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A Deformable Bag Model of Hadrons. I

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As a generalization of the MIT spherical bag model, we construct the spheroidal bag model of hadron with an arbitrary eccentricity. This generalization is made by slightly modifying the MIT linear boundary condition: The linear boundary condition is examined in detail. Our model always satisfies two necessary requirements of the MIT bag model—i.e., \( n \cdot j = 0 \), no quark colour flux leaves the bag, and \( \bar{q}q = 0 \), the scalar density of quark should vanish on the bag surface—and it reduces to the MIT spherical bag model in the limit of zero-eccentricity. Lagrangian formalism of our model is briefly described.

The eigenfrequencies of a single massless quark confined in this spheroidal bag are numerically calculated. We obtain the level-splitting of the excited quark orbits, which is just analogous to the well-known Nilsson's splitting of single particle orbits in deformed nuclei. By using the numerical results of the lowest orbit, the effect of the bag-deformation on the mass of low-lying hadrons is estimated. It is found that, although the spherical bag is stable, the quark bag is extremely soft against the quadrupole deformation. Brief discussions are added on the mechanisms which make the spherical bag more stable.

§ 1. Introduction and summary

As a phenomenological model inspired by quark confinement, the MIT bag model has been remarkably successful in explaining the static properties of low-lying hadrons. These static properties, such as the mass spectrum, the magnetic moments and the axial coupling constant, have been calculated in an approximation in which the bag is considered to be a spherical cavity filled with the coloured quark and vector-gluon fields. Within the bag model, dynamical processes such as scattering and decay of hadrons may be viewed as fusion and fission phenomena of bags. To treat the fission of a bag, it is first necessary to generalize the spherical cavity bag to a deformable cavity bag. Although a general theory of bag deformation was proposed by Rebbi, its actual application was limited to the case of a small \( P_1 \)-deformation to eliminate the spurious mode due to the centre-of-mass motion of the spherical bag.

In this paper, we shall explicitly construct the spheroidal bag model with an arbitrary eccentricity to deal with a non-trivial \( P_2 \)-deformation of the bag. In constructing the model, we have to slightly modify the so-called MIT linear boundary condition. In this connection, we note that, in the quark sector of the MIT bag model, there are two fundamental requirements to be imposed on the bag surface. The first one is \( n \cdot j = 0 \): the component of quark colour current \( j \) in the normal direction \( n \) of the bag surface should be zero, which ensures that no
The colour flux leaves the bag in consistent with the quark confinement. The second one is that the scalar density $\bar{q}q$ of the quark should vanish on the bag surface: $\bar{q}q = 0$. In the conventional formalism, the linear boundary condition has been adopted in such a way that these two requirements are fulfilled by this linear boundary condition alone. In the next section, we examine in detail the problem whether these two requirements uniquely lead to the MIT linear boundary condition or not. It is pointed out that the first requirement —although this is physically the most important— is not necessarily to be followed from the boundary condition, but can be satisfied in each step of model construction. In particular, it is explicitly proved that, if we restrict ourselves to a non-spherical cavity having both an axial symmetry and the reflection symmetry with respect to a plane perpendicular to the symmetry axis —as exemplified by the spheroidal cavity, the first requirement can be easily satisfied almost independently of the boundary condition. Since the sphere is a limiting case of the spheroid, the same remark will, of course, be applicable to the spherical bag model. Indeed, in any solution of the MIT spherical bag model, it can be easily seen that $\mathbf{r} \cdot \mathbf{j} = 0$ is always trivially fulfilled not only on the bag surface, but also everywhere inside the bag, even before explicitly imposing the MIT linear boundary condition. Nevertheless, it should be emphasized here that, from the very compatibility to free Dirac equation, the MIT linear boundary condition in the case of the spherical bag is certainly the unique boundary condition which is consistent to the second requirement $\bar{q}q = 0$.

We formulate in § 3 the spheroidal bag model, in which quark obeys the free Dirac equation inside the spheroid, together with the two requirements $\mathbf{n} \cdot \mathbf{j} = 0$ and $\bar{q}q = 0$ on the spheroidal surface. Lagrangian formalism of our spheroidal bag model is briefly described at the end of the section. To solve the problem, we employ the scale transformation which maps the spheroid onto the volume-conserving sphere. Under this scale transformation, the free Dirac Hamiltonian in the spheroidal bag is transformed into a new Hamiltonian in the transformed spherical bag, which is not the free Dirac Hamiltonian, but contains an additional (induced-interaction) term. We diagonalize this transformed Hamiltonian in the (transformed) spherical bag on the basis of the MIT spherical bag states to get eigenfrequency of a quark in the spheroidal bag model. Namely, we explicitly impose the MIT linear boundary condition, $-i(\gamma \cdot \mathbf{r})q = q$, on our transformed spherical bag surface. Therefore, our modified linear boundary condition of the spheroidal bag model is simply obtained by its inverse scale transformation. Because of the non-conformal property of the volume-conserving scale transformation, our linear boundary condition thus obtained differs from the MIT linear boundary condition, $-i(\gamma \cdot \mathbf{n})q = q$, to be imposed on the spheroidal surface. But, they coincide, of course, with one another in the limit of small deformation.

We perform in § 4 numerical diagonalization of the transformed Hamiltonian
on the basis of the MIT spherical bag states to get the eigenfrequencies of a confined quark in the spheroidal bag as a function of a deformation parameter $\lambda$, which is related to the eccentricity through $\varepsilon=[1-e^{2\lambda}]^{1/2}$. The results are summarized in Fig. 1(a) and (b). For excited states, we observe the splitting of a single quark orbit, just analogous to the well-known Nilsson splitting of a single nucleon orbit in deformed nuclei. Using the numerical result of lowest quark orbit, we estimate how masses of low-lying hadron change with changing deformation parameter. It is found that, although the spherical quark bag is stable, it is extremely soft against quadrupole deformation: Only 10 MeV supply to a proton is sufficient to obtain a well-deformed proton state in which the ratio of the lengths of two semi-axes is larger than 1.5.

Section 5 is devoted to concluding remarks where a brief discussion is added on the mechanisms to make the spherical quark bag more stable.

§ 2. Remarks on the MIT bag model

This section is divided into two parts. In the first part, we shall briefly summarize the MIT bag model which is needed in the present work, while the second part is devoted to a detailed discussion on the MIT linear boundary condition.

2.1. The MIT bag model

Following Chodos and Thorn, we start from the following action of massless quark without gluon

$$W = \int dx \theta_b(x) \left[ \frac{1}{2} i \bar{q} \gamma^\mu \gamma^\mu q - B + \frac{1}{2} \partial_\mu (n^\mu \bar{q} q) \right]$$  \hphantom{(2.1)}

where $\theta_b(x)$ is the 4-function which takes the value unity inside the bag and zero outside. $n^\mu$ is the inward normal of the bag surface. Clearly, $\partial_\mu \theta_b(x) = n_\mu \delta_b(x)$, where $\delta_b$ is the surface delta function including the measure associated with the curvilinear coordinate of the surface. Since the last term is the total divergence, it only affects the boundary condition. The stationary under the variations of $q$ and the bag surface requires that

$$i \gamma^\mu \partial_\mu q = 0 \hspace{1cm} \text{inside the bag}$$ \hphantom{(2.2)}

and

$$i n_\mu \gamma^\mu q = q \hspace{1cm} \text{on the surface}$$ \hphantom{(2.3a)}

$$i (1/2) \bar{q} \gamma^\mu \gamma^\mu \bar{q} - B + (1/2) \partial_\mu (n^\mu \bar{q} q) = 0$$ \hphantom{(2.3b)}

*We adopt the metric $-g^{00}=g^{00}=1$; all the other conventions are the same as those of Bjorken and Drell.
The second equation (2.3a) is the so-called MIT linear boundary condition to be imposed on the free Dirac equation (2.2), while the third one (2.3b) leads to the quadratic boundary condition

$$\frac{1}{2}n_0\partial^n(\bar{q}q) = B$$

which may be interpreted as expressing the pressure balance on the bag surface.

Several remarks are now in order. From the linear boundary condition alone, one obtains the two relations

$$n\cdot j = 0$$
$$\bar{q}q = 0$$

on the bag surface. (2.5)

The first relation tells us that no quark (colour) current $j$ leaves the bag. This is, therefore, the most important, physically necessary requirement from the standpoint of quark confinement. The second relation seems also to be essentially necessary in constructing a model in which relativistic quarks are confined — free from Klein paradox — in a finite region of space-time with a sharp boundary. Its physical meanings are interpreted as follows. Consider the energy density associated with quark mass $m_q(\bar{q}q)$. By quark-confinement, the mass of quark is supposed to be infinite outside the bag, while it is finite inside. Since the mass density will be everywhere continuous, $\bar{q}q$ should be zero on the sharp boundary of the bag. From the above remarks, one observes that there are two physical requirements (2.5) in the MIT bag model. Indeed, MIT group\textsuperscript{1} first adopted the MIT linear boundary condition so as to satisfy these two requirements — without using the surface term in the action (2.1). We shall, therefore, call any bag model the MIT bag model when these two requirements are fulfilled, even if they need not follow from a boundary condition alone.

If the bag is the time-independent sphere (spherical cavity) of a fixed radius $R$, the free Dirac equation (2.2) with the linear boundary condition can easily be solved as follows. The positive frequency solution of the free Dirac equation in spherical coordinate is given by

$$q^{n\ell m} = N_{n\ell}(\hat{j}_i(k_n r)\pm i(\sigma \cdot \hat{r})j_{\ell \pm 1}(k_n r))(\varphi_{\ell m},)$$

where $\varphi_{\ell m}$ is the spinor harmonics

$$\varphi_{\ell m} = \frac{1}{\sqrt{2l+1}}(\pm \sqrt{l+m+1/2} Y_{l m-1/2} - \sqrt{l+m+1/2} Y_{l m+1/2}).$$

The linear boundary condition (2.3a) then requires

$$j_i(k_n R) = (\pm) j_{\ell \pm 1}(k_n R)$$

from which the kinetic energy of the quark confined in the spherical cavity can be
determined for a fixed value of $R$. The energy of quark-systems is then evaluated as an expectation value of the static Hamiltonian obtained from (2-1) and the value of $R$ is finally determined by minimizing the energy with respect to the variation of $R$. For further details and its application to hadron spectroscopy, we refer the pioneering works of MIT group and several review articles.

2.2. Discussions of the boundary condition

As discussed in the preceding subsection, there are two physical requirements in the MIT bag model: $\mathbf{n} \cdot \mathbf{j} = 0$ and $\bar{q}q = 0$ on the bag surface. We shall now examine in some detail these two requirements. We start from the following observations. In any single spherical wave solution (2-6) of the free Dirac equation, there are everywhere no current in the $r$-direction: $\mathbf{r} \cdot \mathbf{j} = 0$. Furthermore, the component of the current in the $\theta$-direction is also everywhere vanishing; $e_\theta \cdot \mathbf{j} = 0$. In other words, the current flows only around the $z$-axis in each spherical wave in (2-6). Therefore, the requirement $\mathbf{n} \cdot \mathbf{j} = 0$ is trivially satisfied in the case of the spherical bag not only on the surface, but also everywhere inside it, even before explicitly imposing the linear boundary condition. Consider, next, a quark confined in a non-spherical bag with an axial symmetry. If we take the $z$-axis as the symmetry axis, the $z$-component of the total angular momentum is clearly a constant of the motion. The wave function of quark can in principle be expanded in terms of the spherical waves (2-6) with a fixed value of $m$. It is then clear from the above observations that the quark current flows still only around the $z$-axis, provided all the expansion coefficients can be taken as real. (We shall explicitly show this later.) This fact suggests a possibility to construct the non-spherical bag model by modifying the MIT linear boundary condition, in which the second requirement $\bar{q}q = 0$ is satisfied by a modified linear boundary condition, while the first requirement $\mathbf{n} \cdot \mathbf{j} = 0$ is fulfilled in each step of model construction.

As already noted, in the conventional formalism of the MIT bag model, these two requirements follow directly from the linear boundary condition alone. In order to seek a possibility to modify the linear boundary condition, it is first necessary to examine whether the converse statement — i.e., these two requirements lead uniquely to the MIT linear boundary condition — is true or not. If it is true, there is, of course, no possibility to alter the linear boundary condition. To study this problem, we decompose the four component spinor $\mathbf{q}$ into two two-component spinors $\Psi_u$ and $\Psi_L$. In terms of these upper and lower components, $\bar{q}q = 0$ can be represented as $|\Psi_u|^2 = |\Psi_L|^2$. Hence, we observe that there exists a unitary transformation $U$ defined on the bag surface which transforms $\Psi_u$ into $\Psi_L$. The most general form of $U$ can be determined as $U = e^{-i\varphi}\{\sin \Theta + i\cos \Theta(\sigma \cdot \hat{e})\}$. We thus obtain

$$\Psi_L = e^{-i\varphi}\{\sin \Theta + i\cos \Theta(\sigma \cdot \hat{e})\}\Psi_u,$$

(2.9a)
where \( \hat{e} \) is an arbitrary unit length vector, and \( \Theta \) and \( \phi \) are any real scalar functions of the coordinate of the surface. This formula — when transcribed back in the four-component spinor \( q \) and the \( \gamma \)-matrices — reads

\[
q = e^{-i\sigma \cdot e} \{ \gamma \sin \Theta - i \cos \Theta (\gamma \cdot \hat{e}) \} q .
\]

This is the most general linear form in \( q \) which is equivalent to the requirement \( \bar{q}q = 0 \) alone. We note that (2.9) reduces to the MIT linear boundary condition when we set \( \Theta = \phi = 0 \) and \( \hat{e} = \hat{n} \).

We next combine the above result to the first requirement \( n \cdot j = 0 \),

\[
n \cdot j = \Psi_{v}^{+}(\sigma \cdot n)\Psi_{l} + \Psi_{l}^{+}(\sigma \cdot n)\Psi_{v} = \Psi_{v}^{+}[(\sigma \cdot n)U + U^{*}(\sigma \cdot n)]\Psi_{v}
= 2\Psi_{v}^{+}[\cos \phi \cdot \sin \Theta (\sigma \cdot n) + \sin \phi \cos \Theta (n \cdot \hat{e})]
- \cos \phi \cos \Theta (\sigma \cdot n \times \hat{e})]\Psi_{v}.
\]

If we let the formula inside the parentheses zero, we obtain the two solutions

(i) \( \Theta = \phi = 0 \), \( \hat{e} = \hat{n} \) and (ii) \( \phi = \pi/2 \), \( \Theta = \text{arbitrary} \), \( \hat{e} \perp \hat{n} \).

The first one is nothing but the MIT linear boundary condition, while the second one seems to be hardly compatible to the free Dirac equation. At first sight, the above statement leads to the conclusion that the MIT linear boundary condition is almost unique consequence from the two requirements. It is, however, noticed that in deriving these two solutions we have imposed the so-called strong condition in which the relation \( (\sigma \cdot n)U + U^{*}(\sigma \cdot n) = 0 \) should hold as an operator identity independently of \( \Psi_{v} \). Instead of the strong condition, the weak condition is sufficient to our problem in which \( \Psi_{v}^{+}[(\sigma \cdot n)U + U^{*}(\sigma \cdot n)]\Psi_{v} \) should vanish in our chosen model state \( \Psi_{v} \). Since it is difficult to present the weak condition in most general way, we restrict ourselves to a concrete example to be used in the next section.

That is, we consider a non-spherical static cavity having the axial symmetry around the \( z \)-axis as well as the reflection symmetry with respect to the \( x \)-\( y \) plane — as typically exemplified by the spheroidal cavity. These symmetries require that \( q \) has a definite parity, which in turn implies that \( \Theta \) in (2.9) should be zero under the reasonable assumption on symmetry property on \( \Theta \), such as \( \Theta(r) = \Theta(-r) \) on the bag surface. We further impose the time-reversal invariant on \( q \). From this requirement, we may take \( \phi = 0 \). We thus obtain the restricted form of (2.9); \( q = -i(r \cdot \hat{e})q \), \( \hat{e} \) still being an arbitrary unit length vector. Next, suppose that we expand \( q \) in terms of the spherical waves. The axial symmetry implies that \( q \) can be expanded by the set of (2.6) with a fixed \( \mathbf{m} \). Further, from
time-reversal invariance, all the expansion coefficients can be taken as real. It is now clear from the explicit forms of (2.6) as well as from that of the spinor harmonics, the upper component spinor $\Psi_U$ of $q$ can be represented as

$$\Psi_U = \left( A e^{i(m-1/2)\theta}, B e^{i(m+1/2)\theta} \right),$$

where both $A$ and $B$ are real functions of $r$ and $\theta$, independent of the azimuthal angle $\phi$. By inserting this into (2.10) and using requirements $\Theta = \psi = 0$ obtained by the symmetry considerations, we have

$$\mathbf{n} \cdot \mathbf{j} = (A e^{-i(m-1/2)\theta}, B e^{-i(m+1/2)\theta}) \mathbf{\sigma} \cdot (\mathbf{\hat{e}} \times \mathbf{n}) (A e^{i(m-1/2)\theta}, B e^{i(m+1/2)\theta})$$

which vanishes obviously when the vector $\mathbf{\hat{e}} \times \mathbf{n}$ is in the $\mathbf{e}_r$-direction, due to the explicit form of $\mathbf{\sigma} = (\mathbf{\sigma} \cdot \mathbf{e}_r)$.

In summary, we have shown that, if we restrict ourselves to a non-spherical static cavity bag having both the axial and reflection symmetries —as exemplified by the spheroidal bag, the two requirements, $\mathbf{n} \cdot \mathbf{j} = 0$ and $\mathbf{\hat{q}} = 0$, on the bag surface can easily be fulfilled by adopting the linear boundary condition of the form: $-i(\mathbf{r} \cdot \mathbf{\hat{e}})q = q$, where the vector $\mathbf{\hat{e}}$ is still an arbitrary unit length vector under the restriction that the vector $\mathbf{\hat{e}} \times \mathbf{n}$ is in the $\mathbf{e}_r$-direction. In the next section, we shall specify this vector by making use of the scale transformation from the MIT linear boundary condition of the spherical bag model. Finally, we note that the most important restriction on any boundary condition is its compatibility to the Dirac equation. This problem is, however, beyond the scope of the present paper.

§ 3. Spheroidal bag model

We consider the static spheroidal cavity which is defined by

$$\frac{x^2}{R^2 e^{-\lambda}} + \frac{y^2}{R^2 e^{-\lambda}} + \frac{z^2}{R^2 e^{-\lambda}} = 1,$$

where $\lambda$ is a deformation parameter, relating to the eccentricity $e = (1 - e^{-3\lambda})^{1/2}$. The volume of the spheroid is given by $V = (4\pi/3)R^3$. We shall, then, construct the spheroidal bag model —quarks confined in this spheroidal cavity— which satisfies all the desired properties of the MIT bag model: The quark field obeys the free Dirac equation inside the bag together with the two-requirements, $\mathbf{\hat{q}} = 0$ and $\mathbf{n} \cdot \mathbf{j} = 0$, on the bag surface. This construction is made by slightly modifying the MIT linear boundary condition. The Lagrangian formalism of our model will be presented later at the end of this section.
Let us start from the following Dirac Hamiltonian of a single massless quark confined in the static spheroidal cavity — a deformed bag,

\[ H_D = \theta(D)(\mathbf{a} \cdot \mathbf{p}), \]  

(3.2)

where \( \theta(D) \) is the \( \theta \)-function which is unity inside the spheroid and zero outside. The eigenmode energy \( \omega_D \) of the quark will be obtained by solving the Dirac equation

\[ \theta(D)(\mathbf{a} \cdot \mathbf{p})q_D = \omega_D q_D \]  

(3.3)

by imposing on it a "suitable boundary condition". The boundary condition to be imposed will be explicitly specified later in (3.8). In order to solve the above equation, we first consider the scale transformation of coordinate space which maps the spheroid to the volume-conserving sphere of the radius \( R; x \rightarrow e^{-R^2}x \), \( y \rightarrow e^{-R^2}y \) and \( z \rightarrow e^Rz \). The unitary transformation which generates this scaling is given by

\[ U(\lambda) = \exp\{i(\lambda/2)S\} \text{ where } S = 2zp_x - xp_y - yp_z. \]  

(3.4)

Under the same scale transformation, the components of the momentum vector transform as \( p_x \rightarrow e^{i\lambda\sigma_3}p_x \), \( p_y \rightarrow e^{i\lambda\sigma_3}p_y \) and \( p_z \rightarrow e^{-i\lambda\sigma_3}p_z \). We thus get

\[ U(\lambda)pU^{-1}(\lambda) = \frac{1}{3}(e^{-i\lambda} + 2e^{i\lambda})\mathbf{p} + \frac{1}{3}(e^{-i\lambda} - e^{i\lambda})\mathbf{q}, \]  

(3.5a)

where \( \mathbf{p} \) is the momentum vector and the vector \( \mathbf{q} \) is defined by

\[ \mathbf{q} = 2e\mathbf{p}_x - e\mathbf{p}_y - e\mathbf{p}_z. \]  

(3.5b)

We now apply the above scale transformation to our problem. Obviously, the \( \theta \)-function defined in the spheroid is transformed into the \( \theta \)-function with respect to the volume conserving sphere; \( U(\lambda)\theta(D)U^{-1}(\lambda) = \theta(S) \), where \( \theta(S) \) is the \( \theta \)-function with respect to the sphere in the transformed "image space". The Dirac equation is transformed into

\[ \theta(S)U(\lambda)(\mathbf{a} \cdot \mathbf{p})U^{-1}(\lambda)\{U(\lambda)q_D U^{-1}(\lambda)\} = \omega_D \{U(\lambda)q_D U^{-1}(\lambda)\}. \]  

(3.6)

Here, \( Uq_DU^{-1} \) should be understood as \( q_D(r, \sigma) \rightarrow q_D(r', \sigma) \), where \( r' = UrU^{-1} \). From (3.5), we obtain the explicit form of Hamiltonian \( H_I \) in the "image spherical cavity":

\[ H_I = \theta(S)\left\{ \frac{1}{3}(e^{-i\lambda} + 2e^{i\lambda})(\mathbf{a} \cdot \mathbf{p}) + \frac{1}{3}(e^{-i\lambda} - e^{i\lambda})(\mathbf{a} \cdot \mathbf{q}) \right\}. \]  

(3.7)

As an illustration of the above procedure, let us suppose the explicit form of Dirac equation as the first-order coupled differential equation written in the cartesian coordinate. The domain of definition of variables (= the cartesian coordinate) is
A Deformable Bag Model of Hadrons. I restricted within the spheroid (3·1). By making the simple change of variables, 

\[ x \rightarrow e^{-i\pi}x, \quad y \rightarrow e^{-i\pi}y, \quad z \rightarrow e^{i\pi}z, \]

the domain of definition of new variables becomes inside the sphere of radius \( R \), while the differential operators \( \partial/\partial x, \partial/\partial y, \partial/\partial z \) are transformed in the obvious way. The functional form of the solution \( q(x, y, z) \) remains, of course, unchanged: We need only the simple replacement \( q(x, y, z) \rightarrow q(e^{-i\pi}x, e^{-i\pi}y, e^{i\pi}z) \). Thus, one sees that our procedure so far employed is nothing but the elementary change of variables to solve the differential equation.

Since everything is mapped on and inside the sphere, we can solve Eq. (3·6) by imposing on it the MIT linear boundary condition. In other words, we perform the diagonalization of \( H_l \) on the basis of the MIT bag states \( q_{\ell m}^{\text{MIT}} \) in this image spherical bag; \( H_l q_l = \omega_l q_l, \quad q_l = \sum a_{\ell m} q_{\ell m}^{\text{MIT}}. \) The resulting eigenvalues give us directly the desired values of \( \omega_l \) in (3·6), while the quark wave function \( q_l \) in the spheroidal bag is obtained through 

\[ q_l = U^{-1} q_{\ell m}. \]

The MIT linear boundary condition which we have imposed on the image spherical bag reads 

\[ -i(\gamma \cdot \hat{r}) q_l = -i(\beta a \cdot \hat{r}) q_l = q_l. \]

This boundary condition, when transformed back onto the spheroidal bag, becomes

\[ -i(\gamma \cdot \hat{\xi}) q_l = q_l, \quad (3·8) \]

where \( \hat{\xi} = U^{-1} \hat{r} U \) is the unit length vector, whose cartesian components are given by 

\[ (e^{-i\pi}x/\sqrt{e^{-i\pi}(x^2+y^2)}+e^{i\pi}z^2, \quad e^{-i\pi}y/\sqrt{e^{-i\pi}(x^2+y^2)}+e^{i\pi}z^2, \quad e^{-i\pi}z/\sqrt{e^{-i\pi}(x^2+y^2)}+e^{i\pi}y^2). \]

We note that this vector differs from the normal vector of the spheroid \( n = (e^{-i\pi}x/\sqrt{e^{-i\pi}(x^2+y^2)}+e^{i\pi}z^2, \quad e^{-i\pi}y/\sqrt{e^{-i\pi}(x^2+y^2)}+e^{i\pi}z^2, \quad e^{-i\pi}z/\sqrt{e^{-i\pi}(x^2+y^2)}+e^{i\pi}y^2). \) These two vectors coincide, of course, in the limit \( \lambda = 0 \). In an another limit \( \lambda = +\infty \), where our spheroid tends to the \( z \)-axis, the vector \( \hat{\xi} \) is in the positive (negative) \( z \) direction for positive (negative) value of \( z \), while \( n \) is perpendicular to the \( z \) direction. For \( \lambda = -\infty \), where the spheroid becomes the \( x-y \) plane, \( \hat{\xi} \) is always on the plane, while \( n \) is perpendicular to this plane.

In summary, the quark sector of our bag model consists of Eq. (3·6) with the boundary condition (3·8). From this boundary condition alone, we immediately obtain \( \bar{q}q = 0 \) on the spheroidal surface. At the same time, we have \( \hat{\xi} \cdot j = 0 \), which does not necessarily imply the desired requirement \( n \cdot j = 0 \). It is, however, evident from the discussion in § 2.2 that our model satisfies this requirement, since the vector \( n \times \hat{\xi} \) is always in the \( \hat{e}_r \) direction.

Finally, we briefly describe Lagrangian formalism of our spheroidal bag model. We define the action

\[ W = \int d^4x \bar{q}_l \gamma^a \hat{e}_d \bar{q}_l - B + \frac{1}{2} \partial_\mu (n^a \bar{q}_l q_l) + \frac{1}{2} \partial_\mu \partial_\nu (\bar{q} \gamma^a q_l), \quad (3·9) \]

where the last term is required to modify the MIT linear boundary condition.
Here, $\alpha_v$ is a vector which will be specified so as to obtain our modified boundary condition. By minimizing this action, we obtain

\begin{align}
\gamma^\nu \partial_\nu q = 0 & \quad \text{inside the bag}, \tag{3\cdot10} \\
\gamma^\nu q = q + (\gamma^\nu \alpha_v)q & \quad \text{on the surface}, \tag{3\cdot11a} \\
\frac{1}{2} \tilde{q} \gamma^\nu \partial_\nu q - B + \frac{1}{2} \partial_\nu (n^\nu q q) + \frac{1}{2} \partial_\nu n^\nu (\tilde{q} \gamma^\nu \alpha_v q) = 0. & \tag{3\cdot11b}
\end{align}

From the last equation, we get our modified quadratic boundary condition,

$$B = \frac{1}{2} n^\nu \partial_\nu (\tilde{q} q) + \frac{1}{2} n^\nu \partial_\nu (\tilde{q} \gamma^\nu \alpha_v q). \tag{3\cdot12}$$

Now, by fixing the vector $\alpha$ to be $\alpha = i(n - \xi)$, we obtain our modified linear boundary condition (3\cdot8). We note that, although the imaginary factor $i$ in this choice seems to be rather artificial, it is always required in order to keep the boundary condition (3\cdot11a) hermitian. Further, as already shown in § 2.2, the last term in the action of our spheroidal bag model always vanishes — with this explicit choice of $\alpha$ — not only on the spheroidal surface, but everywhere inside it. From the same reasonings, the last term appearing in the quadratic boundary condition (3\cdot12) gives no finite contribution. Namely, the quadratic boundary condition remains practically the same as that of the MIT bag model (2\cdot4).

§ 4. Calculations of eigenvalues and numerical results

In this section, we shall perform numerical diagonalization of $H_I$ in (3\cdot8) on the basis of the MIT spherical bag states. We split $H_I$ into two parts;

$$H_I = \theta(S)(H_0 + V), \tag{4\cdot1a}$$

where

$$H_0 = (1/3)(e^{-\beta} + 2e^{i\beta})(\alpha \cdot p) \quad \text{and} \quad V = (1/3)(e^{-\beta} - e^{i\beta})(\alpha \cdot q). \tag{4\cdot1b}$$

The unperturbed energy of our model is then given by

$$\omega_0(\lambda) = (1/3)(e^{-\beta} + 2e^{i\beta})\omega_{\text{MIT}}$$

which is always larger than the corresponding energy $\omega_{\text{MIT}}$ of the MIT spherical bag state for any non-vanishing deformation parameter $\lambda$.

Next, we need to evaluate all the relevant matrix elements of $V$ between various MIT bag states. Representing $(\alpha \cdot q)$ in the form of the two-by-two matrix with vanishing diagonals, we rewrite the off-diagonals in the following form:
Using the explicit wave function of the MIT bag state, \( |n; l \frac{1}{2} jm \rangle \) in (2.6) and (2.8), we have

\[
\langle n'; l' \frac{1}{2} j' m' | \alpha \cdot q | n; l \frac{1}{2} jm \rangle = N^{n l} N^{n' l'}
\]

\[
\times \int_0^\pi \int_0^{2\pi} \int_0^\infty r^2 d\Omega \mathcal{Q} j_{l' m'} \left[ j_{l'}(k_n r) i(r \cdot q) \frac{1}{r} \{ \pm j_{l z}(k_n r) \}
\right.
\]

\[- \{ \pm j_{l' z}(k_n r) \} \frac{1}{r} (r \cdot q) j_{l z}(k_n r)
\]

\[
+j_{l'}(k_n r) \sigma \cdot (r \times q) \frac{1}{r} \{ \pm j_{l z}(k_n r) \}
\]

\[
+ \{ \pm j_{l' z}(k_n r) \} \frac{1}{r} \sigma \cdot (r \times q) j_{l z}(k_n r) \mathcal{Q}_{l m}.
\]

(4.2)

We then employ the standard technique of tensor algebra to obtain

\[
i(\hat{r} \cdot q) = 2 \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) \frac{\partial}{\partial r} - \sqrt{6} \sqrt{\frac{4\pi}{5}} \frac{1}{r} Y^{(2)}(\theta) \times L_{(2)}(r)
\]

(4.3a)

and

\[
\sigma \cdot (\hat{r} \times q) = - \frac{1}{3r}(\sigma \cdot L) - \sqrt{\frac{10}{3}} \sqrt{\frac{4\pi}{5}} \frac{1}{r} Y^{(2)}(\theta) \times [\sigma \times L](r)
\]

\[
- \sqrt{\frac{2}{3}} \sqrt{\frac{4\pi}{5}} Y_{20}(r) \sigma \cdot L
\]

\[
- \sqrt{6} \sqrt{\frac{4\pi}{5}} [\sigma \times L](r) \frac{\partial}{\partial r} - \frac{7}{3} \sqrt{\frac{4\pi}{5}} \frac{1}{r} Y^{(2)}(\theta) \times [\sigma \times L](r).
\]

(4.3b)

For the operators \( i(r \cdot q)/r \) and \( \sigma \cdot (r \times q)/r \) in (4.2), the radial derivatives appearing in the above formulas should be replaced by \( (\partial/\partial r) - (1/r) \), all the other terms remaining unchanged. The spin-angle reduced matrix elements are easily obtained: as expected, the sum of the reduced matrix elements of the first two terms of (4.3b), both of which are rotation scalar, can be proved to vanish. For the second rank tensors, we obtain
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\[ \langle I_1 \frac{1}{2} j' \mid Y^{(2)} \mid I_2 \frac{1}{2} j \rangle = \sqrt{\frac{5}{4\pi}} (-1)^{j' j + j} \left[ 1 + (-1)^{j' j} \right] j \left( j_2 \frac{1}{2} 0 \mid j \frac{1}{2} \right). \]

\[ \langle I_1 \frac{1}{2} j' \mid [Y^{(2)} \times L]^{(2)} \mid I_2 \frac{1}{2} j \rangle = (-1)^{j' j} \sqrt{5} \sum_{j''} \left\{ \begin{array}{ccc} 2 & 1 & 2 \\ j' & j'' & j'' \end{array} \right\} \times \langle I_1 \frac{1}{2} j' \mid Y^{(2)} \mid I_2 \frac{1}{2} j'' \rangle \langle I_2 \frac{1}{2} j'' \mid L \mid I_2 \frac{1}{2} j \rangle, \]

where the reduced matrix element of orbital angular momentum \( L \) between the spinor harmonics is well-known,

\[ \langle I_1 \frac{1}{2} j' \mid L \mid I_2 \frac{1}{2} j \rangle = (-1)^{j' j + j} \sqrt{(I + 1)(2I + 1)} \delta_{2j}^{(j')}(I_j), \]

\[ \langle I_1 \frac{1}{2} j' \mid [\sigma \times L]^{(2)} \mid I_2 \frac{1}{2} j \rangle = \delta_{2j}^{(j')} \sqrt{30} \tilde{j} \tilde{j} \sqrt{(I + 1)(2I + 1)} \delta_{2j}^{(j)} \left( \begin{array}{c} I_j \\ j \\ 1 \\ 1 \end{array} \right), \]

\[ \langle I_1 \frac{1}{2} j' \mid [Y^{(2)} \times [\sigma \times L]^{(2)}]^{(2)} \mid I_2 \frac{1}{2} j \rangle = \sqrt{5} \sum_{j''} (-1)^{j' j''} \left\{ \begin{array}{ccc} 2 & 2 & 2 \\ j' & j'' & j'' \end{array} \right\} \langle I_1 \frac{1}{2} j' \mid Y^{(2)} \mid I_2 \frac{1}{2} j'' \rangle \langle I_2 \frac{1}{2} j'' \mid [\sigma \times L]^{(2)} \mid I_2 \frac{1}{2} j \rangle. \]

Here, \( \tilde{j} = \sqrt{2j + 1} \) and all the other notations are standard ones in nuclear spectroscopy, so that they need not be explained. Further, \([\sigma \times Y^{(2)}]^{(2)}\) can easily be treated by using the identity

\[ \sqrt{6}[\sigma \times Y^{(2)}]^{(2)} = Y_{20}(\sigma \cdot L) - (\sigma \cdot L) Y_{20}. \]  

\[ (44) \]

The spin-angle matrix elements are now ready to numerical calculation. Also, all the radial matrix elements involving spherical Bessel functions and their derivatives can be quickly obtained by numerical integrations.

At this stage, it is important to note that, in contrast to non-relativistic problems, the hermiticities of various operators should carefully be examined in our problem, because any wave function of the relativistic quark confined in the finite sphere has always a non-vanishing boundary value which may destroy the hermiticity of the operators involving radial derivatives. To examine the hermiticity, we first write all the spin-angle matrix elements in obviously symmetric forms by using (4.4) together with the following identities,

\[ [Y^{(2)} \times L]^{(2)} = - [L \times Y^{(2)}]^{(2)} - \sqrt{6} Y_{20}. \]
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and next perform the partial integration of radial matrix elements containing the radial derivatives. In this way, we found that the part in (4·2) arising from

\[ i(\mathbf{\hat{r}} \cdot \mathbf{q}) \] and \[ i(\mathbf{r} \cdot \mathbf{q})/r \] is hermitian, while the term involving the radial derivatives due to \[ \hat{\sigma}(\mathbf{\hat{r}} \times \mathbf{q}) \] and \[ \hat{\sigma}(\mathbf{r} \times \mathbf{q})/r \] is not hermitian; a surface term survives. As clearly seen from (4·4), this non-hermitian part appears only in the off-diagonal matrix elements. Hereafter, we shall symmetrize this part of the off-diagonal matrix elements. This procedure is, of course, equivalent to using the manifestly hermitian operator \((\mathbf{a} \cdot \mathbf{q})\) instead of \((\mathbf{a} \cdot \mathbf{q})\). Therefore, our model Hamiltonian to be diagonalized is given by (4·1), where \((\mathbf{a} \cdot \mathbf{q})\) in \(V\) will be replaced by \((\mathbf{a} \cdot \mathbf{q})\).

We note in passing that by representing the matrix elements in the most symmetric form and using explicit algebraic expressions of various recoupling coefficients, we obtain the diagonal matrix elements of \((\mathbf{a} \cdot \mathbf{q})\) in a closed form

\[
\langle n; l 1/2 jm | \mathbf{a} \cdot \mathbf{q} | n; l 1/2 jm \rangle = 2\omega(\lambda)|l| \langle l 1/2 jm | Y_{2l} | l 1/2 jm \rangle .
\]

This formula gives us the first-order energy-shift for \(j \neq 1/2\) states, which is essentially responsible to the splitting of the single quark orbit in the spheroidal bag, just analogous to the well-known Nilsson splitting of the single nucleon orbit in deformed nuclei, as will be discussed shortly.

Having calculated all the relevant matrix elements of \(H_i\), we perform numerical diagonalization to obtain the eigenvalues \(\omega(\lambda)\) of \(H_i\) as a function of the deformation parameter \(\lambda\). As is clear from the form of Hamiltonian, the dimensionless quantity \(\omega(\lambda)R\) is entirely independent of \(R\). Therefore, \([\omega(\lambda) - \omega_{MIT}]R\) is the most convenient quantity to see how the eigenfrequency of a confined quark in the spheroidal bag depends on the bag deformation. We display in Fig. 1(a) and (b) \([\omega(\lambda) - \omega_{MIT}]R\) vs \(\lambda\) for several low-lying nodeless states in the range \(|\lambda| \leq 0.15\). In this calculation, we have restricted the MIT basis states within \(l \leq 10\), \(j \leq 21/2\) and \(n \leq 10\). In each figure, the value of \(\omega(\lambda=0)R = \omega_{MIT}R\) is indicated together with its quantum number \(lj\). For non-zero deformations, each quark orbit is doubly degenerate according to \(\pm m\); the quantum number \(|m|\) is shown in both the sides of each figure.

From the figures, one observes the splitting of a single quark orbit in our spheroidal bag, which is just analogous to the well-known Nilsson splitting of a single particle orbit in deformed nuclei. This splitting arises essentially from the first-order energy-shift due to (4·5). In the first-order perturbation, all curves in the figure become, of course, straight lines, the deviations from the straight lines arising from higher order effects.

For the lowest energy state, we have adopted the different scale of the
ordinate. From the figure, we have an approximate formula

$$\omega(\lambda) = \frac{\omega_{\text{min}}}{\lambda^2}(1 + 0.1\lambda^3) = 2.0428(1 + 0.1\lambda^3)/R \quad (4.6)$$

in the range $|\lambda| \leq 0.15$. We note that the value of the deformation parameter $|\lambda|$
= 0.15 does not necessarily imply a small deformation, because the ratio of the major to minor semi-axes of the spheroid is 1.252 in this case.

Let us now estimate the effect of the deformation on the mass of low-lying hadrons. In just the same way as in the spherical Bag model, the total energy of the quark bag containing of \( N \) quarks and anti-quarks in its lowest eigenmode may be written as

\[
M(R, \lambda) = N\omega(\lambda) + \frac{4\pi}{3} R^3 B - \frac{Z_0}{R} + \Delta E_g + \Delta E_m ,
\]

where the first term is the energy of \( N \) quarks and antiquarks, the second term is the volume energy arising from the bag constant \( B \), the third one the zero-point energy renormalization (Casimir energy) and the last two terms are the colour interaction energies, which are usually introduced semi-phenomenologically in order to take into account the \( N - \Lambda \) mass splitting. By neglecting the last three terms and minimizing \( M(R, \lambda) \) with respect to \( R \), we obtain

\[
M(R_0, \lambda) = \frac{4}{3} \times 2.0428 \times N (1 + 0.1 \lambda^2) / R_0 ,
\]

where \( R_0^2 = (1/4\pi B) \times 2.0428 \times N (1 + 0.1 \lambda^2) \).

This result shows that, although the spherical quark bag is stable, it is extremely soft against the quadrupole deformation. For examples, the mass of hadron increases only 1% from the spherical to \( (\lambda = 0.3) \) deformation. In other words, the addition of only 10MeV energy to a proton is sufficient to get the well-deformed proton state of \( \lambda = 0.3 \), in which the ratio of the two semi-axes is now 1.56. We note that the magnetic moment will change from the spherical to such well-deformed states.

If we include the Casimir energy \( -Z_0/R \) to the above consideration, the conclusions mentioned above remain unchanged for the usual value \( Z_0 \) such as \( Z_0 = 1.64 \). Strictly speaking, the Casimir energy will depend on the shape of the bag. Although the correct form of the Casimir energy for the spheroidal bag is not known, we may take an approximate form \( -Z_0 (1/3) \left[ 1/R_s + 2(1/R_L) \right] \), where \( R_s \) and \( R_L = R_x = R_y \) are the lengths of semi-axes. Using our parametrization, we have; Casimir Energy \( = -Z_0 (1/3) \left( e^{-\lambda^2} + 2 e^{\lambda^2} \right) / R \approx -(Z_0/R)(1 + 0.25 \lambda^2) \).

Introducing this assumed form into (4.7) and minimizing the total energy with respect to \( R \), we found that the spherical quark bag becomes unstable against the quadrupole deformation, provided that somewhat larger value of \( Z_0 \) (say, \( Z_0 = 2.5 \)) is adopted.

§ 5. Concluding remarks

We have calculated the eigenfrequencies of a single massless quark confined
in the spheroidal bag and found the level splitting of the excited orbits which is just analogous to the well-known Nilsson splitting of single particle orbits in deformed nuclei. Using the numerical result of the lowest orbit, we have estimated the effect of the deformation on the mass of low-lying hadrons. It turns out that, although the spherical quark bag is stable, the quark bag of hadron is extremely soft against quadrupole deformation. Since we have already in hand the detailed wave function of the lowest quark state in the spheroidal bag, the deformation-dependence of various quantities, such as the magnetic moment and \((g_{\mu}/g_{e})\) ratio, will be explicitly calculable; the results will be presented in the second of this series.

The softness of hadron would imply the existence of low-lying vibrational states, which seems to be inconsistent with the experimental fact. We need, therefore, consider some mechanisms which make the hadron more stable against the quadrupole deformation. The simplest way to this direction is to introduce the surface tension into the Lagrangian, as has been done by Budapest group\(^8\) — so-called Budapest bag. Another, more interesting dynamical mechanism to stabilize the spherical hadron might be obtained by introducing pion field outside the bag, as has been done by many authors\(^9\),\(^10\),\(^11\), in several different stand-points. As first pointed out by Brown and Rho,\(^9\) the pion field outside the bag induces an extra pressure to make the bag radius smaller — hence, the small bag — in consistent with common belief of low energy nuclear physicist. Since the source of pion field is the axial current on the bag surface, this extra pressure will depend on the geometrical shape of the bag, which might give rise to a surface-tension-like additional energy to the model. There problems are left to a future investigation.

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