Excitation of normal modes on non-rotating and rotating earth models

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Summary. The method of spectral decomposition for linear operators, formulated in Dirac’s bra-ket notation, gives the excitation formulae for the normal modes of infinitesimal oscillations of a non-rotating earth. The formalism is then extended, in parallel with Lancaster’s λ-matrix treatment, to obtain the corresponding formulae for a rotating earth. The algebraic structure of these formulae is carefully examined based in Chao’s group-theoretical results, and some particular cases of earth models and sources are discussed.

1 Introduction

The problem of the excitation of normal modes of infinitesimal oscillations, or simply the ‘excitation problem’, belonging to a stable, non-dissipative, non-rotating earth model (henceforth referred to as a ‘non-rotating earth’) has been studied by various investigators in the last decade — among them, Gilbert (1970), Dahlen & Smith (1975, henceforth D & S), and Backus & Mulcahy (1976). In this respect, the non-rotating earth is simply a conservative, linear oscillating body whose mathematics are not different from those of a musical instrument. The excitation formula can be readily obtained by means of the common methods of classical mechanics, such as the Laplace transform or Green’s function technique.

Difficulties arise, however, when one deals with the excitation problem with respect to a stable, non-dissipative, uniformly rotating earth model (henceforth referred to as a ‘rotating earth’). The rotation gives rise to a Coriolis force which, in turn, introduces to the mathematics of the normal mode excitation a fundamental complication — the normal modes are no longer truly ‘normal’, or orthogonal, in the ordinary sense. Similar problems have, to the best of the author’s knowledge, received little attention in the classical literatures, probably because a spinning musical instrument is of little practical interest. However, to obtain an exact excitation formula for a rotating earth is important toward the understanding of such problems as the response of the Earth to tidal forces, the excitation of the Chandler wobble and other secular motions, the excitation and behaviour of the gravest elastic modes, and the inversion for the low-frequency behaviour of an earthquake.

Efforts to solve the excitation problem for a rotating earth have been made in recent years by D & S, Dahlen (1977, 1978, 1980) and Wahr (1980). However, as will be shown
in this paper, the method used by D & S and later by Dahlen (1977) for both non-rotating and rotating earth models is not justified and, hence, the formulae obtained there are incorrect. Dahlen (1978, 1980) revised the procedure and, with the help of some plausible assumptions, deduced the correct results. Wahr (1980), proposing an unconventional scalar product, also managed to obtain the equivalent results in an elegant fashion.

The purpose of this paper is to re-examine the whole problem, and to introduce yet another new systematic procedure that leads to exact, closed-form excitation formulae for both non-rotating and rotating earth models. The method used is the spectral decomposition for linear operators, and will be formulated in terms of Dirac’s (1958) bra-ket notation which is found to be elegant and extremely convenient. It should be emphasized that our argument and mathematical formalism are sufficiently general to be applied to any conservative, linear oscillating body. Therefore, we shall preferably speak of an ‘oscillating body’ or simply a ‘body’, non-rotating or rotating; and explicitly refer to the Earth only when so desired. By the same token, we shall use the general term ‘source’ or ‘source distribution’ to mean any forcing function, such as the tidal forces exerted on the Earth by the Sun and the Moon, or the equivalent body force during an earthquake.

In Section 2, the steady-state response of a non-rotating body to a simple-harmonic source is first obtained by means of the method of spectral decomposition. A superposition of such responses in the form of an integral then gives the impulse response; and the response to a general source distribution with arbitrary time history follows immediately from a time convolution. The algebraic structure of these formulae for a non-rotating earth having different symmetries is then examined. Section 3 extends the spectral decomposition formalism to yield the steady-state response of a rotating body. This extension parallels the treatment in Lancaster’s λ-matrix theory (Lancaster 1966). The impulse response and the response to a general source of the rotating body are then obtained in the same way as in Section 2, followed by an examination of the algebraic structure. It should be borne in mind, and will be pointed out from time to time, that the results obtained for the rotating case in Section 3 will, as they should, reduce to those in Section 2 for the non-rotating body in the limit of zero rotation.

2 Excitation of normal modes on a non-rotating earth

2.1 Mathematical formulation

The convenient Dirac bra-ket notation (Dirac 1958) has been extensively used in quantum mechanics. Here we shall develop the associated theory to the extent as to meet our present needs. Thus, we denote by |p⟩ a complex vector function p(r) defined over the volume V of the physical body under inspection (e.g. the Earth); and V, of course, is a subspace of the ordinary three-dimensional Euclidean space. |p⟩ is called a ket function, and is an element in a Hilbert space H. Corresponding to every ket function |p⟩ in H, there is a vector function called a bra function, denoted by ⟨p|, in the dual space of H. Here we will not distinguish between the Hilbert space H and its dual space (see Rietz & Sz.-Nagy 1955). The inner product between a bra function ⟨q| and a ket function |p⟩ is defined as:

⟨q | p⟩ ≡ ∫ V q*(r₀) · p(r₀) ρ(r₀) dV₀,

where the integration is over the volume V, ρ(r) is the mass density distribution of the body (and is everywhere non-negative), and * denotes the complex conjugate. The ket function resulting from a linear operator A acting on the ket function |p⟩ is written as A | p⟩. Simi-
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The bra function resulting from $A$ acting on the bra function $\langle q |$ is written as $\langle q | A$, defined by

$$\langle q | A | p \rangle = \langle q | \{ A | p \} \rangle.$$  \hspace{1cm} (2)

The scalar (2) can thus be written simply as $\langle q | A | p \rangle$ without confusion. Note that the expression $\langle p | q | A \rangle$ can be regarded as a linear operator.

Now consider the linear operator $(A_1 \lambda + A_0)$ formed by two linear operators $A_1, A_0$, and the scalar parameter $\lambda$, in conjunction with some boundary conditions on the boundaries of $V$. Suppose, as we will in this paper, that the spectrum of this operator is discrete, i.e. there exists a denumerable number of solutions $\{ \lambda_n, |p_n\}; n = 1, 2, 3, \ldots$ satisfying

$$(A_1 \lambda_n + A_0) | p_n \rangle = | 0 \rangle, \hspace{1cm} n = 1, 2, 3, \ldots$$  \hspace{1cm} (3a)

where $0$ denotes the null vector. Following Lancaster (1966), we shall call $\lambda_n$ a latent value of $(A_1 \lambda + A_0)$, and its corresponding ket function $| p_n \rangle$ a right latent function. Corresponding to equation (3a), we have

$$\langle p_n | (A_1 \lambda_n + A_0) = \langle 0 |, \hspace{1cm} n = 1, 2, 3, \ldots$$  \hspace{1cm} (3b)

where the bra function $\langle p_n |$ is called the left latent function associated with the latent value $\lambda_n$.

A vital reduction to equations (3a, b) is possible if the operators $A_1$ and $A_0$ are both self-adjoint, as we shall assume in this section. A self-adjoint operator $A$ satisfies the requirement that

$$\langle p | A^\dagger | q \rangle = \langle q | A | p \rangle^* = \langle p | A | q \rangle$$  \hspace{1cm} (4)

for all $| q \rangle$ and $| p \rangle$, where $\dagger$ denotes an adjoint operator. Equation (4) can simply be written as $A = A^\dagger$. It is then easy to see that $\langle p_n | A_1 | p_n \rangle$ and $\langle p_n | A_0 | p_n \rangle$ are both real numbers. Now from equation (3a) we have

$$\langle p_n | (A_1 \lambda_n + A_0) | p_n \rangle = \lambda_n \langle p_n | A_1 | p_n \rangle + \lambda_n \langle p_n | A_0 | p_n \rangle = 0.$$  \hspace{1cm} (5)

Therefore, $\lambda_n$ must be a real number as well. (This, incidentally, is a generalization of the well-known theorem that the eigenvalues of a self-adjoint operator are real.) As a result, if we take the complex conjugate of equation (3a), we get (Dirac 1958):

$$\langle p_n | (A_1^\dagger \lambda_n^* + A_0^\dagger) = \langle p_n | (A_1 \lambda_n + A_0) = | 0 |, \hspace{1cm} n = 1, 2, 3, \ldots$$  \hspace{1cm} (6)

Equation (6) now provides a solution to equation (3b), with $\lambda_n^* = \lambda_n$ and $\lambda_n = \lambda_n$. Suppose further that $A_1$ and $A_0$, in addition to being self-adjoint, are positive-definite, so that $\langle p_n | A_1 | p_n \rangle$ and $\langle p_n | A_0 | p_n \rangle$ are both positive numbers. Then from equation (5) we see that $\lambda_n$ must be negative. Thus, summarizing, we conclude that:

If $A_1$ and $A_0$ are two positive-definite, self-adjoint operators, then

(i) the latent values of $(A_1 \lambda + A_0)$ are all negative numbers,

(ii) the latent values of $(A_1 \lambda + A_0)$ associated with right latent functions are the same as those with left latent functions, and

(iii) the bra function corresponding to any right latent (ket) function is a left latent function belonging to the same latent value, and conversely.

Now suppose we want to solve the inhomogeneous equation

$$(A_1 \lambda + A_0) | S \rangle = | f \rangle$$  \hspace{1cm} (7)

for given $A_1, A_0, \lambda$, and $| f \rangle$. Formally, all we have to do is to find the inverse operator for $(A_1 \lambda + A_0)$. Note that the inverse operator $(A_1 \lambda + A_0)^{-1}$ exists only when $\lambda$ does not
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coincide with any of the latent values \( \lambda_n \) because only then will the operator \((A_1 + A_0)\) send only the null vector into the null vector.

To find \((A_1 + A_0)^{-1}\), we proceed as follows. From equations (3a) and (6), we can show, for properly normalized \(|p_n\rangle\) and \(|p_n\rangle\), that

\[
\langle p_m | A_1 | p_n \rangle = \delta_{mn} \tag{8a}
\]

\[
\langle p_m | A_0 | p_n \rangle = -\lambda_n \delta_{mn}, \tag{8b}
\]

where \(\delta_{mn}\) denotes the Kronecker delta. In Appendix A we show that the right latent functions \(|p_n\rangle; n = 1, 2, 3, \ldots\) from equation (3a) form a complete set for the Hilbert space \(H\). Then, it follows immediately from the orthonormality (8a) that

\[
|u\rangle = \sum_n |p_n\rangle \langle p_n | A_1 | u\rangle, \tag{9}
\]

for any ket function \(|u\rangle\) in \(H\). Equation (9) implies

\[
\sum_n |p_n\rangle \langle p_n | A_1 = I, \tag{10a}
\]

where \(I\) is the identity operator. Since the whole argument is valid with respect to the left latent functions \(|p_n\rangle; n = 1, 2, 3, \ldots\), we can similarly obtain

\[
\sum_n A_1 |p_n\rangle \langle p_n | = I, \tag{10b}
\]

which is nothing but the complex conjugate expression of equation (10a). Equations (10a, b) describe the closure property of the complete, orthonormal set of the latent functions. Finally, using equations (3a), (6), and (10a, b), we get the desired inverse operator for \((A_1 + A_0)\):

\[
(A_1 + A_0)^{-1} = \sum_n \frac{1}{\lambda - \lambda_n} |p_n\rangle \langle p_n |, \tag{11}
\]

in the sense that \((A_1 + A_0)^{-1} (A_1 + A_0) = (A_1 + A_0) (A_1 + A_0)^{-1} = I\), as long as the parameter \(\lambda\) does not equal any latent value \(\lambda_n\).

Equation (11) is called the spectral decomposition of the linear operator \((A_1 + A_0)^{-1}\). From it, we obtain the solution to equation (7):

\[
|S\rangle = (A_1 + A_0)^{-1} |f\rangle = \sum_n \frac{1}{\lambda - \lambda_n} \langle p_n | f \rangle |p_n\rangle. \tag{12}
\]

### 2.2 Formulae for the Excitation of Normal Modes

Having obtained equation (12), we will switch to the conventional notations currently used in the literature, mainly to facilitate our discussion and comparison. Thus, equation (12) becomes

\[
S(r) = \sum_n \frac{1}{\lambda - \lambda_n} (p^*_n, f) p_n(r) \tag{13}
\]

where the inner product, now written as \((,\), is defined by \((q^*, p) = \langle q | p \rangle\).

Now consider a conservative, linear body whose normal modes of infinitesimal oscillations can be described by the following eigenvalue equation:

\[
(L - \omega^2_n) u_n(r) = 0, \quad n = 1, 2, 3, \ldots \tag{14}
\]

where \(L\) is a positive-definite, self-adjoint linear operator, \(\omega_n\) is the eigenfrequency of the \(n\)th mode, and \(u_n(r)\) the corresponding eigenfunction satisfying the boundary conditions. In
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the case of an earth model, \( L \) is a second-order tensor operator. In fact, equation (14) is usually expressed equivalently as a set of simultaneous linear equations (see e.g. Dahlen 1968); and all we have done here to obtain equation (14) and its 'composite' operation \( L \) is simply putting these linear equations together formally. Thus \( L \) has built in it a linear stress-strain relation (Hooke's law), the gravitational law, the principle of conservation of mass, and an earth model. For this reason \( L \) will be called the 'elastic-gravitational' operator. The non-trivial problem when \( L \) is only positive semi-definite will not be addressed in this paper (see Section 4).

Equation (14) is recognized as a special case of equation (3a) in that \( A_1 = I, A_0 = L, \lambda_n = -\omega_n^2, \) and \( p_n(r) = u_n(r). \) Now since both \( A_1 \) and \( A_0 \) are indeed positive-definite and self-adjoint, it follows from Section 2.1 that \( \lambda_n \) must be negative. The eigenfrequencies \( \omega_n \) are therefore real and non-zero. Moreover, equations (8a) and (8b) both reduce to the ordinary orthonormality relation:

\[
(u^*_m, u_n) = \delta_{mn}. \tag{15}
\]

Equation (13), then, immediately gives the solution to the inhomogeneous equation of motion

\[
(L - \omega^2) S(r) = f(r), \tag{16}
\]

which describes the steady-state response of the oscillating body to a simple-harmonic source distribution \( f(r) \exp(i\omega t) \) with constant driving frequency \( \omega \) not equal to any eigenfrequency of the body. Thus, if we denote by \( S(r, t) \) the total displacement field due to this source, then \( S(r, t) \) will have the form

\[
S(r, t) = S(r) \exp(i\omega t)
= \sum_n \frac{1}{\omega_n^2 - \omega^2} (u^*_n, f) u_n(r) \exp(i\omega t), \quad \text{for a simple-harmonic source.} \tag{17}
\]

This equation is directly applicable to the study of, for example, the tidal excitation of normal modes on a non-rotating earth.

By properly superposing the solutions (17), we can obtain the impulse response of the oscillating body, i.e. the response to an impulsive source distribution \( f(r) \delta(t) \), where \( \delta(t) \) denotes the Dirac delta function. The result (see Morse & Feshbach 1953, pp. 849–850) is

\[
S(r, t) = \sum_n \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{\omega_n^2 - \omega^2} d\omega (u^*_n, f) u_n(r). \tag{18}
\]

The singularities \( \pm \omega_n(n = 1, 2, 3, \ldots) \) lie on the real axis on the complex \( \omega \)-plane and must be bypassed in the integration. We choose the integration contour parallel to the real axis and just below it, while closing the contour in the upper half plane for \( t > 0 \) and in the lower half plane for \( t < 0 \). This particular choice of the integration contour is determined solely by the physical principle of causality. It is not required by the mathematics and, furthermore, is independent of any argument involving attenuation of the normal modes. Thus from equation (18) the theorem of residue gives

\[
S(r, t) = \sum_n \frac{\sin \omega_n t}{\omega_n} (u^*_n, f) u_n(r) H(t), \quad \text{for an impulsive source} \tag{19}
\]

where \( H(t) \) is the Heaviside step function. Examples of impulsive forces exerted on the Earth include nuclear explosions and meteorite impacts.

Now suppose that the body is at rest in the equilibrium position prior to time \( t = 0 \), when a source distribution with an arbitrary time history for \( t > 0 \) starts to act. The response of
the oscillating body to such a source distribution $f(r, t)$ is given by the time convolution of the impulse response with the source. Thus,

$$S(r, t) = \sum_n S_n(r, t) = \sum_n u_n(t) \int_0^t \frac{\sin \omega_n (t - \tau)}{\omega_n} f_n(\tau) d\tau, \quad t > 0$$

(20a)

where

$$f_n(t) = (u_n, f) = \int_V u_n^*(r_0) \cdot f(r_0, t) \rho(r_0) dV_0.$$ 

(20b)

and $S_n(r, t)$ denotes the displacement field associated with the $n$th mode. Anticipating its use in the next two sections, let us put $S_n(r, t)$ in a more concise and suggestive form:

$$S_n(r, t) = C_n(t) \ast \int_V f(r_0, t) \cdot G_n(r, r_0) \rho(r_0) dV_0.$$ 

(21)

where $\ast$ denotes the convolution with respect to time, $C_n(t)$ is the impulse response of the $n$th mode of the oscillating body:

$$C_n(t) = \frac{1}{\omega_n} \sin \omega_n t,$$

(21a)

and $G_n(r, r_0)$ is a tensor of rank 2, which we shall call the 'spatial Green tensor' for the $n$th mode:

$$G_n(r, r_0) = u_n^*(r_0) u_n(r).$$

(21b)

Note, in passing, that $G_n(r, r_0)$ satisfies the reciprocity relation: $G_n(r, r_0) = G_n^*(r_0, r)$.

Equations (20a, b) appear widely in the literature of classical acoustics (see, e.g. Rayleigh 1877; Morse & Feshbach 1953). Appendix B gives the alternative (and conventional) ways of derivation by means of the Laplace transform and Green's function technique. Equations (20a, b) describe, for our purpose, the excitation of normal modes on a non-rotating earth due to, say, the body force distribution $f(r, t)$ of an earthquake. They are first introduced to earthquake source study by Backus & Mulcahy (1976).

D & S studied the same problem using the Laplace transformation. However, having obtained the basic equation (B5) (see Appendix B), they employed, instead of the convolution theorem, the inversion formula:

$$a_n(t) = \frac{1}{2\pi i} \int_B \frac{f_n(p)}{p^2 + \omega_n^2} \exp(pt) dp,$$ 

(22)

where $B$ denotes a Bromwich contour on the complex $p$-plane. They then evaluated the integral (22) by finding its residues; and that led to their formula (53), a result inconsistent with equations (20a, b). There are certain objections to their procedure, namely:

1. $f_n(p)$ may well have its own poles. For example, if the time history of the source is described by the function $1 - \exp(-at)$, then $f_n(p)$ has two simple poles, one at $p = 0$ and one at $p = -a$. For an arbitrary time history, however, there is no telling how many and where the poles of $f_n(p)$ will be. The single simple pole at $p = 0$ proposed by D & S is subjective and artificial. Basically the implication of this objection is that while it is legitimate to use formula (22), it is simply not fair to draw any conclusions from there on.

2. Proceeding, D & S derived their final equation (53) for the total displacement field $S(r, t)$. Another erroneous relation, namely $f_n(-i\omega) = f_n^*(i\omega)$ (which is true only when the eigenfunctions $u_n(r)$ are all chosen real, see below) has been used in the derivation. That
being remediable, equation (53) of D \& S still suffers from its involvement with the quantity 
\[
\int_0^\infty f(r, \tau) \exp \left[ \pm i \omega (t - \tau) \right] d\tau,
\]
a relation that is non-causal. For a causal system, \( S(r, t) \) must depend only on the past history of the source function \( f(r, \tau) \) in the time interval \( 0 < \tau < t \) (instead of \( 0 < \tau < \infty \)), exactly as stated by equation (20a). The implication is that D \& S's results is objectionable simply on the ground that it is non-causal and, hence, physically unrealistic.

So far we have taken the eigenfunctions \( u_n(r) \) to be complex. Equations (19) and (20a), therefore, appear to be complex as well. This, of course, is not the case since \( S(r, t) \) is a physically measurable quantity for any real source \( f(r, t) \). We shall discuss this in detail in the next two sections.

### 2.3 An Arbitrary Earth Model: Non-Degenerate Case

Chao (1981), using group theory, has given a thorough investigation on the structure of the normal mode spectra belonging to earth models having different symmetries. It is shown that an arbitrary earth, here defined as a non-rotating earth model having no geometrical symmetries whatsoever, has and only has one-dimensional eigenspaces. That is, all the normal mode eigenfrequencies of an arbitrary earth are non-degenerate. The only possible degeneracies are 'accidental' ones which, being the consequence of some 'hidden' symmetry, are also assumed to be absent.

However, at first glance, because of the reality of the eigenfrequencies and the equation of motion (14) for the normal modes of a non-rotating earth, one may argue that if \( u_n(r) \) is an eigenfunction associated with the eigenfrequency \( \omega_n \), so is its complex conjugate \( u_n^*(r) \). Then one may conclude that each eigenspace belonging to an arbitrary earth is spanned by the pair of eigenfunctions \( u_n(r) \) and \( u_n^*(r) \), and, hence, is generally two-dimensional. This argument, as put forward by D \& S, is in contradiction to Chao's (1981) results. The key to resolve this paradox is that although both \( u_n(r) \) and \( u_n^*(r) \) are indeed decent normal mode solutions to the system, they are not linearly independent. In other words, the real and imaginary parts of \( u_n(r) \) are linearly dependent. The deceptive dimensionality of 2 is the price one pays when one agrees to make the eigenfunctions complex (for whatever reason).

Let us use the sub-index \( n \) to label the eigenspaces. In the present case, \( n \) continues to be the label for the normal mode eigenfunctions because now the eigenspaces are all one-dimensional. We shall allow for complex eigenfunctions and call a complex function trivially complex if its real and imaginary parts are linearly dependent, and non-trivially complex otherwise. Thus, the normal mode eigenfunctions belonging to an arbitrary earth are trivially complex. Any such function \( u_n(r) \), when normalized, differs from a normalized real function, say \( v_n(r) \), only by an unimportant phase factor \( \exp(i\theta_n) \):

\[
u_n(r) = \exp(i\theta_n) v_n(r).
\]  (23)

In the light of this, there is only one way whereby we obtain real seismograms. Associated with every eigenfunction \( u_n(r) \), there is exactly one (non-degenerate) eigenvalue \( \omega_n^2 \), which gives rise to two eigenfrequencies, \( \pm \omega_n \). Thus, within the \( n \)th (one-dimensional) eigenspace \( \{ u_n(r), \pm \omega_n \} \), we must linearly combine the pair \( u_n(r) \exp(i\omega_n t) \) and \( u_n^*(r) \exp(-i\omega_n t) \) in the following way:

\[
\frac{1}{2} \left[ \alpha_n(t) u_n(r) \exp(i\omega_n t) + \alpha_n^*(t) u_n^*(r) \exp(-i\omega_n t) \right] = \text{Re} \left[ \alpha_n(t) u_n(r) \exp(i\omega_n t) \right]
\]  (24)
where $\omega_n(t)$ is the complex amplitude with respect to the eigenfunction $u_n(r)$. Equation (24) represents a standing wave. This can be shown by substituting equation (23) into equation (24), so that the latter can be expressed as a multiplication of a real temporal part and a real spatial part, which, of course, is a standing wave.

Now we are in the position to relate equation (24) to Section 2.2. First we note that because $u_n(r)$ is trivially complex, the spatial Green tensor $G_n(r, r_0)$ (equation 21b) and, hence, $S_n(r, t)$ (equation 21) will always be real – in a rather trivial fashion. Equation (20a) can then be put in a form complying with equation (24). Thus,

$$S_n(r, t) = u_n(r) \int_0^t \text{Re} \left[ \exp \left( \frac{i\omega_n(t - \tau)}{i\omega_n} \right) \right] f_n(\tau) d\tau.$$  

$$= \text{Re} \left[ u_n(r) \left( \int_0^t \frac{\exp(-i\omega_n \tau)}{i\omega_n} f_n(\tau) d\tau \right) \exp(i\omega_n t) \right]. \quad (25)$$

because now the quantity $u_n(r)f_n(t)$ is real. By comparing equation (25) with equation (24), we obtain the complex amplitude of the $n$th normal mode:

$$\alpha_n(t) = \int_0^t \frac{\exp(-i\omega_n \tau)}{i\omega_n} f_n(\tau) d\tau. \quad (26)$$

Equation (26) shows explicitly the dependence of the complex amplitude of the normal modes on time, as well as on the history of the source. Note also that this expression is causal, as it should be.

It is interesting to study the excitation of normal modes due to sources of a particular type. For a source having a step-function time history, i.e. when $f(r, t) = f_\infty(r) H(t)$, equation (20a) reduces to the result first obtained by Gilbert (1970):

$$S_n(r, t) = f_\infty^\infty \frac{1 - \cos \omega_n t}{\omega_n^2} u_n(r), \quad t > 0 \quad (27)$$

where $f_\infty^\infty = (u_n^*, f_\infty)$. Next consider a source which starts to act at $t = 0$ and attains its final static value $f_\infty(r)$ at $t = T$. (In the case of an earthquake, $T$ is its ‘duration’.) For $0 < t < T$, all we can say is that each normal mode undergoes a forced oscillation according to the time history of the source, as formally described by equation (20a). For $t > T$, however, we decompose the time integration of equation (25) into two parts:

$$\int_0^T + \int_T^t.$$  

Equation (25) then becomes

$$S_n(r, t) = \text{Re} \left\{ u_n(r) \left[ \left( \alpha_n(T) - \frac{f_\infty^\infty}{\omega_n^2} \exp(-i\omega_n T) \right) \exp(i\omega_n t, + \frac{f_\infty^\infty}{\omega_n^2}) \right] \right\}, \quad t > T \quad (28)$$

where $\alpha_n(t)$ is given in equation (26). When there is a small amount of dissipation, small enough that it does not appreciably alter the equations of motion, we replace $\omega_n$ by $\omega_n(1 + i/2Q_n)$, with $Q_n > 1$. Then the static displacement as $t \to \infty$, according to equation (28), is

$$S_{\text{stat}}(r) = \sum_n \frac{f_\infty^\infty}{\omega_n^2} u_n(r). \quad (29)$$

Equation (29) is, again, due to Gilbert (1970). It is the static configuration of the body in mechanical equilibrium with the final static force $f_\infty(r)$. Thus, equation (28) states that,
after time $T$, each normal mode undergoes the free oscillation at its natural frequency $\omega_n$ (or $\omega_n(1 + i/2Q_n)$ if dissipation is included) about the equilibrium level $f_n^\infty/\omega_n^2$ with complex amplitude of oscillation $a_n(T) = (f_n^\infty/\omega_n^2) \exp(-i\omega_n T)$.

### 2.4 Symmetrical Earth Models: Degenerate Cases

In the preceding section we saw that the structure of the normal mode spectrum belonging to an arbitrary earth is rather simple. Complications arise, and modifications are necessary when the earth model under consideration possesses some degree of symmetry. Chao (1981) has shown how symmetry gives rise to degeneracies in the spectrum. We shall, again, use the sub-index $n$ to label the eigenspaces, and concentrate on a particular degenerate eigenspace (or ‘multiplet’) of dimension $M$, $M > 1$. The set of $M$ (linearly independent) eigenfunctions belonging to the $n$th degenerate eigenspace will then be

$$\{u_{nm}(r); m = 1, 2, \ldots, M\}.$$  

(30)

Suppose we have orthonormalized the set (30) using, say, the Gram-Schmidt procedure, such that

$$\langle u_{nm_1}, u_{nm_2} \rangle = \delta_{m_1, m_2}.$$  

(31)

In general, these orthonormal eigenfunctions may be made complex, and if so, non-trivially. With this choice, the excitation formula (21) is still valid for the $n$th multiplet with, however, a different form for the spatial Green tensor:

$$G_n(r, r_0) = \sum_{m=1}^{M} u_{nm}(r_0) u_{nm}(r).$$  

(32)

Equation (32) is the generalization of equation (21b) to the degenerate case. It must be real in order that the displacement field $S_n(r, t)$ is real for a real source distribution $f(r, t)$. That this is indeed true can be shown as follows:

Suppose we have been given an orthonormal set of $M$ real eigenfunctions $\{v_{nm}(r); m = 1, 2, \ldots, M\}$ (which always exists) spanning the $n$th eigenspace. Then any orthonormal set (30) can be obtained from it through a unitary transformation:

$$U_n(r) = R V_n(r)$$  

(33)

where

$$U_n(r) = \begin{pmatrix} u_{n1}(r) \\ u_{n2}(r) \\ \vdots \\ u_{nM}(r) \end{pmatrix}, \quad V_n(r) = \begin{pmatrix} v_{n1}(r) \\ v_{n2}(r) \\ \vdots \\ v_{nM}(r) \end{pmatrix}$$  

(34)

and $R$ is an $M \times M$ unitary matrix, i.e. $R^\dagger R = RR^\dagger = I_M$, $I_M$ being the $M \times M$ identity matrix. Then the assertion that $G_n(r, r_0)$ in equation (32) is real follows immediately from the following equation:

$$G_n(r, r_0) = \sum_{m=1}^{M} u_{nm}(r_0) u_{nm}(r) = U_n(r_0)^\dagger U_n(r) = V_n(r_0)^\dagger V_n(r) = \sum_{m=1}^{M} v_{nm}(r_0) v_{nm}(r).$$  

(35)
In fact, equation (35) also shows that, for given \( r_0 \) and \( r \), the spatial Green tensor (32) is \textit{invariant}, as well as real, regardless of the choice of the orthonormal eigenfunctions. This, of course, is to be expected from physical reasoning because the excitation of a degenerate multiplet must not depend upon the basis functions we choose to span the eigenspace. Note also that equation (33) is the generalization of equation (23) to the degenerate case.

Thus, unlike that for the non-degenerate normal modes (equation 21b), the spatial Green tensor for the multiplet (equation 32) is real in a rather non-trivial manner. The orthonormal eigenfunctions (30) are no longer trivially complex; nor do they necessarily appear in complex conjugate pairs, as often mistakenly inferred. Yet the spatial Green tensor (32) formed by them is always real and invariant.

Two particular cases of symmetric non-rotating earth models are of interest:

(1) Spherically symmetric earth: As shown by Chao (1981), the eigenspaces belonging to a spherically symmetric earth are \((2l + 1)\)-dimensional, where \( l \in \{0, 1, 2, \ldots\} \). Because of the nature of the spherical symmetry, each \((2l + 1)\)-dimensional eigenspace is spanned by the \(2l + 1\) vector spherical harmonics of angular order \( l \) with the azimuthal-angular order \( m \) assuming values \( \{0, \pm 1, \ldots \pm l\} \). These eigenfunctions are orthonormal, and they do appear in complex conjugate pairs. Thus, the spatial Green tensor is real in the manner that it is a sum, performed in the \( n \)th degenerate eigenspace, of a real term (corresponding to \( m = 0 \)) and \( l \) pairs of complex conjugate terms (corresponding to \( \pm m \neq 0 \)). Note also that for the same reason, the excitation energy within a multiplet is partitioned symmetrically with respect to \( m \), i.e. the energy associated with \( u_{nm}(r) \) is the same as that with \( u_{n,m}(r) \).

(2) Axial-symmetric earth: again as shown by Chao (1981), the eigenspaces belonging to an axial-symmetric earth are either one-dimensional or two-dimensional. Thus,

\[
G_n(r, r_0) = \sum_{m=1,2} u_{nm}^*(r_0) u_{nm}(r) \quad (36)
\]

is real in the manner that it is a sum, performed in the \( m \)th degenerate eigenspace, of a real term (corresponding to \( m = 0 \)) and \( l \) pairs of complex conjugate terms (corresponding to \( \pm m \neq 0 \)). Note also that for the same reason, the excitation energy within a multiplet is partitioned symmetrically with respect to \( m \), i.e. the energy associated with \( u_{nm}(r) \) is the same as that with \( u_{n,-m}(r) \).

We point out here that while in the latter case it is not necessarily true that \( u_{n1}(r) \) and \( u_{n2}(r) \) form a complex conjugate pair, Dahlen (1968) has shown that they may be made so at least to zeroth order of the spheroidicity. As a result, the excitation energy is partitioned evenly between such two degenerate modes, at least to zeroth order of the spheroidicity.

3 Excitation of normal modes on a rotating earth

3.1 Mathematical formulation

In this section we wish to solve the inhomogeneous equation of an operator \textit{quadratic} in \( \lambda \):

\[
(A_2 \lambda^2 + A_1 \lambda + A_0) |S\rangle = |f\rangle
\]

for given linear operators \( A_2, A_1, A_0 \), scalar parameter \( \lambda \), and \(|f\rangle\). We shall again assume a discrete spectrum and explore the method of spectral decomposition. Specifically, here we are only interested in the case where (1) both \( A_2 \) and \( A_0 \) are positive-definite and self-adjoint, and (2) \( A_1 \) is anti-self-adjoint (or, equivalently, \( iA_1 \) is self-adjoint). Again, the implication of the case where \( A_0 \) is positive \textit{semi}-definite is far from trivial, and will not be discussed in this
paper (see Section 4). Thus, corresponding to equations (3a, b), we have the following quadratic latent value problems:

$$(A_2 \lambda_n^2 + A_1 \lambda_n + A_0) | p_n \rangle = | 0 \rangle \quad (39a)$$

$$\langle p'_n | (A_2 \lambda_n'^2 + A_1 \lambda_n' + A_0) = \langle 0 | \quad (39b)$$

where $\lambda_n$ and $\lambda_n'$ are latent values of $(A_2 \lambda^2 + A_1 \lambda + A_0)$, $| p_n \rangle$ and $\langle p'_n |$ are the corresponding right and left latent functions, respectively. Taking the scalar product of $\langle p_n |$ with equation (39a), we get

$$a_2 \lambda_n^2 + a_1 \lambda_n + a_0 = 0, \quad (40)$$

where $a_j = \langle p_n | A_j | p_n \rangle$, $j = 0, 1, 2$. Now because of the stipulations (1) and (2) above, $a_2$ and $a_0$ are positive numbers, and $a_1$ is pure imaginary. Hence, the solution $\lambda_n$ to equation (40) must be pure imaginary and non-zero:

$$\lambda_n = \left(-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}\right)/2a_2 = i(a_1 \pm \sqrt{-a_1^2 + 4a_2a_0})/2a_2. \quad (41)$$

As a result, taking the complex conjugate of equation (39a) gives

$$\langle p_n | (A_2^\dagger \lambda_n - (iA_1)^\dagger (i\lambda_n) + A_0^\dagger) = \langle p_n | (A_2 \lambda_n^2 + A_1 \lambda_n + A_0) = \langle 0 |. \quad (42)$$

As in Section 2.1, a comparison of equation (42) with equation (39b) thus establishes the one-to-one correspondence between the right and left latent functions $| p_n \rangle$ and $\langle p_n |$ associated with the same (pure imaginary) latent value $\lambda_n$ of $(A_2 \lambda^2 + A_1 \lambda + A_0)$.

Now we are ready to find the inverse operator for $(A_2 \lambda^2 + A_1 \lambda + A_0)$, when $\lambda$ does not equal to any $\lambda_n$, in order to solve equation (38). To accomplish this, we follow the treatment proposed in Lancaster's (1966) $\lambda$-matrix theory, and modify it in reconciliation with the fact that we are dealing with an infinite-dimensional Hilbert space. Thus, the crucial step is to write equations (39a) and (42) in the following matrix form:

$$(B \lambda_n + C) | P_n \rangle = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (43a)$$

$$\langle P_n | (B \lambda_n + C) = \begin{bmatrix} \langle 0 | \\ \langle 0 | \end{bmatrix}, \quad (43b)$$

where $B$ and $C$ are $2 \times 2$ operator matrices:

$$B = \begin{bmatrix} 0 & -A_2 \\ A_2 & A_1 \end{bmatrix}, \quad C = \begin{bmatrix} A_2 & 0 \\ 0 & A_0 \end{bmatrix}, \quad (44a)$$

and $| P_n \rangle$ and $\langle P_n |$ are the bra-ket pair (remember that $\lambda_n$ is pure imaginary) defined as

$$| P_n \rangle \equiv \begin{bmatrix} \lambda_n | p_n \rangle \\ | p_n \rangle \end{bmatrix}, \quad \langle P_n | \equiv \begin{bmatrix} -\lambda_n \langle p_n | \\ \langle p_n | \end{bmatrix}. \quad (44b)$$

From equations (43a, b), we get the orthonormality relations

$$\langle P_m | B | P_n \rangle = 2\lambda_n \delta_{mn} \quad (45a)$$

$$\langle P_m | C | P_n \rangle = -2\lambda_n^2 \delta_{mn} \quad (45b)$$

for properly normalized $\langle P_n |$ and $| P_n \rangle$. Here the normalization is chosen in such a way that the results to be obtained below for a rotating body will reduce to the corresponding results given in Sections 2.2 and 2.3 in the limit of zero rotation (see, particularly, equation 55).
To proceed as in Section 2.1, we now wish to prove the completeness of the set \( \{ |P_n\rangle; n = 1, 2, 3, \ldots \} \) in the product space \( H \times H \). First we note that the two operators \( B \) and \( C \) in equation (43a) are so chosen that \( B \) is anti-self-adjoint while \( C \) is diagonal, self-adjoint and positive-definite. Thus, if we express \( B\lambda_n \) as \( (ib)(-i\lambda_n) \), and note that \( iB \) is self-adjoint (and the non-zero real quantity \( -i\lambda_n \) becomes the latent value), then Appendix A leads readily to the conclusion that \( \{ |P_n\rangle; n = 1, 2, 3, \ldots \} \) (and hence \( \{ \langle P_n |; n = 1, 2, 3, \ldots \} \) form a complete set for \( H \times H \). The closure property of this complete, orthonormal set, i.e. the counterpart of equations (10a, b), becomes

\[
\sum_n \frac{1}{2\lambda_n} |P_n\rangle \langle P_n| B = I_2, \tag{46a}
\]

and its complex conjugate relation (note that the operator \( \frac{1}{\lambda_n} B \) is self-adjoint since \( \lambda_n \) is pure imaginary)

\[
\sum_n \frac{1}{2\lambda_n} B |P_n\rangle \langle P_n| = I_2 \tag{46b}
\]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix. Then, from equations (43a, b) and (46a, b), we obtain the spectral decomposition for the matrix operator \( D \equiv (B \lambda + C)^{-1} \).

\[
D \equiv (B \lambda + C)^{-1} = \sum_n \frac{1}{2\lambda_n(\lambda - \lambda_n)} |P_n\rangle \langle P_n|, \tag{47}
\]

in the sense that

\[
D (B \lambda + C) = (B \lambda + C) D = I_2, \tag{48}
\]

as long as \( \lambda \) does not coincide with any latent value \( \lambda_n \). Equation (48), when written out explicitly, gives

\[
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
A_2 & -A_2 \lambda \\
A_2 \lambda & A_1 + A_0
\end{bmatrix}
= \begin{bmatrix}
A_2 & -A_2 \lambda \\
A_2 \lambda & A_1 + A_0
\end{bmatrix}
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}. \tag{49}
\]

It is an easy exercise to show from equation (49) that

\[
D_{22}(A_2 \lambda^2 + A_1 \lambda + A_0) = (A_2 \lambda^2 + A_1 \lambda + A_0) D_{22} = I. \tag{50}
\]

Then from equations (47) and (50), we obtain the spectral decomposition for \( (A_2 \lambda^2 + A_1 \lambda + A_0)^{-1} \):

\[
D_{22} = (A_2 \lambda^2 + A_1 \lambda + A_0)^{-1} = \sum_n \frac{1}{2\lambda_n(\lambda - \lambda_n)} |P_n\rangle \langle P_n|. \tag{51}
\]

The right side of equation (51), apart from the normalization factor \( 1/2\lambda_n \), is formally identical to that of equation (11). The difference, of course, lies in that the latent value \( \lambda_n \) and the latent functions \( |p_n\rangle \) and \( \langle p_n| \) in equation (51) belong to the operator \( (A_2 \lambda^2 + A_1 \lambda + A_0) \) (equations 39a and 42), while in equation (11) they belong to the operator \( (A_1 \lambda + A_0) \) (equations 3a and 6).

Finally, equation (38) can be solved:

\[
|S\rangle = (A_2 \lambda^2 + A_1 \lambda + A_0)^{-1} |f\rangle = \sum_n \frac{1}{2\lambda_n(\lambda - \lambda_n)} \langle p_n | f \rangle |p_n\rangle. \tag{52}
\]

Equation (52) is, again, formally identical to equation (12).
3.2 FORMULAE FOR THE EXCITATION OF NORMAL MODES

As in Section 2.2, we now rewrite equation (52) in terms of conventional notations. Thus,

$$S(r) = \sum_n \frac{1}{2\lambda_n(\lambda - \lambda_n)} (p_n^*, f) p_n(r). \quad (53)$$

Now consider the case where the conservative, linear oscillating body discussed in Section 2.2 is under a stable rotation with constant angular velocity $\Omega$ (see Section 4). From now on, unless stated otherwise, the oscillating body under consideration is of no geometrical symmetry (such as an arbitrary earth) so that the discussions in Section 2.2 need not be modified as done in Section 2.4 to take into account any possible degeneracies. We shall designate as the 'rotating frame' the reference frame that is rotating uniformly at the angular velocity $\Omega$, one that coincides with the 'body-axes' fixed relative to the uniformly rotating body.

In the rotating frame, the eigenvalue equation for the normal modes of oscillation, i.e. the rotational counterpart of equation (14), is

$$(L + i\omega_n Q - \omega_n^2) u_n(r) = 0, \quad n = 1, 2, 3, \ldots \quad (54)$$

where $Q \equiv 2\Omega x$ is the Coriolis operator. The elastic-gravitational operator $L$ here differs from that in equation (14) by a centrifugal force term. D & S have shown that both $L$ and $iQ$ are self-adjoint and that $L$ is, in addition, positive-definite for elastic modes on a stably rotating body. Thus, equation (54) is a special case of equation (39a) provided that we let $A_2 = I$, $A_1 = Q$, $A_0 = L$, $\lambda_n = i\omega_n$, and $p_n(r) = u_n(r)$. Now that, indeed, (1) $A_2$ and $A_0$ are positive-definite and self-adjoint, and (2) $iA_1$ is self-adjoint, the theory developed in Section 3.1 applies. In particular, we conclude that the eigenfrequencies $\omega_n$ of the rotating body are all real and non-zero. Physically, this is a direct consequence of the observation that neither the centrifugal force nor the Coriolis force is dissipative. Furthermore, some algebra will show that the orthonormality relations (45a, b) both now reduce to the 'quasi-orthonormality' relation for the normal modes of an oscillating body under rotation, first obtained by D & S:

$$(\omega_m + \omega_n) (u_m^*, u_n) - 2i (u_m^*, \Omega \times u_n) = 2\omega_n \delta_{mn}. \quad (55)$$

Equation (55) gives explicitly the normalization we use for later development. It is obvious that equation (55) reduces to the ordinary orthonormality (15) in the limit of zero rotation, $\Omega = 0$.

We are now ready to solve the inhomogeneous equation of motion in the rotating frame, i.e. the rotational counterpart of equation (16), which reads:

$$(L + i\omega Q - \omega^2) S(r) = f(r). \quad (56)$$

This equation describes the steady-state response of the rotating body to a simple-harmonic source distribution $f(r) \exp(i\omega t)$ with driving frequency $\omega$ not equal to any eigenfrequency of the body. The solution is immediately given by equation (53):

$$S(r, t) = S(r) \exp(i\omega t)$$

$$= \sum_n \frac{1}{2\omega_n(\omega_n - \omega)} (u_n^*, f) u_n(r) \exp(i\omega t), \quad \text{for a simple-harmonic source.} \quad (57)$$
Next, as in Section 2.2, we obtain the impulse response of the rotating body by a prescribed contour integration of equation (57). That results in

\[ S(r, t) = \sum_n \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{2\omega_n(\omega_n - \omega)} \, d\omega \left( u_n^*(r, f) \right) u_n(r) \]

\[ = \sum_n \frac{\exp(i\omega_n t)}{2i\omega_n} \left( u_n^*, f \right) u_n(r) H(t), \quad \text{for an impulsive source.} \]  

(58)

Finally, the response of the rotating body to any source distribution \( f(r, t) \) for \( t \geq 0 \) is given by the time convolution:

\[ S(r, t) = \sum_n S_n(r, t) = \sum_n u_n(r) \int_0^t \frac{\exp[i\omega_n(t-\tau)]}{2i\omega_n} f_n(\tau) \, d\tau, \quad t \geq 0 \]  

(59a)

where

\[ f_n(t) = \left( u_n^*, f \right) = \int_V u_n^*(r_0) \cdot f(r_0, t) \rho(r_0) \, dV_0. \]  

(59b)

Here, again, we have assumed that prior to \( t = 0 \) the body is at rest in the equilibrium position relative to the rotating frame. Equations (59a, b) are the rotational counterpart of equations (20a, b).

The treatment of this problem by D & S using Laplace transformation is again incorrect because of the same pitfalls mentioned in Section 2.2. A revised procedure following Dahlen (1978, 1980) is given in Appendix C. It inevitably suffers from a complication due to mode coupling whose root lies in the quasi-orthonormality (55).

Equations (58) and (59a), again, appear to give complex displacement field \( S(r, t) \). The fact that \( S(r, t) \) is yet indeed real is established in the next section.

3.3 Rotating Earth Model: Non-Degenerate Case

Chao (1981) has shown that in the absence of accidental degeneracies, the eigenfrequencies of the normal modes belonging to a rotating earth are in general non-degenerate. Even for an otherwise spherically symmetric earth, the introduction of a rotation will completely remove the degeneracies.

Thus, every eigenspace belonging to a rotating earth is one-dimensional, spanned by a single eigenfunction \( u_n(r) \). However, unlike the non-rotating case, \( u_n(r) \) is in general non-trivially complex because of the presence of the complex Coriolis force term in the equation of motion (54). It has been pointed out by D & S that the eigenspaces appear in natural pairs in the sense that if \( \{ u_n(r), \omega_n \} \) is an eigensolution to equation (54), then so is \( \{ u_n^*(r), -\omega_n \} \). Note, incidentally, that this pair of eigensolutions satisfies the quasi-orthonormality (55) (by setting \( u_n = u_n^*(r), \omega_m = -\omega_n \)) in a trivial manner — both sides vanish identically.

When all the pairs of eigensolutions are grouped together, equation (57) becomes

\[ S(r) \exp(i\omega t) = \sum_n \frac{1}{2\omega_n} \left[ \frac{1}{\omega_n - \omega} (u_n^*(r, f) u_n(r)) + \frac{1}{\omega_n + \omega} (u_n(r, f) u_n^*(r)) \right] \exp(i\omega t) \]  

(60)

for a simple-harmonic source, where the sub-index \( n \) now labels the eigenspace pairs. Equation (60) describes the tidal excitation of the normal modes when observed on a rotating earth. It is the rotational counterpart of equation (17). However, unlike equation (17), the bracketed quantity in equation (60) is generally complex. This, of course, implies a phase difference between the tidal force and each modal amplitude on a rotating earth.
Putting together the eigenspace pairs as above, we get from equation (58)

$$S(r, t) = \sum_n \left[ \frac{\exp(i \omega_n t)}{2i \omega_n} (u^*_n, f) u_n(r) - \frac{\exp(-i \omega_n t)}{-2i \omega_n} (u_n, f) u^*_n(r) \right] H(t)$$

$$= \sum_n \text{Re} \left[ \frac{\exp(i \omega_n t)}{i \omega_n} (u^*_n, f) u_n(r) \right] H(t)$$

(61)

for an impulsive source. Similarly for equation (59a), we get

$$S(r, t) = \sum_n S_n(r, t) = \sum_n \text{Re} \left[ u_n(r) \int_0^t \frac{\exp[i \omega_n(t-\tau)]}{i \omega_n} f_n(\tau) d\tau \right], \quad t > 0$$

(62)

for a general source distribution $f(r, t)$. Equation (62) is the basic excitation formula for the normal modes of oscillation belonging to a rotating body when observed in the rotating frame. It describes, in particular, the response of the rotating earth to the equivalent body force $f(r, t)$ of an earthquake.

Using equation (62), we can readily deduce the rotational counterpart of equations (27) and (29). Thus, we have for a step-function source $f_m(r) H(t)$

$$S_n(r, t) = \text{Re} \left[ f_n^\infty \frac{1 - \exp(i \omega_n t)}{\omega_n^2} u_n(r) \right], \quad t > 0$$

(63)

and for the static displacement as $t \to \infty$

$$S_{\text{stat}} = \sum_n \text{Re} \left[ f_n^\infty u_n(r) \right],$$

(64)

where $f_n^\infty \equiv (u^*_n, f_m)$.

We should point out that to obtain equations (61) and (62) we do not make any $a$ $priori$ decisions to take the real part; instead, that follows naturally from our arguments on the algebraic structure of the spectrum. Note also that equation (62) is of the same formal expression as equation (25); it certainly reduces to the latter, or equivalently, to equation (20a), in the limit of zero rotation. However, the formal similarity is only superficial. Apart from the fact that the summation index $n$ in equation (62) indicates eigenspace pairs, there is a fundamental difference between the eigenfunctions $u_n(r)$ in equation (62) and those in equation (25) — namely, that the eigenfunctions $u_n(r)$ in (62) are not trivially complex as they are in the non-rotating case (25), as pointed out earlier. As a result, each term in the summation (62), unlike in (25), cannot be expressed in the form of equation (24) as the multiplication of a real temporal part with a real spatial part, i.e. it does not represent a standing wave relative to the rotating frame. From the mathematical point of view, the eigenfunctions $u_n(r)$ are merely mathematical objects designed to study the normal mode problem. In the case of a non-rotating body, any eigenfunction $u_n(r)$ as defined in equation (14) is trivially complex and has the physical meaning of a standing wave. On the other hand, an eigenfunction $u_n(r)$ defined in equation (54) for the rotating case is non-trivially complex and its physical meaning is somewhat obscure. Instead of being a standing wave, it represents a so-called ‘generalized travelling wave’ (D & S) when observed in the rotating frame.

It should be emphasized that, in the limit of zero rotation, all the results obtained in Section 3.2 and so far in this section reduce to those given in Section 2.2 and 2.3. This suggests that, when the rotation is ‘turned off’, the pair of eigenspaces $\{u_n(r), \omega_n\}$ and $\{u_n^*, -\omega_n\}$ (here we use primed symbols to indicate quantities with regard to a rotating
body in order to distinguish them from their non-rotating counterparts) coincide to become a single one-dimensional eigenspace \( \{ u_n(r), \pm \omega_n \} \). Or, put in a different way, when the rotation is gradually 'turned on', the trivially complex eigenfunction \( u_n(r) \) belonging to the non-rotating body begins to 'evolve' into a pair of complex conjugate eigenfunctions \( u_n'(r) \) and \( u_n^*(r) \). The corresponding eigenfrequencies \( \pm \omega_n \) are at the same time shifted to \( \pm \omega_n' \), each of which is associated with one of the two eigenfunctions. Symbolically, we write

\[
\{ u_n(r), \pm \omega_n \} \rightarrow \{ u_n'(r), \omega_n' \} \quad \text{or} \quad \{ u_n^*(r), -\omega_n' \} .
\] (65)

Note that the evolution (65) does not imply a sudden double in the number of normal modes at the onset of the rotation. On the contrary, the number of normal modes being summed in equations (60–62), where the summation index \( n \) indicates eigenfunction pairs, is the same as in the non-rotating case. In equations (60–62) the pair of eigenfunctions are linearly combined in a fixed fashion, and only one degree of freedom is associated with each pair of eigenfunctions.

It is interesting to study the case where the rotational speed \( \Omega \) is small compared with the eigenfrequencies \( \omega_n \). Here we make two interesting observations about the first-order perturbation due to a slow rotation. First, the original nodal surfaces of the standing wave in the non-rotating body will, to first order of \( \Omega/\omega_n \), remain as nodal surfaces when observed in the rotating frame. This is because a nodal surface is characterized by the vanishing of not only the displacement but also the total force (e.g. the elastic-gravitational restoring force in a non-rotating earth) exerted on any point on the surface. The introduction of the Coriolis force by a rotation causes no change in the nodal surface pattern because it also vanishes on the nodal surfaces. The only influence, then, comes from the centrifugal force; but that is only of second order in \( \Omega/\omega_n \). In other words, the generalized travelling waves are actually 'drifting' standing waves relative to the rotating frame; and the 'drifting speed' of the standing wave pattern is only of second order. Secondly, because the product \( u_n'(r) \cdot \Omega \times u_n(r) \) vanishes identically for trivially complex \( u_n(r) \), the first-order departure of \( \omega_n' \) from its non-rotating limit \( \omega_n \) also vanishes. The slow rotation has only second-order effects on the eigenfrequencies.

The latter first-order, non-degenerate perturbation argument, however, is of little practical significance. Indeed, because of the Earth's nearly spherically symmetric configuration, the normal modes are grouped into numerous 'multiplets' according to their eigenfrequencies. The frequency differences among members of one multiplet are generally so small compared with the rotational speed \( \Omega \) that the first-order non-degenerate perturbation theory is not adequate. One therefore must resort to the quasi-degenerate theory in order to take into account all the members in a multiplet at the same time. In the extreme case of a spherically symmetric earth, where all members of each multiplet belong to one \((2l+1)-1\) fold degenerate eigenspace, the rotation will give rise to a Zeeman-type, first-order splitting in the eigenfrequency spectrum. The latter problem has been well studied by Backus & Gilbert (1961) using degenerate perturbation theory.

4 Discussion

The method of spectral decomposition is by no means indispensable when attacking the excitation problem for a non-rotating body, in which case the linear operator involved, namely \( (A_1 \lambda + A_0) \), is only linear in the parameter \( \lambda \). Nevertheless, this method is effective in the sense that it can be naturally extended to the study of the same problem for a rotating earth, where the linear operator in question \( (A_2 \lambda^2 + A_1 \lambda + A_0) \), is quadratic in \( \lambda \). In fact,
Lancaster (1966) has shown that this procedure is quite general, at least with respect to oscillating systems with finite degrees of freedom, as to be applicable to any linear operator of the form $A_n \lambda^n + A_{n-1} \lambda^{n-1} + \ldots + A_1 \lambda + A_0$.

It has been pointed out that during our discussions a vital simplification has been made—we have avoided any zero-frequency modes and have confined our attentions to the non-zero-frequency modes only. For instance, in equations (19), (20a), and (57–59a) some detailed discussions would have been necessary if any of the $\omega_n$'s are to vanish. A zero-frequency mode of a stable earth model represents some secular motion. It is characterized by the fact that it does not alter the net elastic-gravitational potential energy at any time. In the case of an everywhere-elastic earth model, the only possible secular motions are the rigid-body translations and rotations. Other secular motions are possible in an earth model containing a fluid core—for example, the rotation of some uniform fluid shell. In any event, the exclusion of these zero-frequency modes from our formalism is a direct consequence of the positive-definiteness of the elastic-gravitational operator $L$. If we allow $L$ to be positive semi-definite, then a summation over all the possible non-zero-frequency modes should be augmented to all the normal-mode summation formulae in Sections 2 and 3, for only then will the normal modes form a complete set.

However, if we only consider the non-zero-frequency, infinitesimal motions of the oscillating body as we did, then the results obtained in Section 2 will be exact when observed from a fixed reference frame. Similarly, the formulae in Section 3 are exact to an observer in a reference frame rotating with the body, if the rotation is uniform. For a rotating earth in a force-free situation, the direction of its angular velocity $\Omega$ can remain fixed in space and relative to the body-axes of the earth only when the earth is rotating about one of its three principal axes, because only then will $\Omega$ be in the same direction as the constant angular momentum vector. In reality, the rotation axis of the Earth is deflected slightly from its figure axis (the principal axis of greatest moment), giving rise to a free precession of $\Omega$ relative to the body-axes of the Earth (as well as in space) known as the Chandler wobble. The astronomical precession and nutation also contribute to the variation in the direction of $\Omega$. Furthermore, the magnitude of $\Omega$, or the rotational speed, also varies with time (the length-of-day variation) owing to physical mechanisms such as the tidal friction and the seasonal loading of the atmosphere and the oceans. Fortunately, the overall variation in $\Omega$ is so small that we can always take the uniformly rotating frame to be coincident with the rotating body-axes of the Earth, as we have practiced in Section 3, without any significant error. Hence, the formulae in Section 3 are trusted as being exact to great precision to an observer on the rotating Earth.

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References


Appendix A

In this appendix we will show, in a schematic fashion, that the latent functions \( \{u_n(r); n = 1, 2, 3, \ldots\} \) of the linear operator \( K\lambda + M \) form a complete set for the Hilbert space \( H \), where \( K \) and \( M \) are two self-adjoint operators and \( M \) is, in addition, positive-definite. For more general cases and more detailed discussions, the reader is referred to Eisenfeld (1968).

By definition, \( u_n(r) \) satisfies
\[
(K\lambda_n + M) u_n(r) = 0, \quad n = 1, 2, 3, \ldots
\]
where \( \lambda_n \) is the latent value associated with \( u_n(r) \). Now because \( M \) is self-adjoint and positive-definite, we can show (see Rietz & Sz.-Nagy 1955) that: (1) the square root of \( M \), denoted by \( M^{1/2} \), is uniquely defined and is also self-adjoint and positive-definite; (2) the inverse of \( M^{1/2} \), denoted by \( M^{-1/2} \), exists and is self-adjoint and positive-definite; and (3) \( \lambda_n \) is non-zero (otherwise 0 will be an eigenvalue of \( M \)). Then equation (A1) can be written in the following equivalent form:
\[
(N - \mu_n I) w_n(r) = 0, \quad n = 1, 2, 3, \ldots
\]
where \( N = M^{-1/2} KM^{-1/2}, I \) is the identity operator, and \( \{\mu_n = -1/\lambda_n, w_n(r) \equiv M^{-1/2} u_n(r)\} \) are the eigensolutions of \( N \). Since \( N \) is self-adjoint, it follows immediately from a well-known theorem in functional analysis (see Rietz & Sz.-Nagy 1955) that the set of all the eigenfunctions of \( N \), i.e. \( \{w_n(r); n = 1, 2, 3, \ldots\} \), is complete in \( H \). It is then easy to show, from the fact that \( M^{-1/2} \) exists, that \( \{u_n(r); n = 1, 2, 3, \ldots\} \) also form a complete set for \( H \).

Appendix B

In this appendix we shall study the problem of normal mode excitation for a non-rotating body using two different, although closely related, classical methods. The source distribution starts to act at time \( t = 0 \), so that the body is at rest in the equilibrium position for \( t < 0 \). We only consider the elastic modes, with non-zero eigenfrequencies.
(1) Laplace transform technique: since the set of eigenfunctions \( \{u_n(r); n = 1, 2, 3, \ldots \} \) satisfying equation (14) is complete in the Hilbert space \( H \) (see Appendix A), we can express the total displacement field as

\[
S(r, t) = \sum_n S_n(r, t) = \sum_n a_n(t) u_n(r), \tag{B1}
\]

and the source distribution as

\[
f(r, t) = \sum_n f_n(t) u_n(r). \tag{B2}
\]

The Laplace-transformed equation of motion to be solved is the inhomogeneous equation:

\[
L S(r, p) + p^2 S(r, p) = f(r, p) \tag{B3}
\]

where \( S(r, p) \) and \( f(r, p) \) are, respectively, the Laplace transform of \( S(r, t) \) and \( f(r, t) \):

\[
S(r, p) = \int_0^\infty \exp(-pt) S(r, t) \, dt, \quad f(r, p) = \int_0^\infty \exp(-pt) f(r, t) \, dt, \quad \text{Re } p > 0. \tag{B4}
\]

From equations (14), (B3), and the orthonormality relation (15), we see that \( a_n(p) \), the Laplace transform of the temporal coefficient \( a_n(t) \), is given by the following well-known equation:

\[
a_n(p) = \int_0^\infty \exp(-pt) a_n(t) \, dt = \frac{f_n(p)}{p^2 + \omega_n^2} \tag{B5}
\]

where \( f_n(p) \) is the Laplace transform of the temporal coefficient \( f_n(t) \). The coefficient \( a_n(t) \) can then be obtained by the inverse Laplace transform of equation (B5) using the convolution theorem; the result is:

\[
a_n(t) = \int_0^t f_n(\tau) \frac{\sin \omega_n(t-\tau)}{\omega_n} \, d\tau. \tag{B6}
\]

Therefore, the excitation of the \( n \)th normal mode is

\[
S_n(r, t) = u_n(r) \left[ \int_0^t f_n(\tau) \frac{\sin \omega_n(t-\tau)}{\omega_n} \, d\tau \right], \quad t > 0. \tag{B7}
\]

It remains to determine \( f_n(r) \). This is accomplished by using, again, the orthonormality (15):

\[
f_n(t) = (u_n^*, f) = \int_V u_n^*(r_0) \cdot f(r_0, t) \, dV_0. \tag{B8}
\]

Equations (B7) and (B8) are identical to equations (20a, b).

(2) Green's function technique: first consider a simple-harmonic point source with unit strength and constant driving frequency \( \omega \) in the direction \( \hat{x}_i(i = 1, 2, 3) \) at point \( r_0 \). The response of the body to such a source, or the simple-harmonic Green function \( G^{(i)}(r, r_0) \), satisfies

\[
L G^{(i)}_\omega(r, r_0) - \omega^2 G^{(i)}_\omega(r, r_0) = \delta(r-r_0) \hat{x}_i, \quad i = 1, 2, 3 \tag{B9}
\]

along with the prescribed boundary conditions. As before, we can expand \( G^{(i)}_\omega(r, r_0) \) in terms of the eigenfunctions \( \{u_n(r); n = 1, 2, 3, \ldots \} \):

\[
G^{(i)}_\omega(r, r_0) = \sum_n g_n^{(i)}(r_0, \omega) u_n(r), \quad i = 1, 2, 3. \tag{B10}
\]
Then from equations (14), (B9), (B10) and the orthonormality (15), we obtain

\[ g_n^{(f)}(r_0, \omega) = \frac{u_n^{(f)}(r_0)}{\omega_n^2 - \omega^2}, \quad i = 1, 2, 3. \]  

(B11)

By properly superposing the simple-harmonic Green function, we can obtain the total Green function, i.e. the response of the oscillating body to an impulsive point source of unit strength. Thus, if we define the tensor of rank 2:

\[ G_\omega(r, r_0) = \sum_n \frac{u_n^*(r_0) u_n(r)}{\omega_n^2 - \omega^2}, \]  

then the total Green function, also a tensor of rank 2, is given by (Morse & Feshbach 1953, pp. 849-850):

\[ G(r, r_0; t, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_\omega(r, r_0) \exp[i\omega(t-t_0)] d\omega \]

\[ = \frac{1}{2\pi} \sum_n u_n^*(r_0) u_n(r) \int_{-\infty}^{\infty} \frac{\exp[i\omega(t-t_0)]}{\omega_n^2 - \omega^2} d\omega. \]  

(B13)

Here, as in equation (8), the choice of the integration contour is dictated by the principle of causality. Thus we get

\[ G(r, r_0; t, t_0) = \sum_n u_n^*(r_0) u_n(r) \frac{\sin \omega_n(t-t_0)}{\omega_n} H(t-t_0). \]  

(B14)

From the total Green function \( G \) we obtain the response of the body to any source \( f(r, t) \) by the double integral:

\[ S(r, t) = \int_0^t dt_0 \int_V f(r_0, t_0) \cdot G(r, r_0; t, t_0) \rho(r_0) dV_0. \]  

(B15)

Substituting equation (B14) in equation (B15), we get for \( S(r, t) \) an expression identical to equations (20a, b).

Appendix C

Appendix B applies the Laplace transformation to the excitation problem for a non-rotating body. The procedure is straightforward. It is less so, however, for the more general case of a rotating body. In this appendix we show that, with some plausible assumptions which may be justified a posteriori, the Laplace transformation also leads to the final equations (59a, b). This development parallels that of Dahlen's (1978, 1980, and private communication).

Assuming that the set of eigenfunctions \( \{u_n(r); n = 1, 2, 3, \ldots\} \) satisfying equation (54) is complete in the Hilbert space \( H \), we can expand the total displacement field

\[ S(r, t) = \sum_n a_n(t) u_n(r). \]  

(C1)

The Laplace-transformed equation of motion to be solved is

\[ L S(r, p) + p Q S(r, p) + p^2 S(r, p) = f(r, p) \]  

(C2)

where \( S(r, p) \) and \( f(r, p) \) are given in equation (B4). From equations (54), (C2), and the quasi-orthonormality relation (55), we can obtain the following equation that expresses
explicitly the coupling among all the Laplace-transformed coefficients \( \{ a_n(p); n = 1, 2, 3, \ldots \} \):

\[
2i\omega_n a_n(p) + \sum_{m} a_m(p) (p - i\omega_m) \langle u_n^*, u_m \rangle = \frac{f_n(p)}{p - i\omega_n}, \tag{C3}
\]

where \( f_n(p) \) is the Laplace transform of \( f_n(t) \), the latter given by equation (59b). Note that equation (C3) reduces to the simple equation (B5) in the limit of zero rotation.

To solve equation (C3), let us find the impulse response of the rotating body, so that the response to any source will follow from a time convolution. For an impulsive force, \( f_n(p) \) becomes a constant \( f_n \). Then equation (C3) gives

\[
a_n(t) = \frac{1}{2\pi i} \int_B a_n(p) \exp(pt) \, dp
= \frac{1}{2\pi i} \frac{1}{2i\omega_n} \left[ f_n \int_B \frac{\exp(pt)}{p - i\omega_n} \, dp - \sum_m \langle u_n^*, u_m \rangle \int_B a_m(p) (p - i\omega_m) \exp(pt) \, dp \right] \tag{C4}
\]

where \( B \) denotes a Bromwich contour. Now, if \( a_n(p) \) has a single, simple pole at \( p = i\omega_n \) for all \( n = 1, 2, 3, \ldots \), then the second term in the bracket of equation (C4) will vanish and all the coefficients become uncoupled. This is an assertion that cannot be proved a priori. However, if it is the case, then equation (C4) reduces to

\[
a_n(t) = \frac{\exp(i\omega_n t)}{2i\omega_n} f_n H(t). \tag{C5}
\]

Equations (C1) and (C5) then lead to the impulse response (58), and equations (59a, b) follow immediately.