The equations of motion including the flexible shaft are then written in terms of the flexibility influence coefficients by expressing the relative deflections and angular rotations as follows:

$$u_i = \alpha_{ij}P_{xj} + \beta_{ij}C_{yj} \quad (25)$$

$$v_i = \alpha_{ij}P_{yj} - \beta_{ij}C_{xj} \quad (26)$$

$$\hat{\theta}_{xi} = \varphi_{ij}P_{xj} + \gamma_{ij}C_{yj} \quad (27)$$

$$\hat{\theta}_{yi} = \varphi_{ij}P_{yj} - \gamma_{ij}C_{xj} \quad (28)$$

where

$$P_{xj} = (-m\ddot{x} - c\dot{x} - Kx + me\omega^2 \cos(\omega t + \varphi) + me\dot{\omega}\sin(\omega t + \varphi) - CI\ddot{u} - \omega CI v - Qy + \hat{F}_x)_j = -m_j\ddot{x}_j + \mathfrak{F}_{xj} \quad (29)$$

$$P_{yj} = (-m\ddot{y} - c\dot{y} - Ky + me\omega^2 \sin(\omega t + \varphi) - me\dot{\omega}\cos(\omega t + \varphi) - CI\ddot{v} + \omega CI u + Qx - W + \hat{F}_y)_j = -m_j\ddot{y}_j + \mathfrak{F}_{yj} \quad (30)$$

$$C_{xj} = (I_T\ddot{\theta}_y - \omega I_P\dot{\theta}_x - \tfrac{1}{2}I_P\dot{\omega}\theta_x - (I_P - I_T)\tau\omega^2 \times \sin(\omega t + \varphi + \beta_\tau) + \tau\dot{\omega}(I_P - I_T)\cos(\omega t + \varphi + \beta_\tau) + C_\theta\dot{\theta}_y + \mathcal{K}\theta_y)_j = I_{Tj}\ddot{\theta}_{yj} + H_{xj} \quad (31)$$

$$C_{yj} = (-I_T\ddot{\theta}_x - \omega I_P\dot{\theta}_y - \tfrac{1}{2}I_P\dot{\omega}\theta_y + (I_P - I_T)\tau\omega^2\cos(\omega t + \varphi + \beta_\tau) + \tau\dot{\omega}(I_P - I_T)\sin(\omega t + \varphi + \beta_\tau) - C_\theta\dot{\theta}_x - \mathcal{K}\theta_x)_j = -I_T\ddot{\theta}_{xj} + H_{yj} \quad (32)$$

By removing the inertia terms from the foregoing equations, the acceleration terms may be expressed as a product of known values for each time step. The equations are expressed as

$$\begin{bmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{bmatrix} \begin{bmatrix} \ddot{X}^{(n)} \\ \ddot{\theta}_x^{(m)} \\ \ddot{Y}^{(m)} \\ \ddot{\theta}_y^{(m)} \end{bmatrix} = \begin{bmatrix} F_x^{(n)} \\ \Theta_x^{(n)} \\ F_y^{(n)} \\ \Theta_y^{(n)} \end{bmatrix} \quad (33)$$

$$A_{ij} = \alpha_{ij}m_j \quad (34)$$

$$B_{ij} = \beta_{ij}I_{Tj} \quad (35)$$

where

$$C_{ij} = \varphi_{ij}m_j \quad (36)$$

$$D_{ij} = \gamma_{ij}I_{Tj} \quad (37)$$

with

$$\ddot{X}^{(n)} \equiv \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n \end{bmatrix}; \quad \ddot{Y}^{(n)} \equiv \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{y}_n \end{bmatrix}; \quad \ddot{\theta}_x^{(m)} \equiv \begin{bmatrix} \ddot{\theta}_{x1} \\ \cdot \\ \cdot \\ \cdot \\ \ddot{\theta}_{xn} \end{bmatrix}; \quad \ddot{\theta}_y^{(m)} \equiv \begin{bmatrix} \ddot{\theta}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \ddot{\theta}_{yn} \end{bmatrix}$$

and (38)

$$F_x^{(n)} \equiv \begin{bmatrix} -u_1 + \alpha_{1j}\mathfrak{F}_{xj} + \beta_{1j}H_{yj} \\ \cdot \\ \cdot \\ \cdot \\ -u_n + \alpha_{nj}\mathfrak{F}_{xj} + \beta_{nj}H_{yj} \end{bmatrix} \quad (39)$$

$$\Theta_x^{(n)} \equiv \begin{bmatrix} -\hat{\theta}_{x1} + \varphi_{1j}\mathfrak{F}_{xj} + \gamma_{1j}H_{yj} \\ \cdot \\ \cdot \\ \cdot \\ -\hat{\theta}_{xn} + \varphi_{nj}\mathfrak{F}_{xj} + \gamma_{nj}H_{yj} \end{bmatrix} \quad (40)$$

$$F_y^{(n)} \equiv \begin{bmatrix} -v_1 + \alpha_{1j}\mathfrak{F}_{yj} - \beta_{1j}H_{xj} \\ \cdot \\ \cdot \\ \cdot \\ -v_n + \alpha_{nj}\mathfrak{F}_{yj} - \beta_{nj}H_{xj} \end{bmatrix} \quad (41)$$

$$\Theta_y^{(n)} = \begin{bmatrix} -\hat{\theta}_{y1} + \varphi_{1j}\mathfrak{F}_{yj} - \gamma_{1j}H_{xj} \\ \cdot \\ \cdot \\ \cdot \\ -\hat{\theta}_{yn} + \varphi_{nj}\mathfrak{F}_{yj} - \gamma_{nj}H_{xj} \end{bmatrix} \quad (42)$$

This formulation then gives the complete equations of motion necessary to express the total dynamic response of the torsionally rigid, flexible, multimass rotor.

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DISCUSSION

D. W. Childs[2]

Kirk and Gunter indicate that the present work is an extension of the analysis by Shen [1] in the sense that Shen employed an approximate solution procedure for the rotational equations of motion. In addition, they have related their experiences concerning the stability problems encountered in numerical integration of the governing differential equations. This comment concerns (*a*) the adequacy and generality of the governing differential equations, and (*b*) some observations concerning numerical instability problems.

There apparently exist some differences of opinion as to the correct form for the dynamic terms of the rotational equations of motion for component rigid bodies. This difference can be clarified by the following restatement of equations (31) and (32)

$$I_T\ddot{\theta}_x + \dot{\phi}I_p\dot{\theta}_y + \ddot{\phi}\{\theta_x I_p/2 + \tau(I_p - I_T)\sin(\phi + \beta_\tau)\} - \dot{\phi}^2\tau(I_p - I_T)\cos(\phi + \beta_\tau) = M_y$$

$$I_T\ddot{\theta}_y - \dot{\phi}I_p\dot{\theta}_x - \ddot{\phi}\{\theta_y I_p/2 + \tau(I_p - I_T)\cos(\phi + \beta_\tau)\} - \dot{\phi}^2\tau(I_p - I_T)\sin(\phi + \beta_\tau) = -M_x \quad (43)$$

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where $\phi = \varphi + \omega t$, and M_x, M_y are the resultant (external plus internal) components of the moment applied to the rigid body about its mass center. The comparable equations of motion from Shen's analysis (equations (28) and (29), Appendix 2) are

$$I_T\ddot{\theta}_X + \dot{\phi}I_p\dot{\theta}_y + \ddot{\phi}I_T\tau \sin(\phi + \beta_\tau) - \dot{\phi}^2\tau(I_p - I_T)\cos(\phi + \beta_\tau) = M_y$$

$$I_T\ddot{\theta}_y - \dot{\phi}I_p\dot{\theta}_x - \ddot{\phi}I_T\tau \cos(\phi + \beta_\tau) - \dot{\phi}^2\tau(I_p - I_T)\sin(\phi + \beta_\tau) = -M_x \qquad (44)$$

My attempts to resolve these two different results yields yet another result, which agrees with neither of the preceding. My derivation follows.

The rotational equations of motion for a rigid body about its mass center are customarily stated in terms of a body-fixed coordinate system as opposed to the "nonspinning" coordinate system used in the results preceding. This derivation will begin with these generally accepted body-fixed equations and a coordinate transformation will be executed to yield comparable nonspinning results.

The body-fixed coordinate system whose origin coincides with the mass center of a component rigid body will be denoted as the $\bar{x}$, $\bar{y}$, $\bar{z}$ system. The component definitions of a vector V in the body-fixed $\bar{x}$, $\bar{y}$, $\bar{z}$ and nonspinning x, y, z coordinate systems are identified, respectively, by $(V)_{\bar{i}}$ and $(V)_i$. The coordinate transformation between these two systems is defined by

$$(V)_{\bar{i}} = [A(\phi)](V)_i,\ [A(\phi)] = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (45)$$

where $c\phi = \cos\phi$, $s\phi = \sin\phi$. The derivative operations with respect to time in these and the inertial X, Y, Z coordinate system are denoted by

$$\dot{V} = \frac{dV}{dt}\Big|_{X,Y,Z},\ \hat{\dot{V}} = \frac{dV}{dt}\Big|_{x,y,z},\ \bar{\dot{V}} = \frac{dV}{dt}\Big|_{\bar{x},\bar{y},\bar{z}} \qquad (46)$$

If the angular velocity of the $\bar{x}$, $\bar{y}$, $\bar{z}$ system (the component rigid body) relative to the inertial X,Y,Z system is denoted by Ω, the derivative operations in equation (46) are related by the following

$$\dot{V} = \bar{\dot{V}} + \Omega x V,\ \hat{\dot{V}} = \bar{\dot{V}} + k\dot{\phi}xV; \qquad (47)$$

hence,

$$\dot{\Omega} = \bar{\dot{\Omega}} = \hat{\dot{\Omega}} - k\dot{\phi}x\Omega \qquad (48)$$

The equations of motion for a rigid body about its mass center may be stated in the $\bar{x}$, $\bar{y}$, $\bar{z}$ system by

$$[J_{\bar{i}}](\dot{\Omega})_{\bar{i}} + [(\Omega)_{\bar{i}}][J_{\bar{i}}](\Omega)_{\bar{i}} = (M)_{\bar{i}} \qquad (49)$$

where $[J_{\bar{i}}]$ is the inertia matrix for the rigid body in the $\bar{x}$, $\bar{y}$, $\bar{z}$ system, and M is the resultant moment vector about the mass center of the body. The notation $[(V)_{\bar{i}}]$ in equation (49) implies

$$[(V)_{\bar{i}}] = \begin{bmatrix} 0 & -V_{\bar{z}} & V_{\bar{y}} \\ V_{\bar{z}} & 0 & -V_{\bar{x}} \\ -V_y & V_{\bar{x}} & 0 \end{bmatrix}$$

and performs the equivalent of the vector-cross-product operation, i.e., $(AxB)_T = [(a)_{\bar{i}}](B)_{\bar{i}}$. From equations (45) and (48), the statement of equation (49) in the nonspinning x, y, z system is

$$[J_i](\hat{\dot{\Omega}} - k\dot{\phi}x\Omega)_i + [(\Omega)_i][J_i](\Omega)_i] = (M)_i \qquad (50)$$

where

$$[J_i] = [A(\phi)]^T[J_{\bar{i}}][A(\phi)] \qquad (51)$$

The required component equations from equation (50) are obtained by assuming that the rigid body is nominally symmetric, but canted relative to the z axis (i.e., $I_{xx} = I_{yy} = I_T$, $I_{zz} = I_p$, $I_{xy} = 0$), with the result

$$I_T(\dot{\Omega}_x + \dot{\phi}\Omega_y) - I_{xz}\dot{\Omega}_z - \Omega_z(I_T\Omega_y - I_{yz}\Omega_z) = M_x$$
$$+ \Omega_y(-I_{zx}\Omega_x - I_{zy}\Omega_y + I_p\Omega_z)$$
$$I_T(\dot{\Omega}_y - \dot{\phi}\Omega_x) - I_{yz}\dot{\Omega}_z + \Omega_z(I_T\Omega_x - I_{xz}\Omega_z) = M_y$$
$$- \Omega_x(-I_{zx}\Omega_x - I_{zy}\Omega_y + I_p\Omega_z) \qquad (52)$$

The components of Ω are related to the Euler angles by

$$\Omega_x = -\dot{\theta}_y \cos\theta_x,\ \Omega_y = \dot{\theta}_x,\ \Omega_z = \dot{\phi} - \dot{\theta}_y \sin\theta_x \qquad (53)$$

Substituting from equation (53) into equation (52), and retaining only those nonlinear terms which contain $\dot{\phi}$ yields

$$I_T\ddot{\theta}_x + \dot{\phi}I_p\dot{\theta}_y + \ddot{\phi}\tau(I_p - I_T)\sin(\phi + \beta_\tau) - \dot{\phi}^2\tau(I_p - I_T)\cos(\phi + \beta_\tau) = M_y \qquad (54)$$

$$I_T\ddot{\theta}_y - \dot{\phi}I_p\dot{\theta}_x - \ddot{\phi}\tau(I_p - I_T)\cos(\phi + \beta_\tau) - \dot{\phi}^2\tau(I_p - I_T)\sin(\phi + \beta_\tau) = -M_x$$

where

$$I_{yz} = \tau(I_p - I_T)\sin(\phi + \beta_\tau) \qquad (55)$$
$$I_{xz} = \tau(I_p - I_T)\cos(\phi + \beta_\tau)$$

The moment equations (54) are stated in the nonspinning x, y, z system; however, the flexibility matrix definition of equations (25) through (28) defines the internal reaction moments in a reference coordinate system, which is denoted here as the $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ system where $\tilde{z}$ coincides with the "rigid rotor axis of main bearings" illustrated in Fig. 13. The coordinate transformation relating the components of a vector V in the x, y, z and $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ coordinate systems is

$$\begin{matrix} V_x \\ V_y \\ V_z \end{matrix} = \begin{matrix} 1 & 0 & -\hat{\theta}_x \\ 0 & 1 & -\hat{\theta}_y \\ \hat{\theta}_x & \theta_y & 1 \end{matrix} \begin{matrix} V_{\tilde{x}} \\ V_{\tilde{y}} \\ V_{\tilde{z}} \end{matrix} \qquad (56)$$

Transformation of the governing equations to the $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ system yields

$$I_T\ddot{\theta}_x + \dot{\phi}I_p\dot{\theta}_y + \ddot{\phi}\{\tau(I_p - I_T)\sin(\phi + \beta_\tau) + \hat{\theta}_y I_p\} - \dot{\phi}^2\tau(I_p - I_T)\cos(\phi + \beta_\tau) = M_{\tilde{y}} \qquad (57)$$

$$I_T\ddot{\theta}_y - \dot{\phi}I_p\dot{\theta}_x - \ddot{\phi}\{\tau(I_p - I_T)\cos(\phi + \beta_\tau) + \hat{\theta}_x I_p\} - \dot{\phi}^2\tau(I_p - I_T)\sin(\phi + \beta_\tau) = -M_{\tilde{x}}$$

All nonlinear terms in θ_x, θ_y, $\hat{\theta}_x$, $\hat{\theta}_y$ and their derivatives were dropped in obtaining equation (57). The moment components $M_{\tilde{x}}$, $M_{\tilde{y}}$ in equation (57) are defined in the same (reference undeflected) coordinate system for which equations (25) through (28) are stated. The left-hand sides of equations (54) and (57) differ only in the $\ddot{\phi}$ coefficients.

As noted earlier, the governing equations of motion (57) do not coincide with either Shen's [1] or Kirk and Gunter's results. I feel that it is necessary to include the coordinate transformation of equation (56) in the derivation. However, if this step is removed, the governing equations reduce to (54), which also fails to coincide with either equation (43) or (45).

After Shen [1], the governing equation for ϕ is obtained via the following steps:

(*a*) A moment-of-momentum equation is stated for each rigid body with moments taken about the elastic axis of the rotor (instead of the mass center).

(*b*) The z component equation of this equation is summed over all n component rigid bodies.

The resultant equation is

$$\ddot{\phi}\, \Sigma(I_{Ti} + m_i e_i^2) = M_c + \Sigma\, m_i e_i \{\ddot{y}_i c(\phi + \phi_i) - \ddot{x}_i s(\phi + \phi_i)\}$$
$$+ \Sigma \tau_i (I_{pi} - I_{Ti})\{\ddot{\theta}_{yi} s(\phi + \beta_{Ti}) - \ddot{\theta}_{xi} c(\phi + \beta_{Ti})\} \quad (58)$$

where $s(\phi + \phi_i) = \sin(\phi + \phi_i)$, etc. This equation couples the transverse (bending) motion to the spin axis rotation. The series of terms on the right-hand side of equation (58) are not included in the model developed by Kirk and Gunter, and the spin acceleration is simply specified. For the cases examined (rapid acceleration with comparatively small bending deflections) this is entirely proper. However, cases arise for which these terms are quite important, e.g., a slow power-limited transition through a critical speed. If the terms are included, one notes that the derivatives $\ddot{\phi}$, $\ddot{x}_i$, $\ddot{y}_i$, $\ddot{\theta}_{xi}$, $\ddot{\theta}_{yi}$ are all coupled via time-varying terms; hence, an exact solution of the governing equations of motion (including equation (58)) requires the repeated solution of a very large ($4n + 1$) number of simultaneous equations.

The solution approach indicated in equations (5) through (8) for the bearing accelerations might account for some of the numerical instabilities encountered by Kirk and Gunter. To clarify this statement, consider the following restatement of equation (5)

$$(b - a) m_{j2} \hat{\ddot{x}}_{j2} = \Sigma \tilde{M}_{yi} + \Sigma(\ell_i - a)\tilde{f}_{xi}$$
$$- \Sigma(\ell_i - a)\ddot{x}_i m_i$$
$$- \Sigma\{I_{Ti}\ddot{\theta}_{yi} + \dot{\phi} I_{pi}\dot{\theta}_{xi} - \ddot{\phi}\tau(I_p - I_T)c(\phi + \beta_{Ti})\} \quad (59)$$

where M_{yi} and f_{xi} are components of the *external* forces and moments applied to the ith component rigid body. The definition of the desired acceleration $\ddot{x}_{j2}$ as a linear sum which includes the acceleration terms $\ddot{x}_i$, $\ddot{\theta}_{yi}$ is an inherently "noisy" procedure. By comparison, equation (29) defines the required acceleration as a summation of lower order derivatives and the external forces applied to the body.

In the first two paragraphs of the section entitled, "Recommendations and Conclusions," Kirk and Gunter provide a lucid and careful explanation of the inherent difficulties (excessive dimensionality and numerical instability) which are encountered in a direct lumped-parameter formulation for the solution of distributed parameter systems. The classical resolution of this difficulty for linear systems is the use of modal coordinates, and modal formulations have recently been employed with good success by several authors [2],[3] [11], [12] for the transient analysis of flexible-rotor systems.

Additional References

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12 J. W. Lund, "Modal Response of a Flexible Rotor in Fluid-Film Bearings," Paper No. 73-DET-103, ASME Design Engineering Technology Conference, Cincinnati, Ohio, Sept., 1973.

[3] Numbers in brackets designate Additional References at end of discussion.

Author's Closure

The authors are grateful for the detailed discussion and derivation furnished by Dr. Childs. The analysis presented in this paper is not an extension of Shen's work but rather a derivation of the equations of motion by energy principles to give a more exact dynamic model. The objective as stated in the description of the analytical model was to present the equations of motion for a torsionally rigid rotor-bearing system. The equations were derived considering an Eulerian coordinate system to describe the disk motion. The assumption of small angular motion allows the kinetic energy of the disk to be expressed as a function of the spin rate, skew angle, and the θ_x, θ_y angular displacements. The derivation is presented here to clarify the equations as presented in the Appendix.

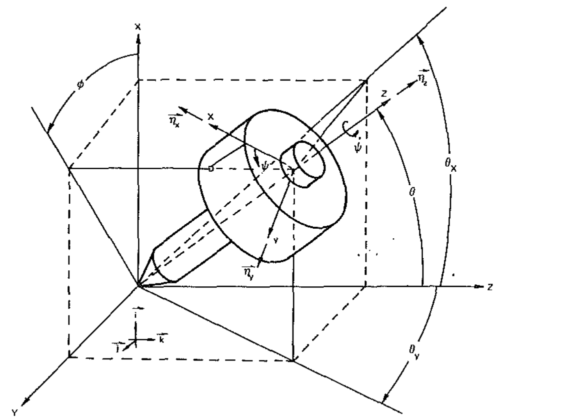

Fig. 14 Typical rotor station showing Eulerian angles and the angular deflection θ_x and θ_y

Consider a single disk as shown in Fig. 14. Considering an Eulerian coordinate system the angular velocity of the disk referenced to the body-fixed coordinate system may be expressed (6) as follows:

$$\boldsymbol{\Omega} = (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\eta_x + (\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)\eta_y + (\dot{\varphi}\cos\theta + \dot{\psi})\eta_z \quad (60)$$

The kinetic energy of the disk may be expressed by the sum of the translational and rotational energies as follows (no cross products of inertia).

$$T_1 = 1/2 m \dot{x}_i \dot{x}_j \delta_{ij} \quad (61)$$

$$T_2 = 1/2 I_{ij}\Omega_i\Omega_j\delta_{ij} \quad (62)$$

Using equation (60) in equation (62) gives

$$T_2 = 1/2 I_T(\boldsymbol{\Omega}\cdot\eta_x)^2 + 1/2 I_T(\boldsymbol{\Omega}\cdot\eta_y)^2 + 1/2 I_P(\boldsymbol{\Omega}\cdot\eta_z)^2$$
$$= 1/2 I_T(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + 1/2 I_P(\dot{\psi} + \dot{\varphi}\cos\theta)^2 \quad (63)$$

For the assumption of small deflections and angles the following expressions are valid and relate the Eulerian angles to the small angles θ_x, θ_y.

$$\omega_z = \dot{\varphi} + \dot{\psi}$$
$$\theta_x = \theta\cos\varphi \quad (64)$$
$$\theta_y = \theta\sin\varphi$$

Inserting equation (64) into equation (63) results in

$$T_2 = 1/2 I_T(\dot{\theta}_x^2 + \dot{\theta}_x^2) + 1/2 I_P(\omega^2 + \omega(\dot{\theta}_x\theta_y - \dot{\theta}_y\theta_x)) \quad (65)$$

The total energy is then expressed by

$$T = T_1 + T_2 = 1/2 m\{(\dot{x} - e\omega\sin(\omega t))^2 + (\dot{y} + e\omega\cos(\omega t))^2\}$$
$$+ 1/2 I_T(\dot{\theta}_x + \dot{\theta}_y^2) + 1/2 I_P\omega^2 + 1/2 I_P\omega(\dot{\theta}_x\theta_y - \dot{\theta}_y\theta_x) \quad (66)$$

Applying Lagranges equation for the θ_x, θ_y coordinates results in the following equations.

$$C_x = I_T\ddot{\theta}_y - \omega I_P\dot{\theta}_x - 1/2 I_P\dot{\omega}\theta_x \quad (67)$$

$$-C_y = I_T\ddot{\theta}_x + \omega I_P\dot{\theta}_y + 1/2 I_P\dot{\omega}\theta_y \quad (68)$$

These equations which express the moments in the fixed x-y coordinate system indicate a coupling between the spin acceleration, $\dot{\omega}$, normal to the disk and the absolute angular deflection of the disk. If the acceleration is not considered then the gyro effects are as normally considered for steady state rotor dynamics applications.

At this point it is desired to include the effects of a small skew of the disk relative to the spin axis as shown in Fig. 15. The skew, τ, is shown to be positive in the direction opposite the angle θ_x at the time $\omega t = 0.0$. The energy expression for the disk may now be expressed including the skew τ by the following procedure. The angular deflection of the axis of the disk is given by

$$\theta_{x0} = \theta_x - \tau\cos(\omega t) \quad (69)$$

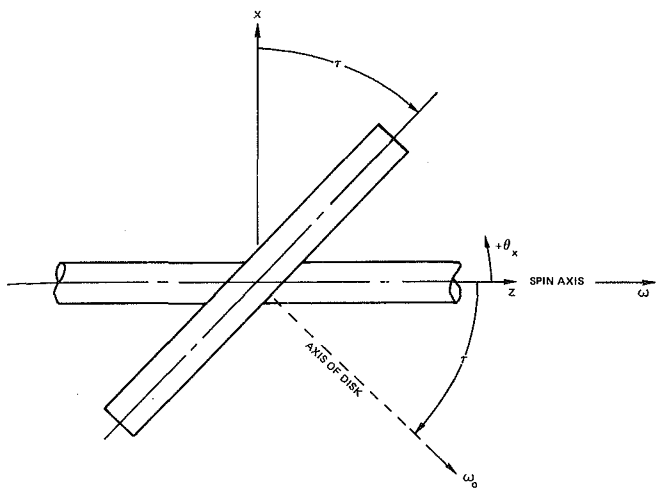

Fig. 15 Rotor station showing the skew angle at the time $\omega t = 0$

$$\theta_{y0} = \theta_y - \tau \sin(\omega t) \quad (70)$$

and hence,

$$\dot{\theta}_{x0} = \dot{\theta}_x + \tau\omega \sin(\omega t) \quad (71)$$

$$\dot{\theta}_{y0} = \dot{\theta}_y - \tau\omega \cos(\omega t) \quad (72)$$

The energy expression for the disk is thus expressed by

$$T_2 = 1/2 I_T(\dot{\theta}_{x0}{}^2 + \dot{\theta}_{y0}{}^2) + 1/2 I_P[\omega_0{}^2 + \omega_0(\dot{\theta}_{x0}\theta_{y0} - \dot{\theta}_{y0}\theta_{x0})] \quad (73)$$

The angular velocity of the disk may be expressed as a function of the spin rate ω, skew, and the angles θ_x, θ_y by an algebraic manipulation of the transformation equations (direction cosines). The result may be expressed as follows:

$$\omega_0{}^2 - \omega^2 - \tau\omega^2(\theta_x \cos(\omega t) + \theta_y \sin(\omega t)) + \tau\omega(\dot{\theta}_y \cos(\omega t) - \dot{\theta}_x \sin(\omega t)) \quad (74)$$

Inserting equation (69), (70), (71), (72), and (74) into equation (73) yields the following expression.

$$T_2 = \frac{\omega^2}{2}[I_P - (I_P - I_T)\tau^2] + 1/2 I_P\omega(\dot{\theta}_x\theta_y - \dot{\theta}_y\theta_x) + 1/2 I_T(\dot{\theta}_x{}^2 + \dot{\theta}_y{}^2) + \tau\omega(I_P - I_T)(\dot{\theta}_y \cos(\omega t) - \dot{\theta}_x \sin(\omega t)) \quad (75)$$

The appropriate moment equations may then be written by Lagranges equation as follows:

$$C_x = I_T\ddot{\theta}_y - I_P\omega\dot{\theta}_x - 1/2 I_P\dot{\omega}\theta_x + \tau\dot{\omega}(I_P - I_T)\cos(\omega t) - \tau\omega^2(I_P - I_T)\sin(\omega t) \quad (76)$$

$$-C_y = I_T\ddot{\theta}_x + I_P\omega\dot{\theta}_y + 1/2 I_P\dot{\omega}\theta_y - \tau\dot{\omega}(I_P - I_T)\sin\omega t - \tau\omega^2(I_P - I_T)\cos(\omega t) \quad (77)$$

The above equations are in agreement with Dr. Childs' derivation with the only exception being the term coupling the spin acceleration to the rotation of the disk (for the case of no skew). The terms relating the skewed disk effects are in full agreement. It should be noted that the expressions in equation (53) need only be written as

$$\Omega_x = -\dot{\theta}_y, \Omega_y = \dot{\theta}_x, \Omega_z = \dot{\phi} \quad (53a)$$

in order to obtain the results shown in equation (54). The additional rotation transformation resulting in equation (57) should not be necessary to obtain the equations for use in the author's analysis. The equations in the appendix were expressed in terms of the absolute angular rotations of the disk and not the rotation relative to the rigid rotor axis. The nomenclature of Fig. 13 could have caused the misinterpretation. However it is interesting to note that Dr. Childs does obtain a coupling term when the additional rotation is allowed. It is the authors' opinion that if the term I_{xz}, I_{yz} were to properly account for the deflection θ_x, θ_y (as Dr. Childs did for the skew) then the equations expressed in the fixed X, Y, Z coordinate system would include the term coupling the spin acceleration normal to the disk and the rotations θ_x, θ_y.

The equation for the torque required may be derived from the energy equation and results in

$$M_C = I_P\dot{\omega}(I_P - I_T)\tau^2\dot{\omega} + 1/2 I_P(\ddot{\theta}_x\theta_y - \ddot{\theta}_y\theta_x) + \tau(I_P - I_T)x (\ddot{\theta}_y \cos(\omega t) - \ddot{\theta}_x \sin(\omega t)) + me^2\dot{\omega} - me(\ddot{x}\sin(\omega t) - \ddot{y}\cos(\omega t)) \quad (78)$$

The torque equation is then seen to be even more complicated than the one proposed by Dr. Childs. The general rotor dynamics model should indeed include the torsional flexibility. There is however a large class of machinery which can be simulated correctly while neglecting the torsional flexibility. This assumption permits a great savings in computation time.