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Over The Top

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A common homework problem[1] in many introductory physics courses is similar to the following.

> *A car drives at constant speed over a hill on a road in the shape of a vertical circular arc. What is the maximum speed the car can have and not lose contact with the road at the crest of the hill?*

Unfortunately this problem is flawed, because if the speed of the car is such that it would lose contact at the crest, then it will lose contact well before that point! In fact, the car would lose traction (i.e., the tires will begin to slip[2]) even earlier, which can lead to a dangerous loss of steering control. The situation is worse still for an unpowered (free rolling) car, because it would travel faster at lower than higher elevations and so it would lose contact with the steeper convex road surface at a yet lower height.[3-4]

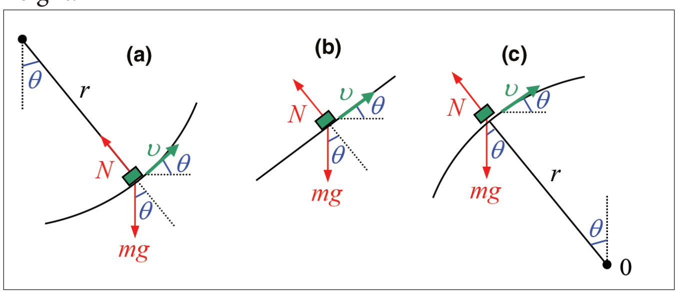

Fig. 1. Free-body diagram for an object on an uphill road whose cross section is (a) concave circular, (b) linear, or (c) convex circular. Only the normal force *N* and weight *mg* are shown, but there may also be purely tangential forces.

The present article analyzes this situation for a generalized object (not necessarily a car) of mass m under the following assumptions. The object is traveling either *up* a hill (everywhere inclined at less than the vertical) or on a (at least instantaneously) *level* portion of the road, so that the angle θ in Fig. 1 is constrained to the range $0^\circ \le \theta < 90^\circ$. For simplicity, it is assumed that there is no rolling friction or air drag.

Three cases will be treated in the sections below: first, a rigid object that slides frictionlessly like a hockey puck on ice or on an air track; second, an unpowered object that rolls without slipping such as a ball; and third, a car that is driven by pressing on the accelerator pedal (but not so hard that the tires slip at their points of contact with the road). The second and third cases require that there be static friction to avoid slipping, but static friction does no work on the object. Instead, in the unpowered case, translational and rotational kinetic energy are progressively converted into gravitational potential energy during the ascent, such that mechanical energy is conserved. For a driven car, on the other hand, there is internal conversion of chemical into mechanical energy by the consumption of fuel.

In treating these cases, however, attention is focused on the possibility of the object losing contact with the road when it is traveling at speed v. That occurs if the normal force N on the object becomes zero. For a spherical hill of radius r, the normal force is in the radial direction. So begin by ignoring any forces that are purely tangential and consider the vertical cross sections sketched in Fig. 1, where panel (b) can be thought of as representing a circle whose radius is infinite. The radial component of Newton's second law (N2L) for panel (a) is

$$N - mg\cos\theta = m\frac{v^2}{r}, \tag{1}$$

where $g = 9.8$ N/kg is the magnitude of Earth's surface gravitational field. Solving Eq. (1) for N reveals that it can never become zero for any $0^\circ \le \theta < 90^\circ$. (It could become zero only if the object was inverted such that $90^\circ \le \theta \le 270^\circ$ as on a loop-the-loop track *and* it was traveling sufficiently slowly.) For panel (b), substitute $r \to \infty$ into Eq. (1) to find $N = mg\cos\theta$, which can also never be zero for $0^\circ \le \theta < 90^\circ$. In other words, an object cannot lose contact when it is upright on a concave or planar track. However it can lift off when upright on a convex track such as in panel (c) for which the radial component of N2L becomes

$$mg\cos\theta - N = m\frac{v^2}{r}, \tag{2}$$

so that contact is lost when $v = (gr\cos\theta)^{1/2}$. Accordingly, in the remainder of this article, the motion will be analyzed only along a *convex* portion of the road. That portion starts at some initial angle $0^\circ < \theta_i < 90^\circ$ below the peak. At larger angles than θ_i (i.e., at lower altitudes) the road may be planar or concave, so that the object can move along that preceding section without risk of flying into the air.

A concrete example of a possible cross-sectional shape of the road makes these ideas clearer. It is helpful to choose one that looks like an actual hill but is constructed out of purely circular or linear segments, for ease of analysis, graphing, and experimentation. With that in mind, model the profile of the road by a plane curve $y(x)$ where y is the vertical height as a function of horizontal position x. The top portion of the road is convex circular with radius r so that $y = (r^2 - x^2)^{1/2}$ over the symmetric range $-cr \le x \le cr$, where c is some constant fraction between 0 and 1 in value. Now duplicate that circular arc, flip it over vertically across its base, and divide it in two at its midpoint to make two new arcs. Shift those two concave arcs leftward and rightward until they smoothly join the ends of the convex arc. The resulting hump-shaped curve will then have zero slope at its two endpoints. Splice them onto two horizontal segments each of length cr. The result is drawn in Fig. 2 for the case of $c = 0.5$. The crest is at height $2r[1 - (1 - c^2)^{1/2}]$ above the horizontal. The angle θ varies between 0° at the crest and $\theta_i = \sin^{-1} c$ at the initial convex portion of the hill where $y(x)$ switches from being concave. (For example, $\theta_i = 30^\circ$ in Fig. 2.) This simple profile has no cusps and is hereby recommended as a track shape for roller coaster motion of

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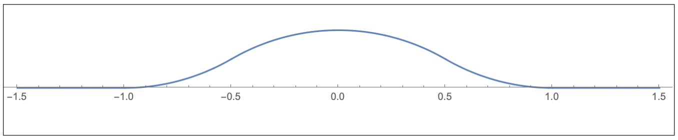

Fig. 2. A possible profile of a road over a hill. The horizontal axis plots *x/r* and the vertical axis is omitted but uses the same scaling to plot *y/r*, where *r* is the radius of curvature of all portions of the profile that are not flat. The road is flat for values of *x/r* from –1.5 to –1, circular concave from –1 to –0.5, circular convex from –0.5 to 0, and then reflected horizontally across the crest of the hill at *x* = 0.

model cars. Nonetheless, it is only an example and many other profiles are possible.

Case 1: Unpowered sliding up a frictionless track

There are no tangential forces in this case. Thus a complete free-body diagram (FBD) on any convex circular portion of the track is depicted in Fig. 1(c). Since mechanical energy is conserved, the speed v of the object monotonically *decreases* as its height $r\cos\theta$ above the origin 0 in this FBD increases. On the other hand, the speed $(gr\cos\theta)^{1/2}$ at which contact with a convex track is lost monotonically *increases* as $r\cos\theta$ increases. Consequently, the object will maintain contact *everywhere* along an uphill convex portion of the track if its speed at the starting angle θ_i of that portion does not exceed

$$v_{max} = \sqrt{gr\cos\theta_i}. \quad (3)$$

To put it another way, if a free-sliding object loses contact with an uphill convex track, it necessarily must do so at angle θ_i at the start of that convex section. The same reasoning holds if the object slides with kinetic friction, which further decreases its speed during the ascent.

Case 2: Unpowered rolling without slipping up a track

The FBD in Fig. 1(c) is now modified to give Fig. 3, which includes a static frictional force f on the ball (or on the set of wheels of a coasting car of total mass m) that produces a torque about its center point C to change its angular speed $\omega = v/R$, where R is the radius of the ball or wheels (not to be confused with the radius of curvature r of the track). In order to decrease ω as the object moves up the hill, f must point *forward*, which surprises students who expect friction to point opposite the direction of the translational velocity v. (In fact, the contact point between the object and track is instantaneously at rest rather than translating forward. More importantly, the contact point is in danger of slipping backward relative to the road, and consequently static friction is directed forward to oppose such slippage.) Thus, if two identical objects start moving up an incline with the same initial translational speed but with one sliding frictionlessly and the other rolling without slipping, the rolling one will travel farther up the incline because f points uphill and thus its net tangential deceleration is smaller than it is for the sliding object. From an energetics view, the rolling object has more kinetic energy (due to its rotation) than does the purely translating object and thus it must end up with more gravitational potential en-

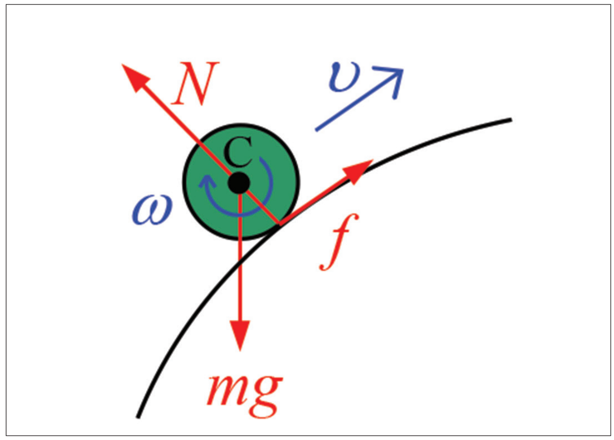

Fig. 3. Free-body diagram of an unpowered object that rolls without slipping up a convex hill. The static friction *f* causes ω to decrease as υ decreases during the ascent.

ergy at its highest point.

In any case, the translational speed v of a free-rolling object monotonically decreases as it ascends a hill, just as it does for a frictionless sliding object (albeit with a different deceleration). Therefore the same conclusions about lift-off apply to case 2 as to case 1: the object will either first lose contact at the initial angle θ_i of an uphill convex portion of the track if its speed exceeds v_{max} given by Eq. (3), or it will not fly off anywhere along that portion. In neither case is it possible for contact to first be lost at the crest of a hill!

Case 3: Car driven up a hill without slipping of the tires

The only way an object can first lose contact with an uphill convex portion of a track above the lowest angle θ_i of that portion is if the object *increases* in speed along it. The object therefore cannot be unpowered, nor is it sufficient to propel the object at constant speed.

For an ordinary car driven by an internal gasoline or electric motor to increase its translational and angular speeds, connected by $v = R\omega$ in the absence of slipping, the FBD representing the entire set of wheels in Fig. 3 is modified by the inclusion of a torque that the driven axle exerts about the center point C in the clockwise direction. (Here R is the radius of a wheel, and m is the total mass of the car including its chassis.) However, the forward force speeding up the car remains the external friction force f exerted by the road. Thus the best chance of getting the car up to a high enough speed v that it loses contact with the road is to make f as large as possible. Accordingly, the friction should be static and at its maximum value $f = \mu N$, where μ is the coefficient of static friction between the tires and the road. In that case, the tangential component of N2L for the car is

$$\mu N - mg\sin\theta = m\frac{dv}{dt}. \quad (4)$$

Consequently a car that does not lift off at angle θ_i when it transitions onto an uphill convex portion of the road *cannot* be driven hard enough to lose contact at a later point along that portion! If the car begins in contact with a convex road, it could only lose contact later if the normal force N fell to zero in value. But as N gets smaller, the frictional force μN gets smaller. Eventually N drops to the value $\mu^{-1}\, mg \sin\theta$ and at that point $dv/dt = 0$ according to Eq. (4). That means the car will not gain any further speed over the next small increment of the road and so it cannot attain the lift-off speed $(gr \cos\theta)^{1/2}$ during that increment. The same is true of all subsequent uphill convex portions (or increments thereof) along the road.

So how does a car perform a jump?

There are a number of websites[5] that give tips on automobile hill jumping, but most of them include strong cautions that it should not be attempted on ordinary roads (to avoid crashing into other cars) or with ordinary vehicles (to avoid damaging an axle or transmission). Nonetheless, trained individuals on closed courses do manage it. How do they get around the limitations of case 3?

The tracks discussed in this article are assumed to be smooth (i.e., differentiable at least once). Clearly one could launch off a ramp (corresponding to a discontinuity in the road surface) or a cusp (such as a track constructed in the shape of the angled roof of a house). But otherwise, any smooth road profile, even if not circular but say gaussian, can be considered to be composed of short circular segments. (Each segment has a curvature that mathematically corresponds to the reciprocal of the radius of an osculating circle[6] fit to it.) How does a car succeed in making a jump on a smooth hill?

One possibility is there could be additional external forces acting on a vehicle. Lift gets a wing airborne (particularly in the presence of a strong headwind) and it can be significant for some cars. Alternatively one could strap a jet or rocket engine onto a car to boost its speed without having to rely on friction between the wheels and the ground.

More importantly, ordinary car jumps on hilly roads are not made while ascending, but just *after* the crest is passed. Once an object is going downhill, the tangential component $mg \sin\theta$ of the gravitational force switches direction from pointing backward to pointing forward, thereby assisting in speeding up the object. In the online appendix,[7] Eqs. (2) and (4) are simultaneously solved to find the speed of a car driven along a convex road initially going uphill until it loses contact with the road on the downhill side. Likewise, even an undriven object can readily lift off a downhill convex surface, depending on its initial speed and on the coefficient of friction.[4]

Acknowledgment

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References

1. For example, https://courses.lumenlearning.com/atd-monroecc-physics/chapter/dynamics-3/ or http://www.physics.usyd.edu.au/~helenj/Mechanics/Problems/L6-car-over-hill.pdf.
2. A similar distinction is that between the point at which a marble on a loop-the-loop loses contact and the earlier point at which static friction becomes too weak to maintain rolling without slipping, discussed in O. Bertran and J. Riba, "A revised solution for a sphere rolling in a vertical loop," *Eur. J. Phys.* **42**, 015008 (Jan. 2021).
3. Thus the AP College Board problem at https://resources.finalsite.net/images/v1594850515/lusdorg/nc5ygm7fot606kqsvucd/answers2.pdf is flawed.
4. Compare the time-reversed problem of starting an object at the top of a hemispherical hill with a nonzero speed and letting it slide down until it loses contact, as in C.E. Mungan, "Sliding on the surface of a rough sphere," *Phys. Teach*. **41**, 326–328 (Sept. 2003).
5. For example, https://www.redbull.com/us-en/how-to-jump-a-rally-car-by-kalle-rovanpera.
6. See https://mathworld.wolfram.com/OsculatingCircle.html.
7. The appendix can be found at *TPT Online* at http://dx.doi.org/10.1119/5.0051081 under the Supplemental tab.

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