Discussion

Wm. J. Carter.[5] Velocity and acceleration analysis of ordinary four-bar mechanisms by the complex number vector method is not new, as the authors state. The writer has used this method for velocity and acceleration analysis and also for the determination of higher time derivatives of motion. The writer shares the author's opinions about the advantages and disadvantages of this method. The wider usage of digital computing equipment should make this method more attractive.

The authors are to be commended for the extension of the complex number vector method to the analysis of direct contact mechanisms. This feature is new so far as the writer is aware.

It is of interest to note that the author's Equation [17] may be established from the known properties of the pitch point P. Let $\overline{DK}$ and $\overline{OH}$ be the lengths of the perpendiculars from D and O respectively to the line N-N. It may be seen that

$$\frac{\overline{OH}}{\overline{DK}} = \frac{r_2 \cos(\theta_2 - \theta_T)}{r_4 \cos(\theta_4 - \theta_T)} = \frac{\overline{OP}}{\overline{DP}} = \frac{\dot\theta_4}{\dot\theta_2} \quad [28]$$

and

$$\dot\theta_4 = \dot\theta_2 \frac{\overline{OP}}{\overline{DP}} = \frac{r_2 \cos(\theta_2 - \theta_T)}{r_4 \cos(\theta_4 - \theta_T)}\, \dot\theta_2 \quad [29]$$

which is the same as the author's Equation [17]. One might now obtain an expression for $\ddot\theta_4$ by differentiation of Equation [17] with respect to time. This would, of course, lead to the same result as that given by the authors, although the physical interpretation would now involve the time derivatives of $\overline{DP}$ and $\overline{OP}$. Koenig[6] has used this method for the analysis of the ordinary four-bar linkage.

Allen H. Candee, in his discussion of a recent paper by Freudenstein[7] has analyzed the direct contact problem by a method which is slightly different from that of the authors or the method of Koenig.

It should be noted that the vector, $r_3 = 0$ in Fig. 6, may not be considered as forming an equivalent four-bar linkage with r_2 and r_4 in the normal sense. If such were the case the crossing of T-T and $\overline{DO}$ would be the pitch point which is obviously incorrect.

Ferdinand Freudenstein.[8] The introduction of complex variable techniques in kinematics in 1940 by S. Sh. Blokh came about as a result of investigations in kinematic synthesis, which field constitutes the major portion of modern kinematics. Up to the present there has been no major effort involving complex variables in mechanisms synthesis work in this country, although the potentialities of the method are great. To the extent that the authors, while concerned with kinematic analysis, focus attention on the use of complex variables, they have performed a service to the profession.

The basic development (i.e., the Introduction) represents no new ideas or principles as is evident from a perusal of pp. 189–195 of R. Beyer's "Kinematische Getriebesynthese" (authors' ref. 1)—as the authors well know—but the derivations using complex variables, which follow, have not previously appeared in Western European or American technical literature as far as is known to this discusser. Some of the velocity equations can be read off at sight by considering the equivalent four-bar linkage and using the well-known theorem relating the angular velocity ratio to the ratio of the perpendiculars drawn from the cranks to intersect the connecting link. Considerable relevant work along the same lines can be found in the writings of S. Sh. Blokh and these have been reviewed in the *Applied Mechanics Reviews*. Some specific comments follow:

(*a*) The description of references (1) and (3) leaves room for improvement. Beyer (1) describes an application of complex variables to the synthesis of four-bar linkages having prescribed values of the angular velocities and angular accelerations of the links. Rosenauer (3) describes an extension of this technique to the synthesis of linkages having prescribed extreme values of the angular velocity ratio of driving and driven links.

(*b*) In using the equations developed by the authors, it may be found desirable to summarize the equations in easy-to-use tabular form. This would permit the use of modern computational facilities.

(*c*) A method used by S. Sh. Blokh and more recently also by K. H. Sieker utilizes complex conjugates to eliminate unwanted unknowns. In a four-bar linkage, for instance, usually only the displacement, θ_2, of the driving crank and the lengths of the links are known; in deriving the angular velocity and acceleration of the driven link it would be convenient, therefore, if no quantities other than these were to enter the equations. This can be done as follows:

Let $r_1 = OC$, Fig. 4. Chain closure is expressed, therefore, by the equation

$$r_2 e^{i\theta_2} + r_3 e^{i\theta_3} + r_4 e^{i\theta_4} = r_1$$

We can write in addition the complex conjugate form of this equation which applies to the mechanism reflected about the fixed link, OC

$$r_2 e^{-i\theta_2} + r_3 e^{-i\theta_3} + r_4 e^{-i\theta_4} = r_1$$

We can eliminate "unwanted" θ_3 from these equations, obtaining

$$\left(\frac{r_1^2 + r_2^2 - r_3^2 + r_4^2}{r_2 r_4}\right) - \left(\frac{r_1}{r_4}\right)$$

$$(e^{i\theta_2} + e^{-i\theta_2}) - \frac{r_1}{r_2}(e^{i\theta_4} + e^{-i\theta_4}) + (e^{i(\theta_2-\theta_4)} + e^{-i(\theta_2-\theta_4)}) = 0$$

Upon differentiation of this equation, it can be shown that

$$\dot\theta_4 = \dot\theta_2 \frac{r_2}{r_4} \frac{r_4 \sin(\theta_2 - \theta_4) - r_1 \sin\theta_2}{r_2 \sin(\theta_2 - \theta_4) + r_1 \sin\theta_4}$$

$$\ddot\theta_4 = \frac{\ddot\theta_2 \dot\theta_4}{\dot\theta_2} - \frac{r_1 r_2 \dot\theta_2^2 \cos\theta_2 + r_1 r_4 \dot\theta_4^2 \cos\theta_4 - r_2 r_4 (\dot\theta_2 - \dot\theta_4)^2 \cos(\theta_2 - \theta_4)}{r_2 r_4 \sin(\theta_2 - \theta_4) + r_1 r_4 \sin\theta_4}$$

of course these equations still include θ_4.

These equations should be compared with Equations [8] and [12] of the authors. This discusser respectfully suggests that the authors consider the possibilities of this technique in their future work, especially in the field of kinematic synthesis.

A. S. Hall, Jr.[9] The increasing availability of digital computing equipment makes it profitable to re-examine our ways of doing things in many fields of engineering. We have barely touched upon the possibilities for using such equipment in kinematic analysis and synthesis. The authors have performed

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[6] "A Uniform Method for Determining Angular Accelerations," by L. R. Koenig, *Journal of Applied Mechanics*, Trans. ASME, vol. 68, 1946, pp. A-41–44.

[7] "On the Maximum and Minimum Velocities and the Accelerations in Four-Link Mechanisms," by F. Freudenstein, Trans. ASME, vol. 78, 1956, pp. 779–787.

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a service in spelling out an approach to writing velocity and acceleration equations which can easily be programmed for automatic computation. If we are to go in for automatic computation of velocities and accelerations, then we may as well include *position*. The authors have assumed position data known in all their example problems, perhaps feeling this problem to be trivial.

In connection with the example on direct contact mechanisms perhaps a word might be added on the subject of "equivalent linkages." It is implied in the paper (as well as in most textbooks) that there is *one* equivalent linkage. Actually there exists an infinite number of four-bar linkages equivalent, insofar as instantaneous velocities and accelerations are concerned, to a given direct contact mechanism. To form such an equivalent four-bar we could choose, quite arbitrarily, a point M on body 2 (see Fig. 6 of the paper) to be the location of the pin joint between 2 and the connecting-rod of the equivalent four-bar. Then, using the Euler-Savary equation, we could solve for the location N of the center of curvature of the path which M_2 traces in the motion of 2 relative to 4. If we then let N be the location of the pin joint between 4 and the connecting-rod we shall have our equivalent mechanism.

The authors have actually found one such equivalent mechanism in their example problem, Fig. 6. It is formed by pinning a connecting-rod between B on body 4 and G on body 2.

This somewhat larger view of the equivalent linkage does make it possible to reduce the direct contact problem to the four-bar problem, even for the case in which one of the direct contact bodies has infinite radius of curvature.

R. T. Hinkle.[10] The authors state that the theory presented in this paper for the determination of velocities and accelerations is not new. However, they have generalized it by extending it to include direct contact and complex mechanisms. Since other analytical methods for determining acceleration become so cumbersome when applied to complex mechanisms as to be almost useless, this paper is a major contribution.

The graphical method will probably remain in wide use for velocity and acceleration analysis, but when synthesis is considered, some form of analytical approach is needed. It may be that this paper will have its greatest value in the development of this field. At the present time, one investigator is applying the theory presented here for the determination of inertia stresses in mechanisms.

Authors' Closure

Professor Carter points out that Equation [17] which applies to direct contact mechanisms may also be established from the known properties of the pitch point P in Fig. 6. Further, he states that this equation may be differentiated to obtain an expression for $\ddot{\theta}_4$, a method which Koenig has used in an analysis of the ordinary four-bar linkage. This is true. However, if we were to follow this suggestion we would find that the resulting expression obtained for $\ddot{\theta}_4$ would contain the quantities $(\dot{\theta}_2 - \dot{\theta}_T)$ and $(\dot{\theta}_4 - \dot{\theta}_T)$. The values of both of these quantities are unknown of course because $\dot{\theta}_T$ is not known. Thus it appears that such a procedure would not provide a solution for $\ddot{\theta}_4$.

In commenting on $r_3 = 0$ in Fig. 6, Professor Carter warns that this is not to be regarded as link 3 in an equivalent four-bar linkage. True, the authors have not used r_3 as such and appreciate that this discusser's comment should help further in preventing any misconception of this kind by the reader.

Professor Freudenstein states that some of the velocity equations can be read off at sight by considering the equivalent four-bar linkage and using the well-known theorem relating the angular velocity ratio to the ratio of the perpendiculars drawn from the cranks to intersect the connecting link. This is the same comment as is made by Professor Carter. The authors have been aware of this method, but throughout the paper the approach to both velocity and acceleration analysis has been by means of summation of relative velocity and acceleration vectors. This has been done to demonstrate the application of the general expressions presented in the Introduction.

The use of complex conjugates as pointed out by Professor Freudenstein for eliminating unknown θ_3 in the four-bar linkage is indeed a further gain even though his equations for $\dot{\theta}_4$ and $\ddot{\theta}_4$ contain more terms than the authors' Equations [8] and [12].

The comments by Professor Hall as to the desirability of expressing position in a form which would be convenient for computational purposes, as the authors have done for velocity and acceleration, are well placed. The authors' equations give the velocity and acceleration of any link in the mechanism in terms of the velocity and acceleration of the driving link alone, and it has been shown how the unknown velocities and accelerations of the other links in the kinematic chain can be eliminated. However, using exponentials, no means has been found to eliminate a sufficient number of unknown angles so that the position of any one link can be expressed in terms of the lengths of the links and the angular position of the driver alone. The method of using complex conjugates, as cited by Professor Freudenstein in his discussion, eliminates "unwanted" θ_3 from the position vector equation for the four-bar linkage. But, as he mentions, unwanted θ_4 still remains. The method commonly used in finding the angular positions of the links has been to use ordinary trigonometry. We know for example that in relating angles and lengths in the four-bar linkage it has been customary to work with the diagonal on the linkage. This is the method the authors have used after devoting considerable effort to find more efficient methods.

The comment by Professor Hall that there is an infinite number of equivalent four-bar linkages for a direct-contact mechanism should be emphasized indeed. Professor Hall has explained how we can obtain equivalent four-bar linkages where all links are of finite length even for cams having flat-faced followers. Thus in addition to the solution of velocities and accelerations for direct-contact mechanisms as expressed by Equations [17] and [21], any direct-contact mechanism may be analyzed using the four-bar-linkage solution as expressed by Equations [7], [8], and [12].

In conclusion, the authors wish to thank Professors Carter, Freudenstein, Hall, and Hinkle for their contributions and for the trouble they have taken in discussing the paper.

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