Quark ALS Structure with Condensed $\pi^0$ Field in Chiral Bag Model

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Quark ALS (Alternating-Layer-Spin) structure is studied from a viewpoint of chiral bag model, as a possible intermediate stage to the transition of matter from hadronic to quark where a condensed $\pi^0$ field and two-dimensional quark matter with a spin-flavor combination of ALS type are realized.

Since quarks have been recognized as real constituents of hadronic matter, it becomes important to study typical problems in hadronic level from the viewpoint of the quark model. Pion condensation is one of such problems. In this note, from a viewpoint of chiral bag model, we study the baryonic ALS (Alternating-Layer-Spin) structure equivalent to a typical $\pi^0$ condensation. This ALS structure is the baryonic phase in which the one-pion-exchange tensor force is most efficiently utilized, among the nonuniform configurations studied earlier. Recently its possible realization at high density has been shown by the calculations taking into account both effects of $\Delta(1232)$ and short-range correlation.

When density $\rho$ becomes much higher than the nuclear density $\rho_n \approx 0.17$ fm$^{-3}$, hadronic matter possibly turns into quark matter. Baym pointed out a phase with an infinite network of interconnected bag prior to this transition. If hadronic matter has such an order as the ALS structure, a partial fusion of baryon bags is expected to take place keeping this order after the ALS aspect is well developed, as illustrated in Fig. 1(a)→(b)→(c). The baryon density in the layer located with equidistance $d$, $\rho_b = \rho_a$, becomes high enough for baryons in the layer to melt into the two-dimensional quark matter of width $2b < d$, an interconnected bag. The condensed $\pi^0$ field with the momentum $\pi/d$ in the $z$-direction persists as the Nambu-Goldstone mode in the region between the adjacent quark layers. In this paper we can show that quarks (u and d) themselves take their spin orientations in the ALS order due to the coupling with the condensed $\pi^0$ field at the layer (bag) surface. This quark ALS structure provides us with an example that chiral
bag model can be applied nonperturbatively. We start with the following action integral

\[ I = \int d^3x \left[ -\frac{1}{2} \gamma^\mu \partial_\mu \phi - B + \partial_\mu (\lambda_3 \phi (\sigma + i e^\mu \pi_3) \psi) \right] \partial_\mu \]

where \( \phi \) represents massless quarks (\( u \) and \( d \)) and the \( \sigma \)-model with the restriction \( \sigma^2 + \pi^2 = f_\pi^2 \) is adopted to describe the pion field \( \pi = (\pi^1, \pi^2, \pi^3) \) with mass \( m_\pi \) existing only in the outside. The step function \( \theta_{in}(\theta_{out}) \) specifies the region inside (outside) the bag; in our case \( \theta_{in} = \theta(b - |z|) \) and \( \theta_{out} = \theta(|z| - b) \), where \( z = z - \ell d \) means the position from the center of the \( \ell \)-th layer. \( B \) is the bag constant of the volume term to prevent the system from expanding. \( A_\pi \) is the Lagrange multiplier to give the boundary condition which assures confinement and continuity of axial-vector current; \( \mu = -n_\pi / 2f_\pi \), where \( n_\pi \) is the outward normal at the bag surface and \( f_\pi \) the pion decay constant \( (f_\pi \approx 93 \text{ MeV}) \). The pion mass term implies the PCAC in the outside. Here we use the notation, \( z = (x, y, z, \ell) \) and \( \gamma_\mu = \gamma_\mu \) in our case. The \( s \)-quark wave functions, \( \phi_s = \exp[iq_s \cdot r_s] / \sqrt{\Omega_s} u(z, s) f_s C e^{-i\omega} \), where the first factor is the two-dimensional plane wave with the momentum \( q_s = (q_x, q_y) \) in \( r_s = (x, y) \) space, the spinor \( \psi(z, s_\pi) \) with the spin index \( s_\pi = 1 \) describes the wave function of \( z \) whose dominant component is the spin-up (down) one if \( q_s \leq q_m \), the flavor state \( L = +1(1) \) for the \( u(d) \) quark and \( C_\pi \) means the color state. Each component of \( u \) is represented by a superposition of \( \sin q_z z \) and \( \cos q_z z \), where \( q_z = m_\pi \) is the eigenvalue related to the energy of \( \phi_s \) as \( \omega = \sqrt{q_z^2 + q^2} \).

The condensed \( \pi^0 \) field realized under the hadronic ALS structure is antisymmetric about each layer center, as seen in Fig. 1(a) and (b). We assume that this symmetry property still holds after the transition to the quark ALS structure; we impose the condition \( \pi^0(2 = -b) = -\pi^0(2 = b) \) assured by \( \theta(z = b) = -\theta(z = b) \). Then the eigenvalue equation for \( q_s \) is given by

\[ q_s(I_\pi \Lambda - I_\pi \Lambda^{-1}) = 0 \]

for \( s = 1 \) and the one for \( s = 2 \) in which \( I_\pi \) and \( I_\pi \) are interchanged, where \( I_\pi \equiv (1 \pm n_\pi \sin \theta_b) / \cos \theta_b \) with the abbreviation, \( \Lambda = \tan q_s b \) and \( \theta_b = \theta(z = b) \). The relation \( I_\pi \Lambda = 1 \) holds. Without the \( \pi^0 \) field \( (\theta_0 = 0) \), because of \( I_\pi - 1 \) we have \( A = 1 \); \( q_m = \pi \ell / 4b \), \( 5\pi \ell / 4b \), ... Presence of the \( \pi^0 \) field is represented by the deviation of \( I_\pi \) from unity. For the quark ALS configuration where all the single-quark states become lower in energy due to the \( \pi^0 \) field (as shown below),
the two eigenvalue equations reduce to the single one,
\[ \tan q_z b = \cos \theta_b / (1 + \sin \theta_b). \] (7)
This leads to a smaller \( q_z \) for a larger \( \theta_b \) \((0 \leq \theta_b \leq \pi / 2)\). Irrespective of the absence or presence of \( \pi^0 \), the first excited state is well separated in energy from the lowest one by about \( 7 \pi / 600 \) MeV for \( b : \sim \) fm. Therefore, we can restrict ourselves to the lowest configuration of the single quark states. The spinor \( \psi(z; \alpha) \) is obtained by solving Eqs. (3) and (4) and represented by using the eigenvalues thus obtained and \( I_{\pm} \).

2. Quark ALS structure

On the basis of the single-quark states, next we consider what configuration of the two-dimensional quark matter is favorable in energy in the presence of the condensed \( \pi^0 \) field. For \( \alpha = 1 \), Eq. (6) gives
\[ \Lambda^2 = I_{\pm} / \alpha, \]
that is, a smaller \( q_z \) than \( \pi / 4 b \) is obtained for the combination of \( \tau_b < 0 \) for \( \theta_b > 0 \) or \( \tau_b > 0 \) for \( \theta_b < 0 \). Conversely for \( \alpha = 2 \), from Eq. (6), the combination of \( \tau_b > 0 \) for \( \theta_b > 0 \) or \( \tau_b < 0 \) for \( \theta_b < 0 \) gives \( q_z < \pi / 4 b \). All the quarks utilize the attractive effect of the \( \pi^0 \) field in the following configurations: (i) in the layer \((\ell = \text{even})\) with \( \pi^0(b) = -\pi^0(-b) > 0 \), \( d \) with \( \alpha = 1 \) and \( u \) with \( \alpha = 2 \) are favorable, and (ii) in the layer \((\ell = \text{odd})\) with \( \pi^0(b) = -\pi^0(-b) < 0 \), \( u \) with \( \alpha = 1 \) and \( d \) with \( \alpha = 2 \) are favorable. We call this configuration the quark ALS structure denoted by \( |\phi_{\text{ALS}}\rangle \), which has the spin-flavor property analogous to the hadronic ALS one\(^6\) and is in \( B \)-equilibrium with electrons in neutron star matter.

3. Properties of \( \pi^0 \) field

The \( \pi^0 \) field in the outside \((b \leq |z| \leq d - b)\) is determined by the field equation and the boundary condition. The former is of the sine-Gordon type,
\[ d^2 \theta / dz^2 - m_{\pi^0}^2 \sin \theta = 0 \] (8)
which results from combining \( \partial_u \phi + f_z m_{\pi^0}^2 \phi = 0 \), where \( \phi \) is the \( \phi \) field at the bag surface,
\[ f_z^2 d\theta / dz \bigg|_{z = \pm b} = \langle \phi_{\text{ALS}} | i (\partial_z \phi_{\text{ALS}}^* \phi_{\text{ALS}} - \phi_{\text{ALS}}^* \partial_z \phi_{\text{ALS}}) | \phi_{\text{ALS}} \rangle. \] (9)

The derivative \( d\theta / dz \) at the \( \ell \)-th layer surface is given by the quark ALS structure; its sign is \((-1)^{\ell+1}\) as expected in Fig. 1(c) and the magnitude is a function of \( d, b, q_x, \) and \( q_\alpha \) \((\ell_b, \alpha, \alpha)\) (the Fermi momenta of the respective quarks). The finite mass of pion is needed to obtain the solution with \( q_\alpha > 0 \).

4. Energy of the system

After solving self-consistently the set of coupled equations, Eqs. (7), (8) and (9), energy calculations are performed by using \( \theta(x) \) and \( q_x \) thus determined. The energy expression per baryon is the sum of three terms; the quark kinetic energy \( E_q / N_B \), the volume energy \( E_v / N_B \) and the pion field energy \( E_\pi / N_B \),
\[ E_q = E_v / N_B = 2 b B / p \theta_d, \]
\[ E_\pi = E_{\pi^0} / N_B = f_{\pi^0} m_{\pi^0} \theta_d \]
\[ E_{\pi^0} = \int_{x_1}^{x_2} dx \left[ \frac{1}{2} \left( \frac{d \theta}{dx} \right)^2 + 1 - \cos \theta(x) \right]. \] (10)

where \( x \equiv m_{\pi^0}, x_1 = m_{\pi^0} b \) and \( x_2 = m_{\pi^0}(d - b) \). In Fig. 2, these quantities and the total energy per baryon \( E = E_q + E_v + E_\pi \) are shown; the case for \( d = 1.2 \) fm at \( \rho = 5 \rho_n \) corresponds to a situation in the hadronic ALS structure\(^7\) and the other two serve to show the dependence on \( d \) and \( p \).

\( E_q \) is essentially determined by the kinetic energy of two-dimensional kinetic energy and the weak \( b \)-dependence reflects drastic reduction of \( q_x \) from \( \pi / 4 b \) due to the strong \( \pi^0 \) field. Without the \( \pi^0 \) field in the outside, the kinetic energy with the \( b \)-dependence of \( q_x = \pi / 4 b \)
balances \( E \propto b \), and the energy minimum appears at a finite \( b \). Far apart from this limit of weak \( \pi^0 \) condensation \((b_0 \sim 0)\), the general aspect of the system obtained by calculations is close to the limit of the fully developed \( \pi^0 \) condensation, i.e., \( b_0 = \pi/2 \). \( E_x \) favors larger \( b \) but its effect is overcome by \( E_v \) which always acts to shrink the system. As a net result, the energy minimum occurs at \( b = 0 \), although the \( b \)-dependence becomes much weaker at higher \( \rho \).

In order to avoid the collapse of the quark layers we need some effects to increase the energy and/or to reduce the attractive \( \pi^0 \) contribution for small \( b \). At least there are three candidates which play such role: (i) the one-gluon-exchange (OGE) contribution in the inside, (ii) the effect of pions substantiating \( q\bar{q} \) pair in the inside\(^{11,11,12}\) and (iii) the \( \omega \)-meson contribution in the outside to restore the stability as shown in the one-baryon problem.\(^{13}\) The OGE contribution in the quark ALS structure is repulsive to favor large \( b \) and increases as \( \rho \) goes high. A preliminary result is shown in Fig. 2. We can see the OGE effect to restore the stability at high \( \rho \). Since the formulation presented here is naturally extended to include the effects (ii) and (iii), to examine such possibilities is tractable. More detailed accounts of this study will be reported elsewhere.

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