author finds $\eta = 0.378$.

In any case, the Galerkin method solution seems to give a better accuracy, even though at the cost of a greater computation effort. Furthermore, it may be readily seen that, contrary to the discusser's claim, there is no difficulty in carrying out the optimization procedure.

The optimum values of N are obtained by solving the equation:

$$(\sinh 2N - 6N)\left(1 - C\frac{\tan h^2 N}{1 + \tanh^2 N}\right)$$

$$+ 6N\frac{dC}{dN}\frac{\tanh^3 N \cosh^2 N}{1 + \tanh^2 N} + 12NC\frac{\tanh^2 N}{(1 + \tanh^2 N)^2} = 0$$

where:

$$\frac{dC}{dN} = \frac{\alpha' C^2 + \beta' C + \gamma'}{2\alpha c + \beta}$$

The primes denote differentiation with respect to N. While the details of the calculation are not reported here, the results are shown plotted in Fig. 1 along with the perturbation solution [2].

The Numerical Prediction of the Turbulent Flow and Heat Transfer in the Entrance Region of a Parallel Plate Duct[1]

T. Cebeci,[2] R. S. Hirsh,[3] and K. C. Chang.[4] The two previous calculations of entry flow in a duct [1, 2][5] have both used rather complex turbulence models, i.e., a mixing-length model with twelve constants, and a k-ϵ model which requires six constants. The present communication describes a much simpler approach to this problem based on the essential physics of the duct flow. As such it is applicable only to this particular flow, but a similar concept has been utilized for the calculation of shock wave/boundary-layer interactions [3].

The entry flow in a duct is shown schematically in Fig. 1. There are three identifiable regions in the flow field. In region I the flow behaves as an external boundary layer interacting with an inviscid flow; this is called the "displacement-interaction" region. By region III, the character of the flow is wholly described by internal concepts; this is the "fully developed" region. Region II is a transition zone from external to internal flow.

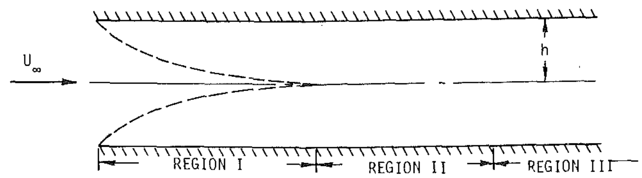

Fig. 1 Schematic representation of duct flow

In region I, we use the well-tested Cebeci-Smith eddy-viscosity model [4] for the boundary-layer-like development. This is given by

$$\epsilon^{\mathrm{I}} = \begin{cases} (0.4y)^2[1 - \exp(-y/A)]^2 \left|\dfrac{\partial u}{\partial y}\right| \gamma_{tr} & (1a) \\ 0.0168 \displaystyle\int_0^\infty (u_e - u)dy \cdot \gamma_{tr} & (1b) \end{cases}$$

In a pipe, the flow in region III can be described by the classical mixing length model of Nikuradse [5]

$$\epsilon^{\mathrm{III}} = \ell^2 \left|\frac{\partial u}{\partial y}\right| [1 - \exp(-y/A)]^2 \quad (2a)$$

$$\ell = r_0\left[0.14 - 0.08\left(1 - \frac{r}{r_0}\right)^2 - 0.06\left(1 - \frac{r}{r_0}\right)^4\right] \quad (2b)$$

where for axisymmetric flow for constant r_0

$$r = r_0 - y$$

so that

$$1 - \frac{r}{r_0} = \frac{y}{r_0}$$

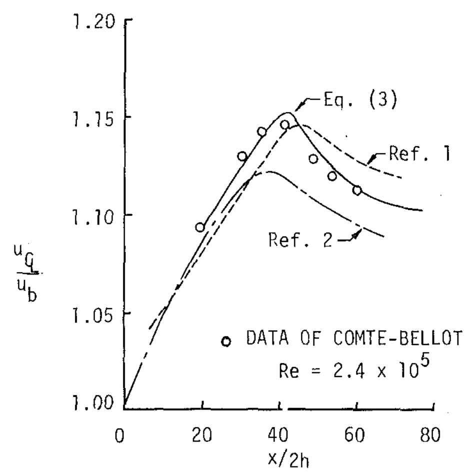

Fig. 2 Comparison of predictions for center-line velocity

To use this for two-dimensional duct flows we simply set $r_0 = h$ to get

$$1 - \frac{r}{r_0} = \frac{y}{h} \quad (2c)$$

Either one of these can be used separately to calculate the developing duct flow using the numerical procedure described below, however, the overall agreement is poor. The solution was to model the entire flow situation based on our knowledge of the physics. A composite eddy viscosity is prescribed which changes smoothly from an external model (ϵ^{I}) to an internal model (ϵ^{III}) as follows

$$\epsilon = \epsilon^{\mathrm{I}} + (\epsilon^{\mathrm{III}} - \epsilon^{\mathrm{I}})\left[1 - \exp - \left(\frac{x - x_0}{20h}\right)\right] \quad (3)$$

where x_0 is the length down the pipe where the two "external" boundary layers merge, and the denominator giving the relaxation scale was prescribed by divine intervention.

Using this model, the "nonlinear eigenvalue" procedure of Cebeci and Keller for laminar flows [6] was used to calculate the developing duct flow. For further details and heat transfer results see reference [7]. Fig. 2 shows the experimental results of Compte-Bellot for center-line velocity decay, along with the predictions of references [1, 2], and equation (3). The composite model provides strikingly improved results. The same comparison of predicted results for the displace-

[1] By A. F. Emery and F. B. Gessner, published in the Nov. 1976 issue of the JOURNAL OF HEAT TRANSFER, TRANS. ASME, Series C, Vol. 98, pp. 594–600.

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[5] Numbers in brackets designate Additional References at end of discussion.

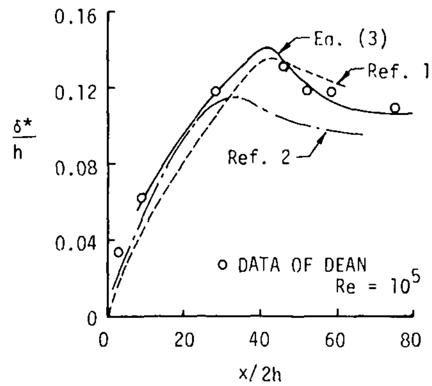

Fig. 3 Comparison of predictions for displacement thickness

ment thickness measured by Dean is shown in Fig. 3. Again, equation (3) gives the best comparison. The typical computation time, for a case using 41 points across the duct, and 35 axial stations down the duct, was less than 2 s on an IBM 370/175. Consequently, we conclude that simple models can be used to predict some complex flows by using a great deal of empiricism.

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4 Cebeci, and Bradshaw, P., *Momentum Transfer in Boundary Layers*, Hemisphere-McGraw Hill, Washington, 1977.

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Authors' Closure

The discussers have proposed a viable alternative to the length-scale model we proposed in our paper [1]. The discussers' claim that their model is "much simpler" because it is based on physical principles is open to question. In reality, the eddy viscosity model which is proposed requires the specification of nine constants (four in equation (1) with two constants implicit in the expression for γ_{tr}, three in equation (2), and two in equation (3)). The total number of constants may, in fact, be even greater than nine if x_0 in equation (3) is not identically constant, but is allowed to vary for matching purposes with experimental data.

Although it may be argued that the constants in the discussers' model are well specified on the basis of previous experience with the expressions for ϵ^{I} and ϵ^{III}, the fact still remains that all of the coefficients are empirically determined quantities. Furthermore, the transition zone model given by equation (3) is not based on "our knowledge of the physics" but, instead, is simply an empirical expression which models experimentally observed behavior. Admittedly, the length-scale model we have proposed involves the specification of twelve constants, but these constants are also used in our three-dimensional length-scale model for developing flow in rectangular ducts of arbitrary aspect ratio [8]. We thus feel that our model offers more flexibility than the one proposed by the discussers, which admittedly is restricted to developing, two-dimensional duct flow.

On the basis of the comparisons shown in Figs. 2 and 3 the discussers conclude that their model gives "strikingly improved results" and "the best comparison." These statements are based, however, on unfair comparisons made in these figures. As we have carefully noted in our paper [1], our predicted distributions for $U_{CL}(x)$ and $\delta^*(x)$ are based on assumed uniform flow at the duct inlet. The corresponding experimental distributions measured by Dean [9] and Comte-Bellot [10] are not based on this condition, however, because the flow was partially developed at the inlet of their experimental configurations. With reference to Dean's results, for example, we stated that it was necessary to match initial conditions at $x/2h = 2.2$ in order to make direct comparisons with his data. If this is done then the distribution for δ^* based on our model (shown in Fig. 3 and labelled "reference [1]") will shift 3.8 duct widths upstream. The level of agreement between our results and Dean's data is then comparable to that observed between the discussers' results and the same data. Similar comments apply for the predicted and experimental profiles shown in Fig. 2. It should also be noted that the discussers' predictions are apparently based on matched initial conditions well downstream of the duct inlet, i.e., at $x/2h = 20$ (Fig. 2) and at $x/2h = 8.3$ (Fig. 3). This simple expedient will always enhance the likelihood of good agreement between predictions and data downstream of the matching location. A preferred procedure is to base comparisons on matched conditions at the first station for which data are available (e.g., at $x/2h = 2.2$ for Dean's data), and this is the approach we took in making our comparisons.

Additional References

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