^{\eta_1} \frac{(\eta_1 - \eta)^{1/2} \, d\eta}{\eta_Q - \eta}$$

$$- \frac{1}{\pi (\eta_Q - \eta_1)^{1/2}} \int_0^{\eta_1} \frac{u_x(\xi, \eta) (\eta_1 - \eta)^{1/2}}{\eta_Q - \eta} \, d\eta, \quad 0 < \xi < \eta_Q, \quad (21)$$

where

$$\eta_1 = \xi, \quad \eta_2 = A\xi, \quad A = (1 - V)/(1 + V) < 1.$$

We introduce the notations $u_x(\xi, \eta; \xi)$ and $u_x(\xi, \eta; \eta)$ as the values of $u_x(\xi, \eta)$, derived from Abel's equations by integration along the lines $\xi = \text{constant} \ (0 < \xi < \eta)$ and $\eta = \text{constant} \ (0 < \eta < A\xi)$ respectively. Then

$$u_x(\xi, \eta; \xi) = \frac{2T_0 (A^{-1} - \eta)^{1/2}}{\pi \mu (\xi - A^{-1} \eta)^{1/2}} \left[ \sqrt{1 - A - \frac{\xi - A^{-1} \eta}{A^{-1} \eta}} \tan^{-1} \sqrt{\frac{(A^{-1} - 1)}{\xi - A^{-1} \eta}} \right]$$

$$- \frac{1}{\pi (\xi - A^{-1} \eta)^{1/2}} \int_0^{\eta} \frac{u_x(\xi', \eta) (A^{-1} \eta - \xi')^{1/2}}{\xi' - \xi} \, d\xi', \quad (22)$$

and

$$u_x(\xi, \eta; \eta) = \frac{2T_0 \xi^{1/2}}{\pi \mu (\eta - \xi)^{1/2}} \left[ \sqrt{1 - A - \frac{\eta - \xi}{\xi}} \tan^{-1} \sqrt{\frac{(1-A)}{\eta - \xi}} \right]$$

$$- \frac{1}{\pi (\eta - \xi)^{1/2}} \int_0^{A\xi} \frac{u_x(\xi, \eta') (\xi - \eta')^{1/2}}{\eta' - \eta'} \, d\eta'. \quad (23)$$

We substitute (23) in (22) and let

$$u_x(\xi, \eta; \eta) = \frac{2}{\pi \mu \sqrt{A\xi - \eta}} f(\xi, \eta). \quad (24)$$
We have
\[ f(\xi, \eta) = T_0 \eta^{1/2} \left[ \sqrt{1 - A} - \sqrt{A \xi^2 / \eta - 1} \tan^{-1} \sqrt{\eta / A \xi} \right] \]
\[ - \frac{T_0 \sqrt{1 - A}}{\pi} \int_{\xi}^{\eta} \frac{\left( \eta - A \xi^2 \right)^{1/2}}{(\xi - \xi') (\eta - \xi')^{1/2}} d\xi' \]
\[ + \frac{T_0}{\pi} \int_{0}^{\eta} \frac{(\eta - A \xi^2)^{1/2}}{\xi - \xi'} \tan^{-1} \sqrt{\xi / (1 - A)} \frac{d\xi'}{\eta - \xi'} \]
\[ + \frac{1}{\pi^2} \int_{0}^{\eta} \frac{(\eta - A \xi^2)^{1/2}}{(\xi - \xi') (\eta - \xi')^{1/2}} \int_{0}^{A \xi'} f(\xi', \eta') (\xi' - \eta')^{1/2} d\eta' \]
\[ = \frac{1}{(1 - \eta) \sqrt{2A}}. \tag{25} \]

Equation (25) relates the stresses at two successive stages of the iteration process along paths \( \eta = \text{constant} \). Equations \((13), (24)\) and \((25)\) define the iteration completely in the sense that \((25)\) can be used repeatedly.

We will calculate \( \alpha(V) \) only up to the second order. In order to terminate the iteration at some stage we require the terminating value of \( f \); from \((16)\) this is given by
\[ f_2 = \frac{\alpha \eta}{(1 - V) \sqrt{2A}}. \tag{26} \]

Equation (25) may be rewritten in a normalized form suitable for numerical computation as follows:
\[ f(\xi, \eta) = T_0 \eta^{1/2} \left[ \sqrt{1 - A} - \sqrt{A \xi^2 / \eta - 1} \tan^{-1} \sqrt{\frac{1 - A}{A \xi^2}} \right] \]
\[ - \frac{2T_0 \sqrt{1 - A} \eta^{1/2}}{\pi} \int_{0}^{1} \frac{(1 - A + At^2)^{1/2}}{t} (1 - t^2)^{1/2} dt \]
\[ + \frac{2T_0 \eta^{1/2}}{\pi} \int_{0}^{1} \frac{t(1 - A + At^2)^{1/2}}{(\xi / \eta - 1 + t^2)} \tan^{-1} \left[ \frac{\sqrt{(1 - A) (1 - t^2)}}{t} \right] dt \]
\[ + \frac{4A^{1/2}}{\pi^2} \int_{0}^{1} \frac{(1 - t_1^2)(1 - A + At_1^2)^{1/2}}{t} \frac{dt_1}{\xi / \eta - 1 + t_1^2} \]
\[ \times \int_{0}^{1} f(\eta (1 - t_2^2), A \eta (1 - t_2^2) (1 - t_2^2) (1 - A + At_2^2)^{1/2} \left[ 1 - A + At_2^2 + At_2^2 (1 - t_2^2) \right] \] \[ \frac{dt_2}{1 - A + At_2^2} \]. \tag{27} \]

6 Numerical results

The evaluation of these expressions is carried out in Appendices B and C. We consider several iterations of the approximate solution \((18)\) and several of the exact numerical solution \((25)\) or \((27)\). For comparison, the exact closed form solution (Appendix A, equation A10) is plotted as the solid curve in Fig. 5.

Five numerical cases are investigated.
Figure 4. The region of slip of a unilateral crack extending with uniform velocity is VOQ. Points M, N, etc. are reference points in the solution procedure.

6.1 CASE I

We integrate the stress drop $T$ along the path QV (Fig. 4) and ignore the problem of the stresses outside the pinned edge $z = 0$, i.e. the influence of cohesive forces at the pinned edge are avoided to start with. From equation (B1) we get the result

$$\dot{a} = a/T_0 = 2\sqrt{1-V}.$$ 

This curve is plotted as the dashed curve in Fig. 5. By comparison with the results of the more intricate calculations in Fig. 5, we see that this result is a good approximation over most of the range of rupture velocities. Only in the region $V < 0.2$ or thereabouts, do we have to be concerned about the accuracy of the calculation. In the region $V < 0.2$ we are

Figure 5. Graphs of coefficients of cohesion versus velocity in the cases of a unilateral crack extending with uniform velocity as obtained from different exact and approximate solution procedures. The identification of these graphs is given in the text.
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dealing with a rupture region confined to a narrow wedge-shaped region between OQ and the t-axis of Fig. 4. For $V < 0.2$ we must begin to be concerned about radiation effects from the pinned edge of the crack. In this simple calculation, there is a limiting value of cohesion; if the rate of rise of the cohesion is too great (varying as the square root of the distance), the crack will not extend, a result consistent with the one-dimensional case (Knopoff et al. 1973). In this case the limiting value is $\tilde{\alpha} = 2.0$, which is greater than the exact limiting value $\pi/2$ (see Appendix A).

6.2 CASE II

The next term in the iteration of the exact solution is evaluated as proposed in equations (25) and (27). Literally this involves the line integral of the stress drop taken over the line QV as in Case I plus the integration over the area OVW. This result, given in (C13), is essentially identical with the exact solution (A10) for values of $V$ in the range $0.1 < V < 1.0$. The values of cohesion coefficients for a given constant rupture velocity are decreased slightly over the entire range. The most obvious contribution of the second term is in the region of small rupture velocities. The estimation of successive terms in the iteration yields a convergent oscillatory series. The two cases I and II represent bounds on the exact iterative solution for the rupture velocity.

6.3 CASE III

This is the first of several calculations making use of the approximate formula (16) for the stress in the region outside the crack tip (or tips). In the present case, we add to the result of Case I, the consequence of taking the approximation (16) in the triangular region OCW (Fig. 4). The result of this calculation is given in formula (B1). The numerical result is plotted as the dot-dashed curve in Fig. 5. In view of the remarks above regarding the resulting nature of the terms in the iteration, the approximation of this case represents an improvement for $1 > V > 0.15$. For velocities $V < 0.15$ the approximation (16) is poor in this order of iteration. The approximation gives no finite value of $\tilde{\alpha}$ for zero rupture velocity.

6.4 CASE IV

In this approximation, we integrate along line QV (Fig. 4) and then carry out the second-order multiple integration as follows: we integrate over the triangle OVW, followed by the integral over line MW, and finally apply the approximation (16) in the region between OM and the axis $\xi = 0$. The calculation is outlined in Appendix C and the result is given in equation (C11). The curve is essentially identical with the second iteration of the exact solution (Case II) for velocities $1 > V > 0.1$. Again the approximation fails for low rupture velocities but represents a significant improvement over the preceding case. We observe that the stress-intensity factor at the pinned edge increases as the square root of time since the initiation of rupture (see equation C5) since $\eta' \propto t_1$.

6.5 CASE V

In the final example of the approximate theory we carry the exact iterative solution out to a greater degree than before and follow this with the approximation, as follows: we integrate over the line QV, the triangle OWV, the line MW and the triangle ONM and follow with the approximation (16) in the triangle ONL. The result is the curve designated by long dashes in Fig. 5. We may assume that the curve is valid to rupture velocities somewhat less than 0.1 in
view of the fact that the exact iterative part is carried out to a greater order than before. As in Case III, the curve again diverges for small $V$, but has significantly smaller values of $\alpha$ than those for Case III.

To summarize these results, the approximation (16) represents a convenient way to get a solution to the problem in those cases for which exact solutions are not to be derived, at least in the region of rupture velocities that are not small. From the exact solution, as well as from the trend of the approximations, we assert that if the modulus of cohesion is large enough, the unilateral crack will not propagate at all, a conclusion that was obtained as well in the one-dimensional case.

In this paper we have not considered the conditions of healing, of bilateral growth, nor of the calculation of theoretical seismograms at locations removed from the crack. These tasks are left for a second contribution on the subject of crack histories under the influence of cohesive forces at the crack tip.

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**References**


Appendix A: exact analytical solution

We present a derivation of the exact analytical solution of the problem of the uniform propagation of cracks discussed in Section 5. The double Fourier transform \( \hat{u}(k, \omega) \) of the function \( u(z, t) \) is

\[
\hat{u}(k, \omega) = \int_0^\infty dt \int_{-\infty}^{\infty} u(z, t) \exp\{i(\omega t + kz)\} \, dz
\]

if \( u(z, t) = 0, \, t < 0 \). The transforms \( \hat{u}(k, \omega) \) and \( \hat{u}_x(k, \omega) \) of the displacement \( u(z, t) \) and the strain \( u_x(z, t) \) on the fault plane are related by

\[
\hat{u}_x(k, \omega) = i(\omega^2 - k^2)^{1/2} \hat{u}(k, \omega)
\]

with the S-wave velocity \( \beta = 1 \).

Since the problem of uniform unilateral propagation under constant stress drop is self-similar, it can be shown that (Willis 1973)

\[
u(z, t) = bt \{ p(\sqrt{\pi - p}) \}
\]

for \( 0 < p = z/t < V \)

\[
u(x, t) = 0 \quad \text{for} \quad p > V, \, p < 0
\]

where \( b \) is a function of \( V \) to be determined. Equations (A1) and (A3) give

\[
\hat{u}_x(k, \omega) = -i\omega \hat{u}_x(k, \omega),
\]

(A4) and (A5) give, after inversion,

\[
u_x(z, t) = \frac{2}{t} \Re \{ z/t [(z/t - 0i)^2 - 1]^{1/2} F''(z/t - 0i) \}
\]

where

\[
F(p) = \frac{ibV^2}{8} [p(p - V)]^{-3/2}.
\]

Hence \( \hat{u}_x(z, t) = 0, \quad 0 < z/t < V \). Integrating (A6), we get

\[
u_x(z, t) = \frac{bV^2}{8} \Re \int_{z/t - 0i}^{\infty} i(p^2 - 1)^{1/2} [p(p - V)]^{-3/2} \, dp.
\]

Inside the torn region, \( \mu u_x(z, t) = -T_0 \) and since \( \hat{u}_x(z, t) = 0 \), (A7) may be evaluated by taking any \( z/t \) in \( 0 < z/t < V \). Taking \( z/t = V/2 \), we have

\[
b(V) = \frac{4T_0}{\mu I(V)}
\]

where

\[
I(V) = \frac{V^2}{\sqrt{2}} \int_0^\infty \frac{[(V^2/4 - q^2 - 1)^2 + q^2 V^2]^{1/2} - (V^2/4 - q^2 - 1)]^{1/2}}{(q^2 + V^2/4)^{3/2}} \, dq.
\]
Also since $\mu_{\alpha}(z, t) = (\alpha z)^{1/2} (z - Vt)^{1/2}/\pi$ as $z \to Vt^*$, we have from (A7)

$$\alpha = \frac{b\pi\mu}{2} (1 - V^2)^{1/2}$$  \hspace{1cm} (A9)

(E8) and (E9) give

$$\alpha = 2\pi (1 - V^2)^{1/2}/I(V).$$  \hspace{1cm} (A10)

It is easy to show that

$$I(V) = 4 - \frac{V^2}{2} (\log V + 3/2 - 6\log 2) + 0(V^4 \log V)$$

so that up to order $V^2$,

$$\tilde{\alpha} = \pi \left[ 1 + \frac{V^2}{8} \log V - V^2 (5 + 12\log 2)/16 \right]$$  \hspace{1cm} (A11)

which approaches $\pi/2$ as $V$ approaches zero.

Appendix B: first-order approximation

We set $f = f_1$ (26) in the second integral in (25). We apply (15) and (24), and get after simplification

$$\tilde{\alpha} = \frac{2 \left( \sqrt{V + \sqrt{2(1 - V)}(I_2 - \sqrt{1 - A I_1})/\pi} \right)}{\sqrt{V/(1 - V)}} (1 - 4I_3/\pi^2)$$  \hspace{1cm} (B1)

where

$$I_1 = \int_0^1 \frac{\sqrt{1 - t^2} dt}{[t^2 + 2V/(1 - V)]^{1/2}}$$

$$I_2 = \int_0^1 \frac{t \tan^{-1} \left( \sqrt{1 - A} (1 - t^2)/t \right) dt}{[t^2 + 2V/(1 - V)]^{1/2}}$$  \hspace{1cm} (B2)

$$I_3 = \int_0^1 \frac{A^{3/2} (1 - t_1^2)^{3/2} dt_1}{[t_1^2 + 2V/(1 - V)]^{1/2}} \int_0^1 \frac{(1 - t_2^2) (1 - A + At_2^2)^{1/2} dt_2}{1 - A + At_1^2 + At_2^2 (1 - t_1^2)}.$$

Second-order approximation

In the second integral of (25) we substitute $f(\xi', \eta')$ as evaluated from (25) and then substitute $f = f_1$ in the multiple integral. We have, after simplification and application of (15), (24),

$$\tilde{\alpha} = \frac{2 \left( \sqrt{V + \sqrt{2(1 - V)}(I_2 - \sqrt{1 - A I_1})/\pi} \right) + I_4 + I_5 + I_6 + I_7}{\sqrt{V/(1 - V)}} (1 - 4I_3/\pi^2)$$  \hspace{1cm} (B3)

where $I_4$ and $I_5$ are given by (B2). If the functions $\phi_1(t)$, $\phi_2(t_1, t_2)$ are defined as

$$\phi_1(t) = (1 - t^2)^{3/2}/[t^2 + 2V/(1 - V)]^{1/2}$$

$$\phi_2(t_1, t_2) = \frac{(1 - A + At_2^2)^{1/2}}{1 - A + At_1^2 + At_2^2 (1 - t_1^2)}$$  \hspace{1cm} (B4)
then

\[ I_3 = \frac{4A^{7/2}}{\pi^2} \int_0^1 \phi_1(t_1) dt_1 \int_0^1 (1 - t_2^3)^{3/2} \phi_2(t_1, t_2) dt_2 \]

\[ \times \int_0^1 (1 - t_2^3)^{3/2} \phi_2(t_2, t_3) dt_2 \int_0^1 (1 - t_4^3) \phi_2(t_3, t_4) dt_4 \]

\[ I_4 = \frac{4A}{\pi^2} \sqrt{2(1-A)(1-V)} \int_0^1 \phi_1(t_1) dt_1 \int_0^1 (1 - t_2^3)^{3/2} \phi_2(t_1, t_2) dt_2 \]

\[ I_5 = -\frac{4A}{\pi^2} \sqrt{2(1-V)} \int_0^1 \phi_1(t_1) dt_1 \int_0^1 t_2 \tan^{-1} \left( \frac{\sqrt{(1-A)(1-t_2^3)}}{t_2} \right) \phi_2(t_1, t_2) dt_2 \]

\[ I_6 = -\frac{8A^2}{\pi^3} \sqrt{2(1-A)(1-V)} \int_0^1 \phi_1(t_1) dt_1 \int_0^1 (1 - t_2^3)^{3/2} \phi_2(t_1, t_2) dt_2 \]

\[ \times \int_0^1 (1 - t_2^3)^{1/2} \phi_2(t_2, t_3) dt_3 \]

\[ I_7 = \frac{8A^2}{\pi^3} \sqrt{2(1-V)} \int_0^1 \phi_1(t_1) dt_1 \int_0^1 (1 - t_2^3)^{3/2} \phi_2(t_1, t_2) dt_2 \]

\[ \times \int_0^1 t_3 \tan^{-1} \left( \frac{\sqrt{(1-A)(1-t_2^3)}}{t_3} \right) \phi_2(t_2, t_3) dt_3. \]  

(B5)

Appendix C: second-order exact solution

We investigate the effect of the approximation used in (16) in determining the stress distribution at the back edge region. To do so, we again start with equations (11), (13) and (19) which give

\[ \alpha \sqrt{\frac{2V}{1-V}} = 2T_0 \sqrt{2V t_2} - \mu \int_0^{\eta P} \frac{u_x(\xi, \eta P; \xi)}{\sqrt{\xi Q - \xi}} d\xi \]  

(C1)

where

\[ t_2 = \frac{1}{2} (\xi Q + \eta P). \]

Equations (23b) and (23a) give

\[ u_x(\xi, \eta P; \xi) = \frac{2T_0 \xi^{1/2}}{\pi \mu (\eta P - \xi)^{1/2}} \left[ \sqrt{1-A} - \sqrt{\frac{\eta P - \xi}{\xi}} \tan^{-1} \sqrt{\frac{\xi (1-A)}{\eta P - \xi}} \right] \]

\[ - \frac{1}{\pi (\eta P - \xi)^{1/2}} \int_0^{A^\xi} \frac{u_x(\xi, \eta'; \eta') (\xi - \eta')^{1/2}}{\eta P - \eta'} d\eta', \quad 0 < \xi < \eta P \]  

(C2a)
and
\[ u_x(\xi, \eta ; \eta') = \frac{2T_0(A^{-1} \eta')^{1/2}}{\pi \mu(\xi - A^{-1} \eta')^{1/2}} \left[ \sqrt{1 - A} - \sqrt{\frac{\xi - A^{-1} \eta'}{A^{-1} \eta'}} \tan^{-1} \sqrt{\frac{\eta'(A^{-1} - 1)}{\xi - A^{-1} \eta'}} \right] \]
\[ - \frac{1}{\pi(\xi - A^{-1} \eta')^{1/2}} \int_0^{\eta'} \frac{u_x(\xi, \eta')}{\xi - \xi'} (A^{-1} \eta' - \xi')^{1/2} d\xi', \ 0 < \eta' < A\xi. \] (C2b)

Letting \( \xi \to A^{-1} \eta' \), we have
\[ \int_0^{\eta'} \frac{u_x(\xi', \eta')}{(A^{-1} \eta' - \xi')^{1/2}} d\xi' = \lambda \sqrt{\eta'} \] (C3)
where
\[ \lambda = \sqrt{\frac{1 - A}{A}} \left( 2T_0 - \frac{\alpha}{\sqrt{1 - V}} \right) / \mu. \] (C4)

Let
\[ u_x(\xi', \eta') = \xi' c(\eta')/[\eta' \sqrt{\eta' - \xi'}], \quad 0 < \xi' < \eta' \] (C5)
where \( c(\eta') \) is a function to be determined. Substituting (C5) in (C3) and simplifying we get
\[ c(\eta') = \lambda \sqrt{\eta'}/f(A) \] (C6)
where
\[ f(A) = \frac{(1 + A)}{2A} \ln \left[ \frac{1 - A}{1 + A - 2A^{1/2}} \right] - A^{-1/2}. \] (C7)

Hence
\[ u_x(\xi', \eta') = (C_T T_0 - C_\alpha \alpha) \xi'/[\mu \sqrt{\eta' (\eta' - \xi')} \] (C8)
where
\[ C_T = 4\sqrt{1 - A} \left[ A^{-1/2} (1 + A) \ln \left\{ \frac{1 - A}{1 + A - 2A^{1/2}} \right\} - 2 \right] \] (C9)
and
\[ C_\alpha = \frac{1}{2} \frac{C_T}{\sqrt{1 - V}} \] (C10)

(C1) to (C10) give
\[ \tilde{\alpha} = \frac{2\sqrt{V + \sqrt{2(1 - V)}} (I_2 - \sqrt{1 - A} I_1)/\pi + I_4 + I_5 - a I_6}{\sqrt{V(1 - V)} - b I_6} \] (C11)
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where \( I_1, I_2, I_4, I_5 \) are the same as described previously and \( I_6, a, b \) are given by

\[
I_6 = \int_0^1 \phi(t_1) \, dt_1 \int_0^1 (1 - t_2^3)^{1/2} \phi_2(t_1, t_2) \, dt_2 \int_0^1 (1 - t_3^3) \phi_2(t_2, t_3) \, dt_3
\]  
\[\text{(C12)}\]

\[
a = 16\sqrt{2}A^2 \sqrt{(1 - V)(1 - A)}/\pi^2 \left\{ \frac{1 + A}{A^{1/2}} \ln \left[ \frac{1 - A}{1 + A - 2A^{1/2}} \right] - 2 \right\}
\]

\[
b = \frac{a}{2\sqrt{1 - V}}.
\]

The second-order exact solution may be obtained by substituting \( I_6 = 0 \) in (C11) so that

\[
\tilde{a} = \frac{2[\sqrt{V} + \sqrt{2(1 - V)}(I_2 - \sqrt{1 - A} I_1)/\pi] + I_4 + I_5}{\sqrt{V}/(1 - V)}
\]  
\[\text{(C13)}\]