The area-preserving baker's transformation is extended in order to include the area non-preserving one, which is the model for a simple horse-shoe structure. The random motion on the horse-shoe is studied precisely, and the invariant density and the correlation function are obtained analytically and numerically.

The random walk on the horse-shoe structure is transformed into the generalized Brownian motion with the attractor, and the non-analytic density of the generalized baker’s transformation is approximated by an analytic function. The topological similarity between the Lorenz chaos and the generalized baker’s transformation is briefly discussed.

§ 1. Introduction

The onset of the homoclinicity or Smale’s horse-shoe is the essential mechanism which generates the chaotic orbit in the dynamical system.11 The chaotic orbit on the horse-shoe structure can be statistically identified by the realization of a symbolic dynamics, and that is proved to be topologically equivalent to the Bernoulli shift. Its invariant set is generally a Cantor-like set in the phase space. This kind of singular set is often observed as the strange attractor in the dissipative dynamical systems.

One of the purposes of the present paper is to elucidate the nature of the random motion on such Cantor-like singular set. The invariant measure is a non-analytic singular function in the phase space under consideration. In order to get the absolutely continuous measure, we must use an adequate coarse-graining method, i.e., a projection onto the restricted space spanned by smaller degrees of freedom. Even in this restricted space, however, the reduced measure is still non-analytic in general. The second purpose of this paper is to propose a useful method to approximate the reduced exact measure by an analytic function.

The discussion of this paper is limited to the analysis of the following generalized baker’s transformation, which is considered to be a simple model of the typical horse-shoe structure. The transformation is defined in an unit cube in the \((x, y, z)\) space as shown in Fig. 1(b),
Generalization of Baker’s Transformation

Fig. 1. Smale’s horse-shoe (a) and generalized baker’s transformation (b).

\[
\begin{align*}
\text{for } 0 \leq x_n \leq 1/2, \\
& \quad x_{n+1} = 2x_n, \\
& \quad y_{n+1} = \lambda_y^{-1} y_n, \\
& \quad z_{n+1} = \lambda_z^{-1} z_n, \\
\text{for } 1/2 < x_n \leq 1, \\
& \quad x_{n+1} = 2x_n - 1, \\
& \quad y_{n+1} = \lambda_y^{-1} (y_n + \lambda_y - 1), \\
& \quad z_{n+1} = \lambda_z^{-1} (z_n + \lambda_z - 1).
\end{align*}
\]

(1.1)

Here \(\lambda_y > 1\) and \(\lambda_z > 1\) are assumed. If \(\lambda_z > 2\) is satisfied, the invariant limiting set becomes a fibrous singular set along the \(x\)-direction. Precisely speaking, the transformation of Eq. (1.1) may be called the ‘broken’ ‘twist’ horse-shoe. In the case \(\lambda_y > 2\), the reduced map on the \((x, y)\) plane is contracting and the invariant measure is singular in the \(y\)-direction. For the case \(\lambda_y < 2\), however, the reduced map is expanding and not uniquely invertible. In what follows, only the reduced dynamics on the \((x, y)\) plane is considered since the situation is the same on the \((x, z)\) plane. The so-called baker’s transformation is the case \(\lambda (=\lambda_y) = 2\). In this paper, \(\lambda\) takes an arbitrary value larger than unity.

The dynamics in the \(x\)-direction is essentially the Bernoulli shift \(B(\frac{1}{2}, \frac{1}{2})\), therefore our main concerns will be limited to the random motion in the \(y\)-direction. The dynamics of \(y\) is represented as

\[
y_{n+1} = \frac{\lambda - 1}{\lambda} \tilde{\sigma}_n + \frac{y_n}{\lambda},
\]

(1.2)

where \(\{\tilde{\sigma}_n\}\) is the symbolic sequence induced by the Bernoulli shift mentioned above; \(\tilde{\sigma}_n = 0\) or \(1\) \((0 \leq x_n \leq 1/2\) or \(1/2 < x_n \leq 1\)). Denoting the distribution density by \(P_n(y)\), Eq. (1.1) is rewritten into
The invariant density is given by \( \lim_{n \to \infty} P_n(y) = P(y) \). For the sake of the latter use, the self-similarity index \( \mu \) of the distribution \( F(y) = \int_0^y P(y') dy' \) is defined as
\[
\mu = \ln 2 / \ln \lambda. \tag{1·4}
\]
When \( F(y) \) is analytic near \( y=0 \) (or \( y=1 \)), the index \( \mu \) corresponds to the power index of \( F(y) \), namely from Eq. (1·3) and \( F(y) \propto y^\mu + o(y^\mu) \).

§ 2. Sample path and invariant measure

2.1. Correlation function

By the change of variables,
\[
\eta_i = 2y_i - 1, \quad \sigma_i = 2\sigma_i - 1, \tag{2·1}
\]
the sample path of Eq. (1·2) is described as
\[
\eta_{n+1} = \frac{1}{\lambda} \eta_n + \left( 1 - \frac{1}{\lambda} \right) \sigma_n
\]
\[
= \frac{\lambda - 1}{\lambda} \left( \sigma_n + \frac{\sigma_{n-1}}{\lambda} + \ldots + \frac{\sigma_0}{\lambda^n} \right) + \frac{\eta_0}{\lambda^{n+1}}, \tag{2·2}
\]
where \( \eta_0 \) and \( \sigma_0 \) are the initial values of \( \eta \) and \( \sigma \). By use of the relation \( \langle \sigma_i \sigma_j \rangle = \delta_{ij} \), the time correlation function is derived as
\[
\langle \eta_n \eta_{n+\nu} \rangle - \langle \eta_n \rangle \langle \eta_{n+\nu} \rangle = \frac{\lambda - 1}{\lambda + 1} \lambda^{-\nu}, \quad (n \to \infty) \tag{2·3}
\]
and \( \lim_{n \to \infty} \langle \eta_n^2 \rangle \) is zero in the limit \( \lambda \to 1 \), therefore the distribution density is \( \delta \)-function in this limit. This is discussed again in the next section.

2.2. Invariant measure

The characteristic function of \( \eta_n \), \( \Phi_n(k; \lambda) \) is given as
\[
\Phi_n(k; \lambda) = \varphi \left( \frac{\lambda - 1}{\lambda} k \right) \varphi \left( \frac{\lambda - 1}{\lambda^2} k \right) \ldots \varphi \left( \frac{\lambda - 1}{\lambda^n} k \right) \exp \left[ \frac{ik\eta_n}{\lambda^n} \right]
\]
\[
= \Phi_{n-1} \left( \frac{k}{\lambda}; \lambda \right) \varphi \left( \frac{\lambda - 1}{\lambda} k \right) \exp \left[ \frac{ik(\lambda - 1)\eta_n}{\lambda^{n+1}} \right]. \tag{2·4}
\]
Here \( \varphi(k) \) is the characteristic function of \( \sigma(=1 \text{ or } -1) \), i.e., \( \varphi(k) = \cos k \).
Therefore the limiting solution \( \Phi(k; \lambda) (= \lim_{n \to \infty} \Phi_n(k; \lambda)) \) yields
The density function $P(r; \lambda)$ may be obtained from the Fourier transformation as

$$P(r; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(k; \lambda) \exp[-i k r] dk.$$  \hspace{1cm} (2·6)

$P(r; \lambda)$ is classified into two cases depending on the parameter $\lambda$.

**Case I** ($\lambda > 2$; singular measure)

The support of the invariant measure is a singular set $S(\lambda) \subset [-1, 1]$. For instance, the set $S(3)$ is Cantor's ternary set. The Hausdroff dimension of $S(\lambda)$ is equal to the self-similarity index defined by Eq. (1·4),

$$\text{Dim}\{S(\lambda)\} = \ln 2 / \ln \lambda < 1.$$  \hspace{1cm} (2·7)

**Case II** ($\lambda \leq 2$; continuous measure)

The support of $P(r; \lambda)$ is the whole region of $[-1, 1]$. In general, $P(r; \lambda)$ seems to be expressed by the convolution of some singular measures and continuous measures. Particularly, when the self-similarity index $\mu$ is integer, $P(r; \lambda)$ becomes an analytic function which is derived from the convolution of some uniform distributions.

Putting $\lambda = 2^{1/m}$ ($m$ = integer), the solution of Eq. (2·5) is given as follows,

$$\Phi(k; \lambda) = \prod_{l=0}^{m-1} \sin\left(\frac{(\lambda - 1)^{l+1} k}{(\lambda - 1)^l k}\right).$$  \hspace{1cm} (2·8)

In the limit $m \to \infty$ (or $\lambda \to 1$), the invariant density function $P(\eta; 1) = \delta(\eta)$, i.e., $\Phi(\eta, 1) = 1$, which is the same relation as discussed in Eq. (2·3). Several simple cases are given as

$$P(\eta; 2) = \frac{1}{2},$$  \hspace{1cm} (2·9)

$$P(\eta; \sqrt{2}) = \begin{cases} \frac{(\sqrt{2} + 1)^2}{4\sqrt{2}} (\eta + 1), & \text{for } -1 < \eta < -1 - \sqrt{2} \\ \frac{1}{2\sqrt{2}(\sqrt{2} - 1)} - (1 - \sqrt{2})^2, & \text{for } -1 - \sqrt{2} < \eta < (\sqrt{2} - 1)^2 \\ \frac{(\sqrt{2} + 1)^2}{4\sqrt{2}} (1 - \eta), & \text{for } (\sqrt{2} - 1)^2 < \eta \leq 1 \end{cases}$$  \hspace{1cm} (2·10)

2.3. **Numerical results**

In the case $m$ = integer, the invariant density function is calculated by the computer. When the initial distribution $P_0(\eta; \lambda)$ is fixed, the successive distribution $P_n(\eta, \lambda)$ is numerically obtained through the functional equation (1·3). For the case $\lambda > 2$, the limiting solution $P(r, \lambda)$ is a singular function, and then it
Fig. 2. Numerical solution of Eq. (1·3) for $\lambda = 3/2$.

Fig. 3. Numerical solutions of the invariant density for $n=12$. 
depends explicitly on the initial distribution $P_0(\eta, \lambda)$. On the other hand, in the
case $\lambda \leq 2$, the limiting distribution is thought to be independent of the initial
distribution.\(^*\)

Figure 2 shows the time evolution of the density function $P_n(\eta, 3/2)$, where
the initial density is taken to be symmetric,
\[
P_0(\eta; \lambda) = \begin{cases} 
\eta + 1, & -1 \leq \eta \leq 0, \\
-\eta + 1, & 0 \leq \eta \leq 1.
\end{cases}
\] (2.11)

$P_n(\eta, \lambda)$ is also symmetric at every step $n$. Figure 3 shows the invariant density
function for some cases. Because of the practical limitation of the machine
calculation, $P_{12}(\eta, \lambda)$ is illustrated. Even when another initial distribution $P_0(\eta, 
\lambda)$ is adopted, the results are almost invariant. The analytical expressions of
Eqs. (2.8)~(2.10) correspond to Figs. 3(c) and 3(e). By increasing the value of
$\lambda(<2)$, it seems that the singular components become dominant and the jagged
structure grows remarkably.

§ 3. Continuous time model-stochastic process with attractor

It is quite difficult to get the analytical form of $P(\eta; \lambda)$ for the case $\mu \neq$
integer, since the process $\eta_n$ is the convolution of the infinitely decomposable
random processes whose measure is singular. In this section we will show the
method that enables us to derive the analytical approximate form of $P(\eta; \lambda)$.

Difference equation (2.2) is approximated by the following stochastic process
$\eta(t)$ with continuous time $t$.
\[
\eta_{n+1} - \eta_n \approx \frac{d\eta}{dt}, \\
= -\beta(\eta - \sigma(t)), \quad (\beta \equiv (\lambda-1)/\lambda).
\] (3.1)

Here $\sigma(t)$ is the dichotomous noise as
$\sigma(t) = 1$ or $-1$
and
$\langle \sigma(t) \rangle = 0$.

In what follows, two types of the stationary random noise $\sigma(t)$ are considered.
The first case is the random telegram noise that is generated by the Pois-

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\(^*\) Recently, Saito discussed the case $\lambda=2$, and showed that an appropriate coarse-graining ope-
ration is necessary for the increase of the entropy.\(^*\) In the case $\lambda<2$, however, the uncertainly in
the initial data is the necessary and enough condition to obtain such irreversable or mixing behaviors.
This point seems to be a crucial difference between two cases for $\lambda<2$ and $\lambda \geq 2$. 
sonian process, i.e.,
\[ \langle \sigma(t)\sigma(t+\tau) \rangle = e^{-2\gamma\tau}, \quad (\gamma > 0) \]  
(3.3)
and the second is the random binary noise,
\[ \langle \sigma(t)\sigma(t+\tau) \rangle = \begin{cases} 1 & (|\tau| \leq \Delta) \\ 0 & (|\tau| > \Delta) \end{cases} \]  
(3.4)
where \( \gamma \) and \( \Delta \) are the adjustable parameters. \( \eta(t) \) given by Eq. (3.1) is a non-
markovian process, and has the quite different nature from the familiar O-U process. Namely, the process \( \eta(t) \) has the attractor with the finite size \((-1 \leq \eta(t) \leq +1)\).

The correlation function \( R(\tau, \lambda) \) and the stationary density \( P(\eta, \lambda) \) are
obtained as follows,
Case I (for Eq. (3.3))
\[ R(\tau, \lambda) = \frac{2(\gamma/\beta)e^{-\gamma\tau}e^{-2\gamma\tau}}{4(\gamma/\beta)^2-1}, \]
\[ P(\eta, \lambda) = \frac{\Gamma(\gamma/\beta+1/2)}{\Gamma(\gamma/\beta)\Gamma(1/2)}(1-\eta^2)^{\gamma/\beta-1}. \]  
(3.5)
Case II (for Eq. (3.4))
\[ R(\tau, \lambda) = \frac{e^{-\gamma\tau}}{(\beta \Delta/2)^2} \sinh^2(\beta \Delta/2) \quad \text{for} \; \Delta \tau. \]  
(3.6)
In this case the compact form of the characteristic function \( \Phi(k) \) is derived
in Eq. (A.9), but the problem concerning the analyticity of \( P(\eta, \lambda) \) is still open.
The simple derivation of \( P(\eta, \lambda) \) is discussed in the Appendix.

So far we have not discussed the correspondence among the parameters \( \gamma \) and \( \lambda \). The correspondence is essential for the adequacy of the modelling by Eq. (3.1). Here we assume that the most reasonable correspondence is obtained when the self-similarity index is the same in both systems, namely,
\[ \mu = \ln 2 / \ln \lambda = \gamma/\beta. \]  
(3.7)
From these correspondence, one can
know that the numerical results shown in Fig. 3 is well reproduced by the analytical solutions of Eq. (3.5) at least qualitatively. For instance, Fig. 4 shows the invariant distribution function $F(\eta)$ for the case $\mu = 1/2$.

§ 4. Discussion

Various kinds of random motions are created after the onset of the homoclinicity. Many of such orbits can be understood in terms of the Lévy process or of the more generalized Brownian motion. Indeed, as was shown in this paper, the motion on the typical horse-shoe structure is well expressed as the stochastic process driven by a dichotomous noise.

The stochastic process with the attractor is closely connected with the chaotic orbit observed on the strange attractor in the dissipative dynamical system. To see this, we will point out the topological similarity between the Lorenz chaos and the generalized baker's transformation studied in this paper.

We assume that the two-dimensional map on the $(x, y)$ plane defined by Eq. (1.1) is a Poincaré map of the continuous orbit in $R^3$ (see Fig. 5). By twisting back the half plane onto itself, the two-dimensional map (Fig. 5(a)) is immersed into the three-dimensional space $(\xi, \eta, \zeta)$ as shown in Fig. 5(b). Here certain new variables, $\xi = 2x-1$, $\eta = 2y-1$ and $\zeta$ are introduced, and these are assumed to be the functions of the time $t$. The transformation is illustrated in Fig. 5(b), and it has the same symmetry as in the Lorenz system, i.e.,

\[
\frac{d\xi}{dt} = \frac{\eta}{\tau}, \quad \frac{d\eta}{dt} = -\frac{\xi}{\tau},
\]

Fig. 5. Immersion of the generalized baker's transformation into $R^3 (\xi, \eta, \zeta)$. 
The suspension model constructed above is thought to be a topological analog of the Lorenz attractor even though the fine topological structure is not identical in every detail of both systems. The contraction rate $\lambda_y^{-1}$ corresponds to the third Lyapunov exponent of the Lorenz chaos, i.e., $\lambda_y^{-1} \approx e^{-14.5}$. The model of Fig. 5 will be discussed again elsewhere in relation to the topological nature of the Lorenz chaos.  

**Appendix**

In this appendix, we give a simple derivation of the analytical form of the characteristic function $\Phi(t, k)$ of $\eta(t)$ for the continuous time model (3·1). The characteristic function is formally written as

$$\Phi(t, k) = \varphi(t, k)e^{-kt}\varphi(0),$$

where

$$\varphi(t, k) = \langle \exp\left[ik\int_0^t e^{\sigma(u)}du\right]\rangle.$$  

and $\eta_0$ is the initial value of $\eta(t)$.

First we take $\sigma(t)$ to be a random telegram noise, i.e., a stationary Markov process which takes two values $\pm 1$ with equal probability and the transition probability is given by

$$P(t + r, \sigma | t, \sigma') = \frac{1}{2}(1 \pm e^{-2r}), \quad \sigma, \sigma' = \pm 1.$$  

From this definition, it follows that for $t_1 \geq t_2 \geq \cdots \geq t_{2n}$

$$\langle \sigma(t_1)\sigma(t_2)\cdots\sigma(t_{2n}) \rangle = e^{-2r(t_1 - t_2)}\langle \sigma(t_3)\sigma(t_4)\cdots\sigma(t_{2n}) \rangle$$

and

$$\langle \sigma(t_1)\sigma(t_2)\cdots\sigma(t_{2n+1}) \rangle = 0.$$  

Using this property, the right-hand side of (A·2) is rewritten as

$$\varphi(t, k) = 1 + \sum_{n=1}^\infty (ik\beta)^n \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \cdots \int_0^{t_{2n-1}} dt_{2n-2} \sigma(t_1)\sigma(t_2)\cdots\sigma(t_{2n})e^{\beta(t_1 + t_2 + \cdots + t_{2n})}$$

$$= 1 - k^2\beta^2 \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 e^{\sigma(t_1 + t_2) - 2r(t_1 - t_2)}\varphi(t_2, k).$$
Generalization of Baker’s Transformation

This is the integral equation for \( \varphi(t, k) \) and can be transformed into an equivalent differential equation

\[
\frac{\partial^\nu}{\partial t^\nu} \varphi(t, k) = (\beta - 2\gamma) \frac{\partial}{\partial t} \varphi(t, k) - k^2 \beta^2 e^{2\beta t} \varphi(t, k)
\]

with the initial condition

\[
\varphi(0, k) = 1 \quad \text{and} \quad \frac{\partial}{\partial t} \varphi(0, k) = 0.
\]

If we replace the variable by \( Z = k e^{\beta t} \) and write as \( g(Z, k) = e^{\beta t} \varphi(t, k) \) with \( \nu = \gamma/\beta - \frac{1}{2} \), then \( g(Z, k) \) satisfies the Bessel equation,

\[
\frac{1}{Z} \frac{\partial}{\partial Z} \left( Z \frac{\partial}{\partial Z} g \right) + \left( 1 - \frac{\nu^2}{Z^2} \right) g = 0.
\]

Hence the solution of Eq. (A·3) which satisfies the initial condition (A·4) is found to be

\[
\varphi(t, k) = \frac{\pi}{2} e^{-\beta t k} [J_{\nu+1}(k) Y_{\nu}(ke^{\beta t}) - J_{\nu+1}(k) J_{\nu}(ke^{\beta t})],
\]

where \( J_{\nu} \) and \( Y_{\nu} \) are Bessel and Neumann functions respectively.

Combining this result (A·5) and (A·1), we obtain the characteristic function \( \Phi(t, k) \), and its Fourier transform is the probability density \( P(t, \eta) \) of \( \eta(t) \). In particular, it is easy to see

\[
\lim_{t \to \infty} \Phi(t, k) = \Gamma(\nu + 1) \left( \frac{2}{k} \right) J_{\nu}(k).
\]

From this and the integral representation of the Bessel function

\[
\sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right) \left( \frac{2}{k} \right)^\nu J_{\nu}(k) = \int_{-1}^{1} e^{i\xi(1-\eta)} e^{-1/2} d\eta,
\]

we find that the stationary density \( P(\eta) = \lim_{t \to \infty} P(t, \eta) \) is given by (3·5). As an example of the transient behavior of \( P(t, \eta) \), let us consider the special case \( \nu = \frac{1}{2} \) where the stationary distribution has a uniform density on \([-1, 1]\). In this case we have

\[
\Phi(t, k) = \left[ e^{-\beta k} \cos k + \frac{1}{k} \sin k (1 - e^{-\beta t}) \right] \exp[ike^{-\beta t} \eta_0].
\]

Therefore, \( P(t, \eta) \) is the superposition of a uniform distribution and singular distributions (two \( \delta \)-functions). In the limit \( t \to \infty \), however, singular distributions vanish and only the uniform distribution survives.

Next we consider the case where \( \sigma(t) \) is a binary noise. Let \( \{ \sigma_n \} \) be a
sequence of identically distributed independent random variables with Prob. \( \{\sigma_n = \pm 1\} = \frac{1}{2} \), and \( \tau \) be another random variable independent of \( \{\sigma_n\} \) which is uniformly distributed over the interval \([0, \Delta]\). Then \( \sigma(t) \) can be represented as

\[
\sigma(t) = \sum_{n=-\infty}^{\infty} \sigma_n \theta(t-n\Delta - \tau), \tag{A.7}
\]

where \( \theta(t) \) is the indicator function of the interval \([0, \Delta]\), i.e., \( \theta(t) = 1 \) for \( 0 \leq t \leq \Delta \) and \( \theta(t) = 0 \) otherwise. Hence the time interval of two successive jumps of \( \sigma(t) \) is always a multiple of \( \Delta \).

In this case, from (3.1) and (A.4) we obtain

\[
\eta(t) = \eta_0 e^{-\beta t} + \sum_{n=0}^{\infty} \sigma_n g(t - n\Delta - \tau), \tag{A.8}
\]

where \( g(t) = 0 \) for \( t < 0 \), \( g(t) = (1 - e^{-\beta t}) \) for \( 0 \leq t < \Delta \) and \( g(t) = (e^{-\beta t} - 1)e^{-\beta t} \) for \( t \geq \Delta \). Thus we find

\[
\varphi(t, ke^{-\beta t}) = \frac{1}{\Delta} \int_0^t dt \prod_{n=0}^{\infty} \cos(kg(t - n\Delta - \tau)),
\]

so that

\[
\lim_{N \to \infty} \Theta(N, k) = \int_0^t ds \prod_{n=1}^{\infty} \cos(kg(n\Delta - s\Delta)). \tag{A.9}
\]

References

2) Y. Aizawa, Prog. Theor. Phys. 68 (1982), 64.