A mechanism is presented for producing explicitly the general solutions to the one-dimensional Toda molecule equations associated with the classical complex Lie algebras of rank $N$: $\mathfrak{sl}(N+1, \mathbb{C})$, $\mathfrak{so}(2N+1, \mathbb{C})$, $\mathfrak{sp}(N, \mathbb{C})$ and $\mathfrak{so}(2N, \mathbb{C})$. The method itself is applicable, however, to any semi-simple Lie algebra. A particular feature of our solutions is their description in terms of the “initial data”, which is consistent with the fact that the one-dimensional Toda molecule equation is a Newtonian equation of motion. The solutions for the algebras associated with $SU(2)$, $SU(3)$ and $Sp(2)$ are given explicitly.

§ 1. Introduction

In a previous paper,\(^1\) (referred to as [A]), we showed how the two-dimensional Toda lattice equations\(^2\)\(^3\) are related to self-dual Yang-Mills equations in Yang’s $R$-gauge,\(^4\) and furthermore how the solutions are derivable by using the Iwasawa decomposition when the gauge group is a classical simple Lie group, $SU(N+1)$, $SO(2N+1)$, $Sp(N)$ or $SO(2N)$.

Solutions of the two-dimensional Toda lattice equations correspond to instantons with cylindrical symmetry.\(^5\)\(^-\)\(^9\) On the other hand, solutions of the one-dimensional equations are spherically symmetric monopole solutions in a spontaneously broken, non-abelian gauge theory [see Ref. 9] and references therein].

This paper is the first of two in which we study in general the one-dimensional Toda lattice or Toda molecule equation, by which we mean

$$dt^2 \Psi_a = -\exp[- \sum_{\neq \beta} K_{a\beta} \Psi_\beta], \quad a=1, 2, \cdots N,$$

where $(K_{a\beta})$ denotes the Cartan matrix determining the appropriate rank $N$ algebra. This paper contains a mechanism for producing the general solution to $(1\cdot 1)$, while the second will be concerned specifically with the monopole application.

The methods of solution of the one-dimensional and two-dimensional equa-

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tions are not quite the same: The method presented here is much easier and more penetrable than the method for the two-dimensional case. One particular difference lies in the fact that the one-dimensional equations are Newtonian equations of motion, and therefore their solutions may be described in terms of the "initial values"

\[ \exp(\mathcal{V}_a)|_{t=0} = a_a, \quad d_t \mathcal{V}_a|_{t=0} = b_a, \quad a = 1, 2, \cdots N, \]  

(1.2)

where \(a_a\) and \(b_a\) are the input data.

The background of the problem has been described in § 4 of [A] in reference to Olshanetsky and Perelomov's method.\(^1\) From the work of Manakov\(^1\) and Flaschka,\(^2\) it is widely known that the system of (1.1) is equivalent to the Lax pair equation

\[ \dot{L} = [L, A], \]  

(1.3)

where the dot is used in place of \(d_t\) to denote differentiation with respect to \(t\). By applying the Iwasawa decomposition to the complexification of the group associated with the Toda molecule equation, we relate the exponentials of the solutions \(\mathcal{V}_a, a = 1, 2, \cdots N\) of (1.1) to the initial data via the variable \(L\) in (1.3).

We shall see that \(\exp(\mathcal{V}_a)\) is given by a semi-principal minor of a matrix \(x(U)\), written in terms of \(a_a, b_a\) and the eigenvalues \(\lambda_i\) of \(L\). When the eigenvalues are distinct, the \(t\) dependence of the solution is confined to exponentials of the form \(e^{\pm \lambda_i t}\).

In § 2, we shall provide a résumé of the algebraic techniques and results developed in [A], specifically those demonstrating the relationship between \(\exp(\mathcal{V}_a)\) and \(\chi(t)\). The remainder of the section is devoted to describing how, in principle, we can determine the semi-principal minors of \(\chi(t)\) and hence the solutions, \(\exp(\mathcal{V}_a)\). In §§ 3 and 4, we turn our attention to particular algebras. Firstly in § 3, we consider the case of \(g_N\), the algebra associated with gauge group \(SU(N+1)\) — the method of solution is typical and highly symmetric. By way of illustration, the solutions of the \(SU(2)\) and \(SU(3)\) Toda molecule equations are given explicitly. In § 4, we then consider the other classical algebras \(b_N, c_N\) and \(b_N\), associated with \(SO(2N+1), Sp(N)\) and \(SO(2N)\) respectively. As an example, the \(Sp(2)\) solution is explicitly constructed. Finally, in § 5, we shall make some concluding remarks.

§ 2. Method of solution

To determine \(\exp(\mathcal{V}_a), a = 1, 2, \cdots N\), we firstly consider the Lax pair equation (1.3), where the matrices \(L\) and \(A\) are not those defined in Refs. 11) and 12), but are given by\(^1\)
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\[ L = B - A, \quad (2\cdot1) \]
\[ A = R^{-1}d_t R, \quad B = \tilde{R}^{-1}d_t \tilde{R}. \quad (2\cdot2) \]

\( R \) and \( \tilde{R} \) belong to a symmetric space \( G/Gc, \ G \) being the maximal compact subgroup of \( Gc. \ Gc \) is the Lie group associated with a semi-simple, complex valued Lie algebra \( g_c \) in which the potentials \( A \) and \( B \) take their values.

For each algebra \( g_c \), there exists a maximal toral subalgebra \( h \) from which can be defined the root spaces \( g_{\pm a} \) by

\[ g_{\pm a} = \{ E_{\pm a} \in g_c : [H, E_{\pm a}] = \pm a(H)E_{\pm a} \forall H \in h \}. \quad (2\cdot3) \]

The set \( \{ a(H) \} \) is known as the positive root system, \( \Delta^+ \). It contains a subset

\[ \Pi^+ = \{ a_i, i = 1, 2, \ldots, N = \text{rank } g_c \} \]

of simple roots with the property that any other root can be written as a linear combination of the \( \{ a_i \} \) with coefficients either all non-negative or all non-positive. (We shall sometimes use the notation \( a = 1, 2, \ldots, N \) to imply summation over the simple roots \( a_i, i = 1, 2, \ldots, N. \))

The algebra \( g_c \) of rank \( N \) can be decomposed according to Cartan's root space decomposition

\[ g_c = h + \sum_{\alpha \neq 0} (g_{+\alpha} + g_{-\alpha}). \quad (2\cdot4) \]

So the \( 3N \) elements of the bases \( h_a, E_{\pm a} (a \in \Pi^+) \) for \( h \) and \( g_{\pm a} \) respectively comprise a set of generators for \( g_c \) itself. The basis elements satisfy\(^{12}\)

\[ [H_a, H_b] = 0, \quad [E_{-a}, E_{-b}] = H_a \delta_{ab}, \quad [H_a, E_{+b}] = \pm K_{ab} E_{+b}, \]

\[ \text{ad}(E_{\pm a})K_a(E_{\pm a}) = 0 \quad \text{if } a \neq b. \quad (2\cdot5) \]

The Cartan matrix \( K \) which appears in the Toda molecule equation (1\cdot1) and in the commutation relations (2\cdot5) determines the algebra \( g_c \).

We can use the generators of \( g_c \) to expand \( R \) and \( \tilde{R} \) in (2\cdot2) as

\[ R = na, \quad \tilde{R} = \tilde{n} \tilde{a} \quad (2\cdot6) \]

with

\[ n = \exp \left( \sum_{a \neq 0} z_a E_{-a} \right), \quad a = \exp \left( \sum_{a \neq 0} \phi_a H_a \right), \]

\[ \tilde{n} = \exp \left( - \sum_{a \neq 0} \tilde{z}_a E_{+a} \right), \quad \tilde{a} = \exp \left( - \sum_{a \neq 0} \tilde{\phi}_a H_a \right), \quad (2\cdot7) \]

where \( z_a, \tilde{z}_a \) are complex functions and \( \phi_a, \tilde{\phi}_a \) are real functions of \( t. \) \( n \) and \( \tilde{n} \)
in (2\cdot7) are subject to the constraints
that is,
\[ y_a = \bar{y}_a = 0 \quad \text{for} \quad a \in \mathbb{A}^+ \setminus \Pi^+ , \] (2.9)
\[ y_a = d_1 z_a , \quad \bar{y}_a = d_1 \bar{z}_a \quad \text{for} \quad a \in \Pi^+ . \] (2.10)

\( R, \hat{R} \mathbb{E}G_c/G \) can be written in the form shown in (2.6), since, regarded as a non-compact Lie algebra over \( \mathbb{R} \), the algebra \( \mathfrak{g}_c \) and hence its associated group \( G_c \), can be decomposed by the Iwasawa decomposition. The details of its application are given in [A]. By using (2.6), we determine \( A \) and \( B \) and then substitute into (1.3) to obtain the “main equations” and “subsidiary equations” similar to (2.22) and (2.23) of [A]. The latter is solved to give
\[ \varepsilon y_a = \bar{\varepsilon} \bar{y}_a = \exp \left[ - \sum_{\alpha \in \Pi^+} K_{as} (\phi_s + \bar{\phi}_s) \right] , \] (2.11)
where \( \varepsilon \) and \( \bar{\varepsilon} \) are constants. Here we choose
\[ \varepsilon = - \bar{\varepsilon} = 1 \] (2.12)
and put
\[ \phi_s + \bar{\phi}_s = \Psi_a ; \] (2.13)
so that
\[ L = \sum_{\alpha \in \Pi^+} [ - \Psi_a H_a + \exp ( - \sum_{\beta \in \Pi^+} K_{as} \Psi_\beta ) E_{-a} - \exp ( - \sum_{\beta \in \Pi^+} K_{as} \Psi_\beta ) E_{-a} ] , \] (2.14)
and the main equation reduces to
\[ d_1 \Psi_a = - \exp ( - \sum_{\alpha \in \Pi^+} K_{as} \Psi_s ) . \] (2.15)

Equation (2.15) is the Toda molecule equation associated with the group \( G \).

We note that, by using (2.11)\textendash{}(2.13), the constraints (2.10) can be integrated to give
\[ z_a = - \bar{z}_a = - \Psi_a \quad \text{for} \quad a \in \Pi^+ \] (2.16)
up to the addition of an arbitrary constant.

We shall now detail a method of obtaining the solution to Eq. (2.15). From (1.3), we can infer that\textsuperscript{12)}
\[ L(t) = R^{-1} L(0) R , \quad A = R^{-1} \dot{R} , \] (2.17)
where \( L(0) \) is constant.\textsuperscript{*}

\textsuperscript{*} We should like to point out here that \( L(0) \) in (4.7)\textendash{}(4.11) of [A] should be read as \( L_{00} \), since \( L(t) \) does not necessarily become \( R(t) R^{-1} \) at \( t = 0 \).
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\[ L_{(0)} = R(B - A)R^{-1} \]
\[ = -\dot{\chi} \chi^{-1}, \]

where

\[ \chi = RR^{-1}. \]  \hspace{1cm} (2·18)

Now (2·18) can be solved for \( \chi(t) \) giving

\[ \chi(t) = \exp(-L_{(0)}t)\chi(0), \]  \hspace{1cm} (2·20)

while the identification (2·19) implies that

\[ \chi(t) = nh\bar{n}^{-1} \]  \hspace{1cm} (2·21)

with \( h = a\bar{a}^{-1} \).

It is possible to find convenient \( M \times M \) matrix representations for the generators \( H_a, E_{\pm a} \) satisfying (2·3), where for the algebras \( \mathfrak{g}_N = \mathfrak{s}(N + 1, \mathbb{C}), \mathfrak{b}_N = \mathfrak{s}(2N + 1, \mathbb{C}), \mathfrak{c}_N = \mathfrak{o}(N, \mathbb{C}) \) and \( \mathfrak{b}_N = \mathfrak{s}(2N, \mathbb{C}) \), \( M \) is \( N + 1, 2N + 1, 2N \) and \( 2N \) respectively. The representations are such that, in (2·21), \( h \) is diagonal and \( n \) and \( \bar{n} \) either have triangular form or have some \( N \times N \) submatrix with this property. In that case, we can show that (2·21) becomes

\[
\exp(\mathcal{V}_n) = \begin{cases} 
\chi(t)\begin{bmatrix} 1 & 2 & \cdots & m \end{bmatrix} & m = 1, 2, \ldots, N \text{ for } \mathfrak{g}_N, \mathfrak{c}_N, \\
\chi(t)\begin{bmatrix} 1 & 2 & \cdots & m+1 \end{bmatrix} & m = 1, 2, \ldots, N-2 \text{ for } \mathfrak{b}_N, \\
\chi(t)\begin{bmatrix} 1 & 2 & \cdots & m+1 \end{bmatrix} & m = 1, 2, \ldots, N \text{ for } \mathfrak{b}_N, 
\end{cases}
\]

\[
\exp(\mathcal{V}_{N-1}) = \left\{ \chi(t)\begin{bmatrix} 1 & 2 & \cdots & N-1 \end{bmatrix}/\chi(t)\begin{bmatrix} 1 & 2 & \cdots & N \end{bmatrix}\right\}^{1/2} \text{ for } \mathfrak{b}_N. \]  \hspace{1cm} (2·22)

Here we have used the notation

\[ X \begin{bmatrix} k_1 & k_2 & \cdots & k_m \\ r_1 & r_2 & \cdots & r_m \end{bmatrix} \]

to denote the minor of the matrix \( X \) formed by striking out all but rows \( k_1, k_2, \ldots, k_m \) and columns \( r_1, r_2, \ldots, r_m \).

The derivation of (2·22) is applicable to any other semi-simple algebra, since it is always possible to choose a basis for \( \mathfrak{g}_\mathfrak{c} \) such that in the adjoint representation the generators \( E_{\pm a} \) have triangular form with zeros in the diagonal.\( ^{14} \)
However, we shall consider only $a_N, b_N, c_N, d_N$ because they have convenient representations.

Having shown that the Toda variables $\Psi_m$ are related to the principal minors of $\chi(t)$ by (2.22), we must now determine these minors in terms of $L(0)$ and $\chi(0)$ using (2.20). We take an $M \times M$ matrix representation for $L(0)$ and assume that its eigenvalues are all different and given by $\lambda_1, \lambda_2, \ldots, \lambda_M$. Then there exist matrices $V$ and $V^{-1}$ such that

$$L(0) = VH(0)V^{-1} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M)$$

and hence

$$e^{-L(0)t} = V \{\text{diag}(e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots, e^{-\lambda_M t})\} V^{-1}. \quad (2.24)$$

By substituting (2.24) into (2.20) and using the Binet-Cauchy formula, we obtain

$$\chi(t) \begin{bmatrix} 1 & 2 & \cdots & m \\ 1 & 2 & \cdots & m \end{bmatrix} = \sum_{1 \leq r_1 < r_2 < \cdots < r_m \leq M} W(r_1, r_2, \ldots, r_m) \chi(0) \begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ 1 & 2 & \cdots & m \end{bmatrix}, \quad (2.25)$$

where

$$W(r_1, r_2, \ldots, r_m) = V \begin{bmatrix} 1 & \cdots & m \\ r_1 & \cdots & r_m \end{bmatrix} \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq M} V^{-1} \begin{bmatrix} r_1 & \cdots & r_m \\ k_1 & \cdots & k_m \end{bmatrix} \chi(0) \begin{bmatrix} k_1 & \cdots & k_m \\ 1 & \cdots & m \end{bmatrix}. \quad (2.26)$$

To evaluate (2.25), firstly the matrices $V$ and $V^{-1}$ must be found by determining the eigenvectors of $L(0)$ and $L(0)^T$ respectively. We note that, from (2.17), $L(0)$ can be expanded in terms of the generators of $g_c$ as

$$L(0) = nXn^{-1} = X + \sum_{\alpha \in \mathbb{F}^*} z_\alpha [E_\alpha, X] + \frac{1}{2} \sum_{\alpha, \beta \in \mathbb{F}^*} z_\alpha z_\beta [E_\alpha, [E_\beta, X]] + \ldots, \quad (2.27)$$

where

$$X = \sum_{\alpha \in \mathbb{F}_0} [-\Psi_\alpha H_\alpha + E_\alpha - \exp(-\sum_{\beta \in \mathbb{F}_0} K_{\alpha \beta} \Psi_\beta^2) E_\alpha].$$

Now, for $\alpha \in \mathbb{F}^*$, by using (2.6),

$$[E_\alpha, X] = -\sum_{\beta \in \mathbb{F}_0} (\delta_{\alpha \beta} H_\alpha + \Psi_\beta K_{\alpha \beta} E_\alpha + f_\alpha \exp(-\sum_{\gamma \in \mathbb{F}_0} K_{\alpha \gamma} \Psi_\gamma) E_{-(\alpha + \beta)}),$$
where the \( f_s \) are zero or constant. All other commutators in (2.27) produce \( E_{-a}, \ a \in \mathcal{A}^* \), terms only and thus
\[
L(0) = \sum_{a \in \mathcal{A}^*} \{ - (\bar{\Psi}_a + z_s) H_a + E_{-a} \} + \sum_{a \in \mathcal{A}^*} F_a(\Psi) E_{-a},
\]
(2.28)
where the functions \( F_a \) are unspecified. Hence, in \( L(0) \), the coefficient of \( E_{+a} \) is unity for \( a \in \mathcal{A}^* \) and zero for \( a \in \mathcal{A}^* \). Furthermore, using (2.16), the coefficient of \( H_a \) in (2.28) is also zero.

When written in terms of an \( M \times M \) matrix representation, the form of \( L(0) \), given by (2.28) for \( a_N \) has zero entries on the leading diagonal and unit entries on the next line parallel to and above it. All the other entries in the upper triangular part are zero and are arbitrary in the strictly lower triangular part. Such a form facilitates the determination of the eigenvectors. For the other classical simple algebras, \( L(0) \) does not have this convenient form, but for \( b_N \) and \( c_N \), it is possible to find a matrix \( L(0) \) which does; \( L(0) \) being given by
\[
L(0) = \text{sign} \ S L(0) S^{-1}.
\]
(2.29)

If we use \( e_{ij} \), \( 1 \leq i, j \leq M \) to denote the \( M \times M \) matrix in which the only non-zero entry is unity where the \( i \)-th row and \( j \)-th column intersect, then \( S \) in (2.29) is given by
\[
S = \begin{cases} 
\sum_{i} (e_{N+i+1 \in i+2} + e_{i+i}) + e_{N+1} & \text{for } b_N, \\
\sum_{i} e_{i+i} + e_{N+i+1} & \text{for } c_N,
\end{cases}
\]
(2.30)
and sign denotes the operation of reversing the signs of the elements in all rows \( k \) where the entry in the \( (k+1) \)-th column is \(-1\).

If, for \( a_N \), we identify \( S \) with \( I_{N+1} \), then for the algebras \( a_N, b_N, c_N \), we can define two secular polynomials in \( \Lambda \) by
\[
P_{k+1}(\Lambda) = (-1)^{s(a, b)} (\Lambda I_M - SL(0) S^{-1})^{[1 \ 2 \ ... \ k]},
\]
\[
Q_{M-k}(\Lambda) = (-1)^{s(M-kM)} (\Lambda I_M - SL(0) S^{-1})^{[M-k+1 \ ... \ M]}.
\]
(2.31a)
for \( k = 1, 2, \cdots M \) and we choose
\[
P_k(\Lambda) = Q_M(\Lambda) = 1.
\]
(2.31b)
In (2.31a), \( s(a, b) \) indicates the number of rows between row \( a \) and row \( b \) inclusive which have undergone sign reversal. Then the column vector
\[
C_i(P) = S^T(P_1(\Lambda_i) \ P_2(\Lambda_i) \cdots \ P_M(\Lambda_i))^T
\]
(2.32)
is an eigenvector of $L_{(0)}$ corresponding to eigenvalue $\lambda_i$. The corresponding eigenvector of $L_{(0)}^T$ is given by replacing $P_\kappa$ in (2·32) by $Q_\kappa$.

Hence for the three algebras $a_N$, $b_N$, $c_N$, we choose the matrices $V$ and $V^{-1}$ by

$$V = A^{-1}(C_1(P) \ C_2(P) \cdots C_M(P)) = A^{-1}P, \quad (2·33a)$$

$$V^{-1} = A\Xi(C_1(Q) \ C_2(Q) \cdots C_M(Q))^T = A\Xi Q, \quad (2·33b)$$

where

$$\Xi = \text{diag}(\xi_1, \xi_2, \cdots \xi_M) \quad (2·34a)$$

with

$$\delta_{ij}\xi_j^{-1} = \sum_{\kappa=1}^M P_\kappa(\lambda_\kappa) Q_\kappa(\lambda_j) \quad (2·34b)$$

and

$$A^M = \det P = \det S^T \prod_{j=1}^{M-1} (-1)^{p(t,j)} \prod_{k=1}^{M} (\lambda_k - \lambda_\kappa), \quad (2·34c)$$

where $(-1)^{p(t,j)}$ is that factor appearing in the definition of $F_{\kappa+1}(\lambda)$ in (2·31). From $VV^{-1} = I_M$, it follows that

$$\sum_{\kappa=1}^M \xi_\kappa \lambda_\kappa^p = \delta_{pM-1} (-1)^{p(t,M-1)}, \quad (p = 0, 1, \cdots M-1) \quad (2·35)$$

which are solved as

$$\xi_\kappa = (-1)^{M-p+\#(1,M-1)} \prod_{j>\kappa} (\lambda_j - \lambda_\kappa)^{-1} \prod_{\kappa<i} (\lambda_i - \lambda_\kappa)^{-1} \quad (2·36)$$

Then we may deduce that

$$\prod_{\kappa=1}^M \xi_\kappa = (-1)^{M(M-1)/2 + \#(1,M-1)} A^{-2M} \quad (2·37)$$

$\det Q$ can be determined consistently either from (2·37) and $\det V^{-1} = 1$ or directly from definition (2·33b) of $Q$.

At this point we should also note that the eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_M$ are not independent. Since the matrix representing an element in any of the classical algebras is traceless, $L_{(0)}$ and hence $H_{(0)}$ in (2·23) are traceless; that is,

$$\sum_{\kappa=1}^M \lambda_\kappa = 0 \quad (2·38)$$

There are additional restrictions on $\lambda_i$ for the algebras $b_N$, $c_N$ and $b_N$, which will be considered in § 4.

The second stage of the evaluation of (2·25) involves determining minors of $\chi(0)$, which is how the initial data $a_\kappa$ and $b_\kappa$ of (1·2) feature in the solution.
From (2.21), we know that, as a matrix, \( \chi(t) \) has entries involving \( z_a, z_b, \beta \epsilon \Delta^+ \) and \( \Psi_a, a \epsilon H^+ \). Now, by using (2.9), \( z_a, z_b, \beta \epsilon \Delta^+ \) can be expressed in terms of \( z_a, z_a, a \epsilon H^+ \), which in turn can be written in terms of \( \Psi_a \) using the intermediate integrals (2.16). Consequently \( \chi(t) \) is expressible entirely in terms of \( \Psi_a \) and \( \Psi_a \) and hence \( \chi(0) \) only involves \( a_a \) and \( b_a \).

The initial data enter into (2.25) also via the eigenvalues, \( \lambda_i \). The eigenvalues are related to the infinite series of constants of the motion \( c_a \) by

\[
\sum_{i=1}^{N} \lambda_i = \text{Tr} \, L^k = k c_a
\]

(2.39)

and the constants \( c_a \) can naturally be written in terms of \( a_a \) and \( b_a \).

We have described above a method for determining the principal minors of \( \chi(t) \), when the eigenvalues of \( L(0) \) are distinct, in terms of the initial data \( a_a \) and \( b_a \). Hence, by (2.22) we have determined \( \exp(\Psi_a) \); that is, we have solved the Toda molecule equation (2.15). The method of determining \( L(0) \) by (2.23), and hence the minors of \( \chi(t) \) by (2.25), is not applicable when the eigenvalues of \( L(0) \) are degenerate. The solution of (2.16) for \( G = SU(N+1) \) under these circumstances will be determined in § 3.

As a conclusion of this section, we should like to include some material on reflection symmetry to enable us subsequently to consider the invariance of the solutions under a permutation of the eigenvalues. We define \( w_a \) by

\[
H(0) = w_a H_a
\]

(2.40)

with \( H(0) \) and \( H_a \) given by (2.23) and (2.5) respectively. Then, introducing a root \( a(H(0)) \) by

\[
[H(0), E_{+a}] = a(H(0)) E_{+a},
\]

(2.41)

the definitions (2.40) and (2.5) imply that

\[
\sum_{b \in \Delta^+} w_a K_{ba} = a(H(0)),
\]

(2.42)

which is equivalent to

\[
(w_a, a_j^\vee) = \delta_{ij},
\]

(2.43)

where \( a_j^\vee \) is defined by

\[
(a_s, a_j^\vee) = K_{ij}.
\]

(2.44)

\( w_a \) is thus a fundamental weight.\(^{15}\)

The Weyl subgroup of \( G \) is generated by the reflections \( s_a \) with

\[
s_a(\lambda_s) = \lambda_s - 2[(\lambda_s, a) / (a, a)]a,
\]

(2.45)
that is, the reflection of \( \lambda_\alpha \) with respect to the hyperplane perpendicular to the root \( \alpha \) of \( G \). In matrix form (2·45) is equivalent to

\[ s_\alpha(H_\alpha) = H_\alpha - 2 \tilde{H}_\alpha \alpha(H_\alpha) / \alpha(\tilde{H}_\alpha), \]

where \( \tilde{H}_\alpha \) and \( \alpha(\tilde{H}_\alpha) \) are defined respectively by

\[ B(H_\alpha, \tilde{H}_\alpha) = \alpha(H_\alpha), \]
\[ \alpha(\tilde{H}_\alpha) = B(\tilde{H}_\alpha, \tilde{H}_\alpha), \]

\( B \) being the Killing form. Then it follows that

\[ 2 \tilde{H}_\alpha / \alpha(\tilde{H}_\alpha) = H_\alpha \]

and hence

\[ s_\alpha(H_\alpha) = H_\alpha - \alpha(H_\alpha) H_\alpha. \]

When \( \alpha_i(H_\alpha) = \lambda_i - \lambda_i, \) as is generally the case, the right-hand side of (2·50) has the same form as \( H_\alpha \) except with \( \lambda_i \) and \( \lambda_i+1 \) interchanged.

Incidentally, \( w = s_1 s_2 \cdots s_\alpha \) for \( a_n \) corresponds to \( \tilde{E} = E^{-1} \) used in § 6 of Ref. 16). The smallest number \( h \) such that \( w^h = 1 \) is \( N + 1 \) for \( a_n \), \( 2N \) for \( b_n \) and \( c_n \) and \( 2N - 2 \) for \( b_n \). Moreover, the smallest numbers \( m_{\alpha \beta} \) such that

\[ (s_\alpha s_\beta)^{m_{\alpha \beta}} = 1 \]

determine, in a similar way to \( K_{\alpha \beta} \), the Dynkin-Coxeter diagram; \( m_{\alpha \beta} = 1 \), and, if \( m_{\alpha \beta} = 2 \), \( s_\alpha \) and \( s_\beta \) commute. 13)

§ 3. General solution for \( G = SU(N + 1) \)

In § 2 we described in principle the various stages involved in determining the principal minors of \( \chi(t) \) given by (2·20) and hence, from (2·22), the exponentials of the solutions \( \Psi_m \), \( m = 1, 2, \cdots, N \), of the one-dimensional Toda molecule equation (1·1) associated with arbitrary groups. As the Toda molecule equations associated with \( SU(N + 1) \) are endowed with typical properties, they deserve explicit study. \( SU(N + 1) \) is the maximal compact subgroup of \( G_e \equiv SL \) \( (N + 1, C) \) with associated algebra \( a_n \), therefore in this section we shall illustrate the method of § 2 in this specific case.

The determination of the minors of \( \chi(t) \) involves determining a) \( L(t) \); b) the secular polynomials (2·31) and hence \( V \) and \( V^{-1} \) in the similarity transformation (2·23); and finally c) \( \chi(0) \). We are then able to use the formulae (2·25) and (2·26) to obtain the solutions. We shall describe the main points of the steps a), b) and c), firstly for \( a_n \) in general and then specifically for \( N = 1 \) and \( N = 2 \). The method for \( N = 2 \) is representative of the method for arbitrary \( N \).
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From the remarks in § 2, we know that

$$L_{(0)} = \sum_{i=1}^{N-1} e_{i,i+1} + \sum_{i=1}^{N} l_{i,i} e_{i,i} ,$$  \hspace{1cm} (3·1)$$

where the $l_{i,i}$ are constant by definition and may be determined as follows. If we denote $\{z_{a}, a \in A\}$ in (2·7) by $\{z_{ij}, 1 \leq j < i \leq N + 1\}$ and define

$$Z_{i,j} = z_{ij} \mp \frac{1}{2} \sum_{k=j+1}^{i-1} z_{ik} z_{kj} ,$$  \hspace{1cm} (3·2)$$

then, from (2·7)

$$n = I + \sum_{1 \leq i,j \leq N+1} Z_{i,j} e_{i,j} , \quad n^{-1} = I - \sum_{1 \leq i,j \leq N+1} Z_{i,j} e_{i,j} .$$  \hspace{1cm} (3·3)$$

By using (3·3), we can write $l_{ij}$ as

$$l_{ij} = - \sum_{j \leq k \leq i-1} (\delta_{i,k+1} + Z_{i,k+1})(\delta_{j,k} - Z_{j,k}) \exp(-\sum_{m=1}^{i} K_{km} \Psi_{m})$$

$$+ \sum_{j \leq k \leq i} (\delta_{i,k} + Z_{i,k})(\delta_{j,k} - Z_{j,k})(\Psi_{k-1} - \Psi_{k})$$

$$+ \sum_{j \leq k \leq i} (\delta_{i,k} + Z_{i,k})(\delta_{k+1,j} - Z_{k+1,j}) \exp(-\sum_{m=1}^{i} K_{km} \Psi_{m})$$  \hspace{1cm} (3·4)$$

with the proviso that

$$Z_{i,j} - \sum_{k=j+1}^{i-1} Z_{ik} Z_{kj} = 0 , \quad i > j + 1 ,$$  \hspace{1cm} (3·5)$$

which follows from (2·9). It is a consequence of (3·5), (3·2) and (2·16), that $l_{ii}$, given by (3·4), can be written in terms of $a_{ii}$ and $b_{ii}$.

Since for $a_{ii}$, $L_{(0)} = L_{(0)}$, the characteristic equations for $L_{(0)}$ and $L_{(0)}^T$, with eigenvectors of the form (2·32), imply that the secular polynomials $P_{k+1}(\lambda)$ and $Q_{N-k+1}(\lambda)$ satisfy the following recurrence relations:

$$P_{k+2} = \lambda P_{k+1} - \sum_{j=1}^{k} l_{j,k+1} P_{j} ,$$

$$Q_{N-k} = \lambda Q_{N-k+1} - \sum_{j=k+2}^{N-1} l_{j,N-k+1} Q_{j}$$  \hspace{1cm} (3·6)$$

with

$$P_{1}(\lambda) = Q_{N+1}(\lambda) = 1 , \quad P_{k}(\lambda) = Q_{N}(\lambda) = \lambda .$$  \hspace{1cm} (3·7)$$

Therefore $P_{k+1}(\lambda)$ and $Q_{N-k+1}(\lambda)$, $k = 0, 1, \cdots N + 1$, are monic polynomials of degree $k$ in $\lambda$ and can be expressed as

$$P_{k+1}(\lambda) = \sum_{m=0}^{k} \alpha_{k+1}^{(m)} \lambda^{m} , \quad Q_{N-k+1}(\lambda) = \sum_{m=0}^{k} \beta_{N-k+1}^{(m)} \lambda^{m}$$  \hspace{1cm} (3·8)$$
with $\alpha^{(m)}_{k+1} = \beta^{(m)}_{N-k+1} = 1$. Furthermore,

$$\alpha^{(k-1)}_{k+1} = \beta^{(k-1)}_{N-k+1} = 0,$$

$$\alpha^{(k-1)}_{k+1} = - \sum_{j=1}^{k-1} I_{j+1} j, \quad \beta^{(k-1)}_{N-k+1} = - \sum_{j=1}^{N-k+2} I_{j+1} j.$$

The recurrence relations (3-6) then imply

$$\alpha^{(m)}_{k+2} = \alpha^{(m-1)}_{k+1} - \sum_{j=1}^{k} I_{j+1} \beta^{(m)}_{j},$$

$$\beta^{(m)}_{N-k} = \beta^{(m-1)}_{N-k+1} - \sum_{j=k}^{N-k+2} I_{j} \alpha^{(m)}_{j+1},$$  \hspace{1cm} (3-10)

which successively determine $\alpha^{(m)}$ and $\beta^{(m)}$. Having determined the secular polynomials, the explicit form of the matrices $P$ and $\mathbb{E}$ in (2-33) can be found. In addition, from (2-34c)

$$A^{(m)} = \sum_{j=1}^{N+1} (\lambda_l - \lambda_j),$$

hence $V$ and $V^{-1}$ can be determined explicitly.

$\tilde{n}^{-1}$ enters into $\chi(0)$, given by (2-21) evaluated at $t=0$, and thus we next determine it. By writing the set \{${\tilde{z}}_a, a \in A^+$\} as \{${\tilde{z}}_{ji}, 1 \leq j < i \leq N+1$\} and using $\tilde{Z}_h$ to denote

$$\tilde{Z}_h = \tilde{Z}_{ji} = \sum_{k=j+1}^{i} \tilde{Z}_{jk} \tilde{Z}_{ki},$$

$\tilde{n}^{-1}$ becomes

$$\tilde{n}^{-1} = I + \sum_{1 \leq j \leq N+1} \tilde{Z}_{ji} e_{ji},$$

subject to the condition

$$\tilde{Z}_{ji} + \sum_{k=j+1}^{i} \tilde{Z}_{jk} \tilde{Z}_{ki} = 0, \quad i > j + 1.$$  \hspace{1cm} (3-13)

All the ingredients of (2-26) necessary to evaluate the principal minors of $\chi(t)$ for the algebra $a_{\omega}$ have now been specified. We should like to stress that $W_{(i-r\omega)}$ of (2-25) is expressible in terms of the initial data, $a_{\omega}$ and $b_{\omega}$. To see this point, let us consider the infinite series $c_{\omega}$ given by (2-39) in the particular case of $a_{\omega}$. We shall show that $c_{\omega}$ is determined by $l_{ij}$ or $\alpha^{(m)}$.

From (2-31),

$$P_{N+1}(\lambda) = \det(\lambda I_{N+1} - L_{(m)}) = \prod_{i=1}^{N+1} (\lambda - \lambda_i) = 0,$$  \hspace{1cm} (3-15)
and, by comparison of the coefficients of $\lambda^{N-1}$ in (3·15) and (3·8), we find that
\[ \sum_{1 \leq i < j \leq N+1} \lambda_{ij} = \alpha^{(N+1)}_{N+2} = -\sum_{j=1}^{N} l_{j+1,j}. \] (3·16)

It then follows from the square of (2·39) that
\[ c_2 = \frac{1}{2} \sum_{i=1}^{N+1} \lambda_i^2 = \sum_{j=1}^{N} l_{j+1,j}. \] (3·17)

Now $L_{(0)}$ satisfies its characteristic equation (3·15), therefore the polynomial $P_{N+2}(L_{(0)})$ vanishes. By taking the trace of this expression, we obtain
\[ \text{Tr } P_{N+2}(L_{(0)}) = \text{Tr } P_{N+2}(L) = 0 \]
and hence, using (2·39),
\[ (N + 1) c_{N+1} + \sum_{m=2}^{N+1} \alpha^{(m)}_{N+2} m c_m + (N + 1) \alpha^{(0)}_{N+2} = 0. \] (3·18)

For example, for $a_2$ (3·18) becomes
\[ 3c_3 + 3a^{(0)}_2 = 0 \]
and, since $a^{(0)}_2 = -l_{31}$,
\[ c_3 = l_{31}. \] (3·19)

We shall now use the above method to determine the general solution to the one-dimensional Toda molecule equations in the specific cases of SU(2) and SU(3).

(i) SU(2): $L_{(0)} = e_{12} + c_2 e_{21}$, where $l_{21} = c_2 = \lambda_1^2$. Definition (3·4) and the intermediate integral (2·16) enable us to determine $l_{21}$ in terms of the Toda variable $\Psi_1$ as
\[ l_{21} = \Psi_1^2 - \exp(-2 \Psi_1). \] (3·20)

But, since $l_{21}$ is constant, (3·20) is true at $t = 0$, that is,
\[ c_1 = b_1 - a_1^2. \] (3·21)

The general solution is constructed in stages. Firstly, having determined $V$ and $V^{-1}$, from (2·24)
\[ \exp(-L_{(0)} t) = I_z \cosh \lambda_1 t - L_{(0)} \lambda_1^{-1} \sinh \lambda_1 t. \] (3·22)

Secondly, since from (2·19)
\[ \chi(t) = \exp(\Psi_1) e_{11} + [\exp(-\Psi_1) - \Psi_1^2 \exp(\Psi_1)] e_{22} + \Psi_1 \exp(\Psi_1)(e_{12} - e_{21}), \]
\[ \chi(0) = a_1 e_{11} + [a_1^{-1} - b_1^2 a_1] e_{22} + a_1 b_1 (e_{12} - e_{21}). \] (3·23)
Therefore, by (2·22) and (2·25), we obtain as the general solution
\[
\exp(\varphi_t) = \exp(-L_{\omega t}) \chi(0)[1]
\]
\[
= a_1 \cosh \lambda_1 t + \lambda_1^{-1} a_1 b_1 \sinh \lambda_1 t .
\]  
(3·24)
We note that \(\lambda_1\) in (3·24) can be expressed in terms of \(a_1\) and \(b_1\) by using (3·21).
Further, if we introduce the new variable \(\xi\) by
\[
a_1 \lambda_1 = \frac{1}{2}(\xi - \xi^{-1}) ,
b_1 \lambda_1^{-1} = (\xi^2 + 1)/(\xi^2 - 1) ,
\]
then (3·24) has the neater form
\[
\exp(\varphi_t) = \frac{1}{2} \lambda_1^{-1} [(\xi - \xi^{-1}) \cosh \lambda_1 t + (\xi + \xi^{-1}) \sinh \lambda_1 t] .
\]  
(3·25)

(ii) \(SU(3)\): In this case we can calculate the \(l_{ij}\) entries from (3·4) as
\[
\begin{align*}
l_{21} &= -\dot{z}_{21} + \dot{z}_{32} - \frac{1}{2} z_{22} z_{21} - z_{31} , \\
l_{32} &= -\dot{z}_{32} + \dot{z}_{33} - \frac{1}{2} z_{32} z_{31} + z_{31} , \\
l_{31} &= z_{21} \dot{z}_{32} - z_{32} \dot{z}_{21} + z_{33} z_{21} (\dot{z}_{21} - \dot{z}_{32})
\end{align*}
\]  
(3·26)
with condition (3·5) that
\[
\dot{z}_{31} = \frac{1}{2} \dot{z}_{21} z_{32} - \frac{1}{2} z_{21} \dot{z}_{32} .
\]  
(3·27)
As a solution of (3·27) we choose
\[
z_{31} = \frac{1}{2} (\dot{z}_{32} - \dot{z}_{21} - z_{32}^2 + z_{21}^2) .
\]  
(3·28)
The constants of the motion are, by (3·17) and (3·19),
\[
c_2 = l_{21} + l_{32} , \quad c_3 = l_{31} .
\]  
(3·29)
Having chosen (3·28), then, from (3·26),
\[
l_{21} = l_{32} = \frac{1}{2} c_2 .
\]  
(3·30)
We can use (2·11) and (2·16) to find \(z_{ij}\) and \(\dot{z}_{ij}\) at \(t = 0\) in terms of the initial data, and then
\[ c_2 = -\left[ (a_2/a_1^2) + (a_1/a_2^2) \right] + b_1^2 + b_2^2 - b_1b_2, \]
\[ c_3 = (a_2b_2/a_1^2) - (a_1b_1/a_2^2) + b_1b_2(b_2 - b_1). \]  

\( V \) and \( V^{-1} \) can easily be determined by first calculating \( P \) and \( Q \) from (3.6); we find
\[ V = \Lambda^{-1}(C_1(P) \ C_2(P) \ C_3(P)), \]
\[ V^{-1} = \Lambda \Xi(C_1(Q) \ C_2(Q) \ C_3(Q))^T, \]
where
\[ C_1(P) = \begin{pmatrix} 1 & \lambda_i & \lambda_i^2 - \frac{1}{2} c_2 \\ \end{pmatrix}^T, \]
\[ C_1(Q) = \begin{pmatrix} \lambda_i^2 - \frac{1}{2} c_2 & \lambda_i & 1 \\ \end{pmatrix}^T, \]
\[ \Lambda = (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1) \]
and
\[ \Xi = \frac{1}{\Lambda} \text{diag}(\lambda_3 - \lambda_2, \lambda_1 - \lambda_3, \lambda_2 - \lambda_1). \]

Now we determine the explicit form for \( \chi(0) \):
\[ \chi(0) = \begin{pmatrix} a_1 & a_1b_1 & \tilde{Z}_{13}(0)a_1 \\ -a_1b_1 & -b_1^2a_1 + (a_2/a_1) & -\tilde{Z}_{13}(0)a_1b_1 + ((b_2a_2)/a_1) \\ a_1Z_{11}(0) & a_1b_1Z_{11}(0) - ((b_2a_2)/a_1) & Z_{11}(0)\tilde{Z}_{13}(0)a_1 - ((b_2a_2)/a_1) + a_1^{-1} \\ \end{pmatrix}, \]
where
\[ Z_{11}(0) = \left( \frac{1}{2} z_{22}z_{21} + z_{21} \right) \bigg|_{t=0}, \]
\[ \tilde{Z}_{13}(0) = \left( \frac{1}{2} z_{12}z_{23} + z_{12} \right) \bigg|_{t=0}. \]

Having determined \( V \), \( V^{-1} \) and \( \chi(0) \), we are able to calculate (2.25) to obtain the explicit form of the general solutions. These are
\[ \exp(\Psi_i) = \sum_{t=1}^{2} W_{ij} \exp(-\lambda_i t) \]
with
\[ W_{ij} = (\lambda_i - \lambda_j)\Lambda^{-2}\{a_2(\lambda_i^2 - c_2) - a_1b_1\lambda_i + a_1b_2^2 - (a_2/a_1) \} \]
and
\[ \exp(\Psi_t) = \sum_{ijkl} W_{ijkl} \exp(\lambda_{ij}t) \]  

with \n
\[ W_{ijkl} = (\lambda_k - \lambda_j) \Lambda^{-3} \left( a_3 \lambda_i^2 + a_3 b_2 \lambda_i + (a_2^2/a_1^2) + a_2 b_1 b_2 - a_3 b_2^2 \right). \]  

In (3.37) to (3.39) the subscripts \((ijk)\) correspond to the cycle \((123)\).

Up until now, we have only described methods for determining the general solution when the eigenvalues of \(L_0\) are distinct. We should now like to turn our attention to the case when the eigenvalues are degenerate, at least when \(L_0\) takes its values in the algebra \(\omega_k\). In this case, \(L_0\) is no longer diagonalizable, but is similar to a matrix in Jordan normal form; that is, \(H_0\) in (2.23) must be replaced by the more general \(H_0\), a matrix with the eigenvalues in the leading diagonal, 0 or 1 entries in the line parallel to and above it and zero entries elsewhere. We arrange the eigenvalues of \(L_0\) into groups of distinct values so that the \(i\)-th distinct eigenvalue \(\lambda_i\) occurs \(r_i\) times, \(i = 1, 2, \ldots, k < N + 1\), and \(\sum_{i=1}^{k} r_i = N + 1\). Since the only free parameter in the eigenvector (2.32) is \(P_i(\lambda_i)\), the dimension of the eigenspace corresponding to a particular eigenvalue \(\lambda_i\) is one. Hence we can take the form of \(H_0\) to be

\[ H_0 = \sum_{i=1}^{k} \left\{ \lambda_i \sum_{j=r_i+1}^{r_i} \lambda_j e_{ji} + \sum_{j=r_i+1}^{r_i-1} e_{ji-1} \right\}, \]  

with \(r_0 = 0\). Of course, the similarity transformation is no longer \(V\) as given by (2.33a); the eigenvectors corresponding to a repeated eigenvalue are linearly dependent and therefore, if

\[ L_0 = VH_0V^{-1}, \]  

the columns of \(V\) are not all eigenvectors of \(L_0\). We choose \(V\) to be the nonsingular matrix

\[ V = \Lambda^{-1} (C_1(P) \ C'^1(P) \ \ldots \ C'^{r_i}(P) \ C_2(P) \ \ldots \ C_r(P) \ \ldots \ C^{N+1}(P))^T, \]  

where \(C_j(P)\) denotes the \(j\)-th derivative with respect to \(\lambda_i\) of \(C_i(P)\) given by (2.32) with \(S = I_{N+1}\). To determine \(V^{-1}\), we must invert \(V\) by conventional methods.

From (3.40) and (3.41), we obtain

\[ \exp(-L_0 t) = \sum_{i=1}^{k} \left\{ \lambda_i \sum_{j=r_i+1}^{r_i} e^{-\lambda_i t} e_{ji} - \sum_{j=r_i+1}^{r_i-1} t^j e^{-\lambda_i t} e_{ji-1} \right\} V^{-1}. \]  

Hence, having found the explicit form of \(V\) and \(V^{-1}\), we can proceed to find the general solution of the Toda molecule equation associated with \(SU(N+1)\) even when the eigenvalues are degenerate. Once again we calculate minors of \(\chi(t)\) as
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minors of the product $\exp(-L(0)t)\chi(0)$, where $\chi(0)$ will be the same as that determined above for the distinct case. We note that $r_i \neq N+1$, otherwise from (2.38) $\lambda_i = 0$, which is not a valid eigensolution — since it corresponds to the case $l_i = 0$ in (3.1).

We shall illustrate the method by considering the general solution for the $SU(3)$ Toda molecule when $L(0)$ has only two distinct eigenvalues. Equations (3.26)–(3.31) are still relevant to this example, but now $V$ is replaced by

$$V = \Lambda^{-1}(C_i(P) \bar{C}_i(P) C_i(P)),$$

(3.44)

where $C_i(P)$, $i=1, 2$ are given by (3.33a), but

$$C_i(P) = (0 \ 1 \ 2\lambda_i)^T$$

(3.45a)

and

$$\Lambda^2 = (\lambda_2 - \lambda_1)^2.$$

(3.45b)

Inverting $V$ produces

$$V^{-1} = \Lambda^{2} \begin{pmatrix} \lambda_2^2 - 2\lambda_1\lambda_2 - \frac{1}{2}c_2 & 2\lambda_1 & -1 \\ (\lambda_1 - \lambda_2)(\lambda_1\lambda_2 + \frac{1}{2}c_2) & \lambda_2^2 - \lambda_1^2 & \lambda_1 - \lambda_2 \\ \lambda_1^2 + \frac{1}{2}c_2 & -2\lambda_1 & 1 \end{pmatrix}.$$

(3.46)

Since $V$ and $V^{-1}$ are determined, we can find $\exp(-L(0)t)$ from (3.43) as

$$\exp(-L(0)t) = V[e^{-i\epsilon_1(e_{11} + e_{22} - t\epsilon_{12})} + e^{-i\epsilon_2}e_{33}] V^{-1}.$$  

(3.47)

By using $\chi(0)$ given by (3.34), we now determine the minors $\chi(t)\{1\}$ and $\chi(t)\{1, 2\}$ to obtain the solutions $\exp(\Psi_1)$ and $\exp(\Psi_2)$. They are explicitly

$$\exp(\Psi_1) = (D_1 - tD_1')e^{-i\epsilon_1} + D_2 e^{-i\epsilon_2}$$

(3.48)

with

$$D_1 = \Lambda^{-1} \left[ a_3(5\lambda_2^2 + c_2) - 2a_1b_1\lambda_1 + (a_2/a_1) - a_1b_2^2 \right],$$

$$D_1' = \Lambda^{-2} \left[ \frac{1}{2}a_1(\lambda_2^2 + c_2) + a_1b_1\lambda_1 + (a_2/a_1) - a_1b_2^2 \right],$$

$$D_2 = \Lambda^{-1} \left[ a_3(\lambda_2^2 - \frac{1}{2}c_2) - a_1b_1\lambda_2 - (a_2/a_1) + a_1b_2^2 \right]$$

(3.49)

and
\[ \exp(\Psi_2) = (\tilde{D}_1 + t\tilde{D}_1')e^{s_1} + \tilde{D}_2 e^{s_1} \quad (3 \cdot 50) \]

with

\[ \tilde{D}_1 = \Lambda^{-1}[a_2 a_2 a_1 + 2a_2 b_2 a_1 - (a_2^2/a_1^2) - a_2 b_2 a_1 + a_2 b_2], \]
\[ \tilde{D}_1' = \Lambda^{-1}[a_2 a_2 a_1 - a_2 b_2 a_1 - (a_2^2/a_1^2) - a_2 b_2 a_1 + a_2 b_2], \]
\[ \tilde{D}_2 = \Lambda^{-1}[a_2 a_2 a_2 + a_2 b_2 a_2 - a_2 b_2 a_2 + (a_2^2/a_1^2) + a_2 b_2], \quad (3 \cdot 51) \]

where \( c_2 \) is given by \((3 \cdot 31)\).

To conclude this section, we consider a symmetry of the solution \((2 \cdot 25)\) for \( a_\alpha \) under permutations of \( \lambda_1, \lambda_2, \ldots, \lambda_{N+1} \). The permutation group \( S_{N+1} \) of \( \lambda_i \) is isomorphic to the Weyl subgroup of \( SU(N+1) \) of order \((N+1)!\).

As an example, we shall discuss the interchange of \( \lambda_i \) and \( \lambda_j \), by using \((2 \cdot 45)\) to introduce the reflection \( s_{\alpha\beta} \) in the hyperplane perpendicular to \( \lambda_\alpha - \lambda_\beta \) as

\[ s_{\alpha\beta}(\lambda_i) = \lambda_i - \frac{2(\lambda_\alpha - \lambda_\beta)}{(\lambda_i - \lambda_\alpha)(\lambda_i - \lambda_\beta)}. \quad (3 \cdot 52) \]

Here \( s_{\alpha\beta} \) has been generated by successive application of \((2 \cdot 45)\) as

\[ s_{\alpha\beta} = (s_{\alpha\beta} - s_{\alpha\beta - 2} \cdots s_{\alpha\beta - 2})s_{\alpha\beta - 1}(s_{\alpha\beta - 2} \cdots s_{\alpha\beta - 2}). \quad (3 \cdot 53) \]

or

\[ s_{\alpha\beta} = (s_{\alpha\beta - 1} s_{\alpha\beta - 2} \cdots s_{\alpha\beta + 1}) s_{\alpha\beta + 1} \cdots s_{\alpha\beta - 1}. \quad (3 \cdot 54) \]

The alternative form \((3 \cdot 54)\) is derivable from \((3 \cdot 53)\) by the use of \((2 \cdot 51)\), since \( m_{\alpha\beta\alpha+1} = 3, m_{\alpha\beta\beta+1} = 2, j \neq i \pm 1 \) in \( \alpha_N \).

In particular, when \( a = a_1 \) and \( \beta = a_j, i < j \), in \((3 \cdot 52)\), \( s_{\alpha\beta} \equiv s_{ij} \) interchanges \( \lambda_i \) and \( \lambda_j \) and leaves \( \lambda_k, k \neq i, j, \) fixed. The effect of the reflection on \( H(o) \) can thus be written as

\[ s_{ij}(H(o)) = S_{ij}^1 H(o) S_{ij} \]
\[ = \text{diag}(\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{N+1}), \quad (3 \cdot 55) \]

where

\[ S_{ij} = \sum_{k=1}^{N+1} e_{ik} + e_{ij} + e_{ji}. \quad (3 \cdot 56) \]

We note that if we conversely use \((3 \cdot 52)\) to define \( s_{\alpha\beta} \) by

\[ s_{\alpha\beta} = s_{i+1} \quad (3 \cdot 57) \]

then the product \( w = \prod_{k=1}^{N+1} s_{\alpha\beta} \) is equivalent to the cyclic permutation matrix \( E = \sum_{k=1}^{N+1} e_{i+1} \) with periodic indices.\(^{16}\)
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By premultiplying equation (3-55) by $S_{ij}$ and postmultiplying by $S^{-1}_{ij}$, we can rewrite $\exp(-L(0)t)$ in (2-24) as

$$\exp(-L(0)t) = VS_{ij} \text{diag}(e^{-\lambda_1 t}, \cdots e^{-\lambda_N t})S^{-1}_{ij} V^{-1}. \quad (3\cdot58)$$

Now $V S_{ij}$ is the matrix obtained by interchanging the $i$-th and $j$-th columns of $V$ and $S^{-1}_{ij} V^{-1}$ is that obtained by interchanging $i$-th and $j$-th rows of $V^{-1}$. Hence the general solution for $a_N$ is invariant under the interchange of any two eigenvalues. If we generalize $s_{ab}$ to any permutation $\sigma$ of the eigenvalues, then it is also true that

$$\sum_{\sigma} W_{[i_1 \cdots i_N]} \exp[-(\lambda_{i_1} + \cdots + \lambda_{i_N})t] = \sum_{\sigma} W_{(\sigma(i_1) \cdots \sigma(i_N))} \exp[-(\lambda_{\sigma(i_1)} + \cdots + \lambda_{\sigma(i_N)})t], \quad (3\cdot59)$$

that is, the general solution is invariant under all permutations of the eigenvalues.

$\sigma(H(0))$ as exemplified by (3-55) is also "physically" attainable. If we assume

$$\lambda_{\sigma(N+1)} > \lambda_{\sigma(N)} > \cdots > \lambda_{\sigma(1)}, \quad (3\cdot60)$$

then we can easily show that

$$\sum_{\sigma} \Psi_{\sigma(H(N))} H_a = -\text{diag}(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(N+1)})$$

$$= -\sigma(H(0)). \quad (3\cdot61)$$

Equation (3·61) is equivalent to saying that the boundary condition as $t \to \infty$ determines the eigenvalues $\lambda_i$ or, more explicitly, that the first time derivative of the Toda variables as $t \to \infty$ determines the fundamental weights, where

$$\Psi_{\sigma(H(N))} = -w_{\sigma(H)}. \quad (3\cdot62)$$

§ 4. Remarks on general solution for other classical groups and explicit solution for Sp(2) as prototype

By using the comments on the form of $L(0)$ in § 2 and the matrix representations for $H_a, E_{ij}$ given in the Appendix of [A], we have

for $b_N: L(0) = \sum_{\ell=1}^{N} \{ e_{i+1,i+2} - e_{i+N+1,i+N+2} + e_{i+N+1} \}$

$$+ \sum_{1 \leq \ell \leq N} \{ l_{i+1,j+1} (e_{i+1,j+1} - e_{j+N+1,i+N+1})$$

$$+ l_{n+1,j+1} (e_{n+1,j+1} - e_{n+1,j+1}) \} + (e_{12N+1} - e_{N+1}); \quad (4\cdot1)$$

for $c_N: L(0) = \sum_{\ell=1}^{N} \{ e_{i+1,i+N+1} + l_{n+1,i+N+1} \} + e_{N} 2N$
The definitions of the algebras $b_N, c_N, d_N$ enable us to deduce restrictions on the eigenvalues of $L_{(0)}$. In particular, we have for $b_N$

$$\lambda_i = 0, \quad \lambda_{i+1} = -\lambda_{1+i+N}$$

(4.4)

and for $c_N$ and $b_N$

$$\lambda_i = -\lambda_{1+i+N},$$

(4.5)

where $i=1, 2, \cdots N$. As a consequence of (4.4) and (4.5), condition (2.38) is automatically satisfied.

Conditions (4.4) and (4.5) introduce a new feature to the general solutions; namely, when $N$ is even, there is a term in the general solution which is independent of $t$. To show this aspect and to illustrate the determination of $V$ and $V^{-1}$ via $L_{(0)}$ given in (2.29), we shall now demonstrate the method of solution for $C_2$ (corresponding to the $Sp(2)$ Toda molecule).

Obviously the solutions are obtained from (2.25) by inserting the appropriate matrices with values in $C_2$. We can generate $L_{(0)}$ from (4.2) with $N=2$ and with

$$l_{31} = -Z - z_i(z_2 - z_i) - \exp\left(-\sum_{j=1}^{3} K_{ij} \Psi_j\right),$$

$$l_{32} = -2Z - \exp\left(-\sum_{j=1}^{3} K_{ij} \Psi_j\right) + \sum_{j=1}^{3} K_{ij} \Psi_j - z_2 Exp(-\sum_{j=1}^{3} K_{ij} \Psi_j),$$

$$l_{41} = Z - z_i + z_i \exp\left(-\sum_{j=1}^{3} K_{ij} \Psi_j\right) - z_2 \exp(-\sum_{j=1}^{3} K_{ij} \Psi_j) + z_2(z_1 - z_2),$$

$$l_{42} = 2Z - \exp\left(-\sum_{j=1}^{3} K_{ij} \Psi_j\right) + z_2^2 - 2z_i z_2.$$

(4.6)

In (4.6), $z_i$ and $Z_i$ are defined via (2.7) as

$$n = l_i + z_i(e_{31} - e_{34}) + z_2 e_{31} + z_3 e_{32} + Z_+ e_{41} + z_2 e_{42}$$

(4.7)

and

$$Z_i = z_i - \frac{1}{6} z_i^2 z_2, \quad Z_2 = z_4 + \frac{1}{2} z_1 z_2.$$

(4.8)
The non-trivial constants of the motion can now be determined by
\[ c_2 = \frac{1}{2} \text{Tr} L_{l_0}^2 = 2l_{12} + l_{42}, \]
\[ c_4 = \frac{1}{4} \text{Tr} L_{l_0}^4 = c_2^2 - l_{21}^2 - l_{31}, \] (4.9)

which are written in terms of the initial data as
\[ c_2 = 2B_1 + B_2 - 2b_1b_2, \]
\[ c_4 = \frac{1}{2} (2B_1 + B_2 - 2b_1b_2)^2 - (b_1b_2 - B_1) + (b_2^2 - B_2)b_1^2, \] (4.10)

where
\[ B_1 = b_1^2 - (a_2/a_5^2), \quad B_2 = b_2^2 - (a_6/a_5^2). \] (4.11)

Now to determine \( V \) and \( V^{-1} \), we must calculate the secular polynomials
(2.31), where
\[ S = e_{31} + e_{22} + e_{43} + e_{34}. \]

We find that
\[ P_1(\lambda) = Q_4(\lambda) = 1, \quad P_2(\lambda) = -Q_3(\lambda) = \lambda, \quad P_3(\lambda) = -Q_2(\lambda) = \lambda^2 - l_{21}, \]
\[ P_4(\lambda) = -\lambda^3 + m\lambda + n, \quad Q_1(\lambda) = -\lambda^3 + m\lambda - n, \] (4.12)

where
\[ m = l_{31} + l_{42}, \quad n = l_{41}, \quad l = l_{21}, \] (4.13)

and consequently
\[ C_i(P) = (1 - \lambda_i - \lambda_i^2 + m\lambda_i + n \lambda_i^2 - l)^T, \]
\[ C_i(Q) = (-\lambda_i^3 + m\lambda_i - n - \lambda_i^2 + l \ 1 \ -\lambda_i)^T. \] (4.14)

Thus, by using (4.14), together with
\[ \Lambda^4 = -4\lambda_1\lambda_2(\lambda_2^2 - \lambda_1^2)^3 \] (4.15)

and
\[ \Xi = \text{diag}\left\{ \left( \frac{1}{2}(\lambda_1^2 - \lambda_2^2)^{-1}(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_1^{-1}, -\lambda_2^{-1}) \right) \right\}, \] (4.16)
in (2.33) we can determine \( V \) and \( V^{-1} \).

We must also determine \( \chi(0) \) before we can find the general solutions. However, since \( \chi(0) [\lambda] \) and \( \chi(0) [\lambda] \) are only required in (2.26), it is sufficient
to specify only the following entries:

\[
\begin{align*}
\chi(0)_{11} &= a_1, & \chi(0)_{21} &= -a_1 b_1, & \chi(0)_{31} &= Z_{20} a_1, & \chi(0)_{41} &= Z_{+0} a_1, \\
\chi(0)_{12} &= a_1 b_1, & \chi(0)_{22} &= -a_1 b_1, & \chi(0)_{32} &= a_1 b_1 Z_{20} + Z_{-0} a_2 / a_1, \\
\chi(0)_{42} &= a_1 b_1 Z_{+0} - (a_2 b_2) / a_1, 
\end{align*}
\]

where \( Z_{i0} = Z_{i|t=0} \). Condition (2.9) implies that

\[
\begin{align*}
\dot{Z}_3 &= Z_z \dot{z}_1 - z_1 \dot{Z}_1, \\
\dot{Z}_4 &= \dot{z}_1 z_2, & \dot{Z}_5 &= -z_1 \dot{z}_2. 
\end{align*}
\]

These equations can be solved by rewriting the Toda molecule equation for \( Sp(2) \) in terms of \( z_1 \) and \( z_2 \) using (2.10)\textasciitilde(2.12) and (2.16). The solutions of (4.19) are easily determined as

\[
Z_z = \frac{1}{2} \left( \frac{1}{2} \dot{z}_2 - \frac{1}{2} \dot{z}_1 + z_1^2 \pm z_2 \right),
\]

while a solution of (4.18) can be found from (4.6), namely

\[
Z_2 = \frac{1}{4} (2B_1 - B_2 \pm 2b_1 b_2).
\]

Hence

\[
\begin{align*}
Z_{00} &= \frac{1}{4} (2B_1 - B_2 \pm 2b_1 b_2), \\
Z_{20} &= n - b_1 Z_{-0} + (b_2 B_1 - b_1 B_2).
\end{align*}
\]

Having determined (4.22), all the relevant entries in \( \chi(0) \) are now given in terms of the initial data. So we are finally able to evaluate the minors of \( \chi(t) \) by (2.25) and hence the solutions of the \( Sp(2) \) Toda molecule equations in one dimension, at least when the eigenvalues of \( L(t) \) are distinct. The solutions are:

\[
\exp(\Psi_t) = W_{11} e^{-4it} + W_{21} e^{-4it} + W_{31} e^{4it} + W_{41} e^{4it}
\]

with

\[
W_{11} \equiv W(\lambda_1), & W_{21} \equiv -W(\lambda_2), & W_{31} \equiv -W(-\lambda_1), & W_{41} \equiv W(-\lambda_2),
\]

where

\[
W(x) = \frac{1}{2} a_1 [(\lambda_1^2 - \lambda_2^2) x]^{-1} [x^5 - b_1 x^3 - (B_1 + B_2 - 2b_1 b_2) x + C_1],
\]

\[
C_1 = B_1 (b_1 - b_2) + b_1 (B_2 - b_1 b_2),
\]
and

\[
\exp(\Psi_t) = W_{11t} + W_{12t} + W_{13t}e^{(\lambda_1 - \lambda_2)t} + W_{14t}e^{(\lambda_2 - \lambda_3)t} + W_{123}e^{(\lambda_1 - \lambda_4)t}
\]  

(4·26)

with

\[
W_{11t} + W_{12t} = 2a_2(\lambda_2^2 - \lambda_1^2)^{-2}[(B_1 - b_1b_2)(2B_1 + B_2 - 2b_1b_2) + C_2],
\]

\[
W_{12t} = A_2(\lambda_2 + \lambda_1)^{-2}[-b_2\lambda_1\lambda_2(\lambda_1 + \lambda_2) + B_2\lambda_1\lambda_2 + (b_2B_1 - b_1B_2)(\lambda_1 + \lambda_2) + C_2],
\]

\[
W_{13t} = A_2(\lambda_2 + \lambda_1)^{-2}[b_2\lambda_1\lambda_2(\lambda_1 + \lambda_2) + B_2\lambda_1\lambda_2 - (b_2B_1 - b_1B_2)(\lambda_1 + \lambda_2) + C_2],
\]

\[
W_{14t} = -A_2(\lambda_1 - \lambda_2)^{-2}[b_2\lambda_1\lambda_2(\lambda_1 - \lambda_2) - B_2\lambda_1\lambda_2 + (b_2B_1 - b_1B_2)(\lambda_1 - \lambda_2) + C_2],
\]

\[
W_{123} = -A_2(\lambda_1 - \lambda_2)^{-2}[-b_2\lambda_1\lambda_2(\lambda_1 - \lambda_2) - B_2\lambda_1\lambda_2 - (b_2B_1 - b_1B_2)(\lambda_1 - \lambda_2) + C_2],
\]  

(4·27)

where

\[
C_2 - b_1(b_1 + b_2)B_2 - b_2^2(B_1 + b_1^2), \quad A_2 - \frac{1}{4} a_2(\lambda_1 \lambda_2)^{-1}.
\]  

(4·28)

We shall not discuss in any detail the case when the eigenvalues of \(L(0)\) are degenerate. As in the corresponding situation for \(a_N\), we would perform a similarity transformation on \(L(0)\) to Jordan normal form. Here conditions (4·4) and (4·5) imply that the resultant \(\tilde{H}(0)\) would be block diagonal, with two \(N \times N\) matrices in normal form.

Finally we shall consider the symmetry of the solutions under permutations of the eigenvalues \(\lambda_i\). The Weyl subgroup of every classical compact group other than \(SU(N+1)\) is a semi-direct product of the permutation group \(S_N\) and some other.\(^{10}\) For example, for \(c_N\) the latter is \((\mathbb{Z}/\mathbb{Z}_2)^N\).

In the case of \(c_N\), the permutation of \(\lambda_i\) and \(\lambda_j\), \(1 \leq i < j \leq N\), implies by (4·5) also the permutation of \(\lambda_{i+N}\) and \(\lambda_{j+N}\). Hence it corresponds to an even permutation belonging to an alternating group, which can be viewed as a subgroup of a permutation group of higher order.

The proof of the invariance of the solutions under such permutations is similar to that given for \(a_N\) in § 3. However, a new feature will occur related to
the definition of $s_N$ by (2·50): for $c_N$, $s_N$ is the operation which changes the sign of $\lambda_N$, that is $\lambda_N \rightarrow -\lambda_N$. In the example of $c_2$ above, the solutions (4·23) and (4·26) are clearly symmetric under $s_1$ ($\lambda_1 \rightarrow -\lambda_1$). We note that they are also symmetric under $A_2 \rightarrow -A_2$.

§ 5. Conclusions

We have presented a procedure for obtaining the general solutions to the one-dimensional Toda molecule equations. We have outlined the method for the rank $N$ classical complex Lie algebras. However, as indicated in § 2, the method is applicable to any semi-simple Lie algebra.

In this paper, we have concentrated on the generation of the solutions in both general and specific cases. Consequently we have omitted any remarks on the use of the solutions and on other aspects of the theory, such as the relationship with the Hamiltonian formalism and dynamical mechanics. We shall consider these in a sequel to this paper, where we shall be concerned mainly with the relevance of the general solutions to spontaneously broken non-abelian gauge theories, specifically to generate spherically symmetric monopole solutions.\footnote{19}

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