Discussion: "Common Errors on Mapping of Nonelliptic Curves in Anisotropic Elasticity" (Ting, T. C. T., 2000, ASME J. Appl. Mech., 67, pp. 655–657)

C. Q. Ru
Department of Mechanical Engineering, University of Alberta, Edmonton, AB T6G 2GB, Canada
e-mail: c.ru@ualberta.ca

Professor Ting's paper ([1]) clearly clarifies several simple but important concepts on conformal mapping techniques applied to anisotropic plane elasticity. Here, I would like to add my own comments on these interesting issues.

(1) First, it should be stated that conformal mapping techniques, combined with the Stroh's method, have been successfully applied in some important cases to anisotropic elasticity with nonelliptical curves. An example is the Eshelby's problem for an inclusion of arbitrary shape in an anisotropic medium ([2]), or in a piezoelectric medium [3], of the same material constants. As stated by Prof. Ting in [1], and also by some other authors elsewhere, because a point z on Γ will be transformed, under three different mappings $w_\alpha(\xi)$ $(\alpha=1,2,3)$, to three different points ξ_α on the unit circle in ζ-plane, the transformed boundary conditions on the unit circle in the ζ-plane will contain three unknown Stroh's functions which take values at three different points. Therefore, unless the boundary conditions are decoupled for the three Stroh's functions, one cannot solve the transformed boundary value problem in the ζ-plane. The key fact associated with the problem studied in [2] is that the three interface conditions (in complex form) for an arbitrarily shaped inclusion, surrounded by an anisotropic medium of the same material constants, can be written in a decoupled form in which the three unknown Stroh's functions are completely decoupled to each other. It is this fact that allows one to apply conformal mapping techniques to each of the three Stroh's functions and the associated curve independently of the other two. For a similar result for piezoelectric materials, see [3].

The second key result of ([2]) is that for each of the three closed curves Γ_α $(\alpha=1,2,3)$ (that is Γ^α defined in [1]), one can construct an auxiliary function $D_\alpha(z)$ which satisfies the condition

$$\overline{z}=D_\alpha(z),\quad z\in\Gamma_\alpha \tag{1}$$

and is analytic and single-valued in the exterior of the curve Γ_α, except at infinity where $D_\alpha(z)$ tends to a polynomial $P_\alpha(z)$. As shown in [2] (and [3] for piezoelectric materials), with aid of these auxiliary functions, the techniques of analytic continuation can be applied to the inclusion of arbitrary shape to get an analytic solution for the Stroh's functions.

The above key result (which has been questioned by someone!) can be shown, clearly and rigorously, as follows. Assume that the exterior of Γ_α is mapped onto the exterior of the unit circle in the ξ-plane by a polynomial conformal mapping

$$z=w_\alpha(\xi)=\lambda_\alpha\xi+\sum_{k=0}^{N}c_{\alpha k}\xi^{-k},\quad \alpha=1,2,3 \tag{2}$$

where λ_α is a real number, $c_{\alpha k}$ are some complex constants, and N is a finite integer. It is emphasized that the definition of the conformal mapping (2) implies that it has a unique inverse conformal mapping $w_\alpha^{-1}(z)$ which is well defined on the exterior of the curve Γ_α and maps the exterior of the curve Γ_α (without any branch cut!) on to the exterior of the unit circle. Evidently, this means that the inverse mapping $w_\alpha^{-1}(z)$ is analytic, single-valued and nonzero in the exterior of the curve Γ_α. This is just part of the definition (2)—not any further "proof" is needed. Here, similar to all other conformal mapping methods, the inverse mapping $w_\alpha^{-1}(z)$ is treated as the known, and we need not discuss how to construct an explicit expression for the inverse mapping $w_\alpha^{-1}(z)$ from the single-valued branches of the multivalued inverse function of (2). In particular, all branch points of the inverse function of (2) fall inside the interior of the curve Γ_α in the z-plane, or inside the interior of the unit circle in the ξ-plane. For example, for a hypotrochoidal curve, it is readily seen from (A11) of [4] that all singularity points of the conformal mapping (at which the derivative of the mapping function vanishes, as described by the condition (3) or (6) of [1]) fall inside the interior of the unit circle in the ξ-plane and thus do not trouble the single-valued inverse mapping for the exterior.

Based on these facts, it is easily verified that the desired function $D_\alpha(z)$ is given by

$$D_\alpha(z)=\overline{w_\alpha}\left(\frac{1}{w_\alpha^{-1}(z)}\right)=\frac{\lambda_\alpha}{w_\alpha^{-1}(z)}+\sum_{k=0}^{N}\overline{c_{\alpha k}}[w_\alpha^{-1}(z)]^k \tag{3}$$

where $w_\alpha^{-1}(z)$ is the (unique) inverse mapping of the polynomial mapping (2). First, $D_\alpha(z)$ given by (3) meets the condition (1) on the curve Γ_α. Second, because $w_\alpha^{-1}(z)$ is analytic, single-valued and nonzero outside the curve Γ_α, $1/w_\alpha^{-1}(z)$ and $[w_\alpha^{-1}(z)]^k$ (k is any integer not larger than N) are analytic and single-valued outside the curve Γ_α. Thus, $D_\alpha(z)$ given by the right-hand side of (3) is obviously analytic and single-valued in the exterior of the curve Γ_α, except at infinity where $D_\alpha(z)$ tends to a polynomial of degree N. Therefore, the auxiliary function $D_\alpha(z)$ complying with the conditions (1) can be constructed by (3) in terms of the associated polynomial mapping which maps the exterior to the curve Γ_α onto the exterior of the unit circle. Similar auxiliary functions have been applied to isotropic elasticity [4] and piezoelectric materials [3].

(2) Finally, as stated in [1], the mapping (15) of [1], although provides a one-to-one mapping for the boundaries, does not always offer a one-to-one mapping for the exteriors of the boundaries. Regarding this issue, as stated in [2,3], the boundary correspondence principle of conformal mappings for exterior domains ([5]) can be used to identify the conditions under which a one-to-one mapping for the boundaries automatically offers a one-to-one mapping for the exteriors. For instance, for an elliptical boundary Γ, because the right-hand side of (15) of [1] is analytic outside the unit circle and has a simple pole (of degree one) at infinity in the ξ-plane, it follows from the boundary correspondence principle [5] that the expression (15) of [1] provides a one-to-one conformal mapping between the exterior of the curve Γ_α and the exterior of the unit circle in the ξ-plane, not any pointwise verification is needed. I believe that this comment offers a valuable insight to this interesting issue.

References

[1] Ting, T. C. T., 2000, "Common Errors on Mapping of Nonelliptic Curves in Anisotropic Elasticity," ASME J. Appl. Mech., **67**, pp. 655–657.
[2] Ru, C. Q., 2001, "Analytic Solution for an Inclusion of Arbitrary Shape in an Anisotropic Plane or Half-Plane" (submitted for publication).
[3] Ru, C. Q., 2000, "Eshelby's Problem for Two-Dimensional Piezoelectric Inclusions of Arbitrary Shape," Proc. R. Soc. London, Ser. A, **A456**, pp. 1051–1068.
[4] Ru, C. Q., 1999, "Analytic Solution for Eshelby's Problem of an Inclusion of Arbitrary Shape in a Plane or Half-Plane," ASME J. Appl. Mech., **66**, pp. 315–322.
[5] Ivanov, V. I., and Trubetakov, M. K., 1995, *Handbook of Conformal Mapping With Computer-Aided Visualization*, CRC Press, Boca Raton, FL.

Closure to "Discussion of 'Combinations for the Free-Vibration Behavior of Anisotropic Rectangular Plates Under General Edge Conditions'" (2001, ASME J. Appl. Mech., 68, p. 687)

T. C. T. Ting
Department of Civil and Materials Engineering,
University of Illinois at Chicago, 842 Taylor Street, M/C 246, Chicago, IL 60607-7023
e-mail: tting@uic.edu

Contrary to what Professor Ru stated in his first sentence, the paper did not discuss "conformal mapping" techniques applied to anisotropic plane elasticity. The paper discussed "mapping" in anisotropic elasticity. As emphasized in Section 4 of the paper, mapping in anisotropic elasticity is not conformal. Many papers that dealt with mapping in anisotropic elasticity used the word conformal mapping indiscriminately.

I have presented clearly what I wanted to say in the paper. I have no further comments.

