$$\theta_2 = \sin^{-1}\frac{b}{A} \qquad b < A \tag{75}$$

Substituting from equations (73) and (74) in equation (68) yields equation (27) in the text.

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DISCUSSION

P. K. C. Wang[2]

In this paper, the authors applied the statistical linearization method to study the stability of a nonlinear control system subjected to Gaussian random noise. The basic idea of statistical linearization was introduced in the early works of Booton [7, 8][3] and Kazakov [9]. Since then, extensions have been made to include composite inputs [10–14]. Here, composite equivalent gain expressions for zero-memory nonlinearities will be derived. Their applications will become evident in the subsequent discussions.

For clarity, the notations adapted by Davenport and Root [15] will be used. The input $x(t)$ to the nonlinearity $(y(t) = f[x(t)])$ is written as:

$$x(t) = A\cos\omega_0 t + n(t) \tag{76}$$

$n(t)$ is a Gaussian random process with zero mean.

The input-output cross-correlation function $R_{yx}(\tau)$ has the form:

$$R_{yx}(\tau) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \epsilon_m \cos m\omega_0\tau \,\frac{R_{nn}{}^k(\tau)}{k!}\,[h_{mk}\cdot g_{mk}] \tag{77}$$

where:

$$h_{mk} = \frac{1}{2\pi}\int_c (ju)^k F(ju) I_m(uA)\exp\left[-R_{nn}(0)u^2/2\right]du \tag{78}$$

$$g_{mk} = \frac{1}{2\pi}\int_{c'} (jv)^{k-2} I_m(vA)\exp\left[-R_{nn}(0)v^2/2\right]dv \tag{79}$$

and

$$F(ju) = \int_{-\infty}^{+\infty} f(x)\exp(-jux)dx \tag{80}$$

$I_m(uA)$ denotes mth order modified Bessel function and ϵ_m is the Neumann factor. c and c' are appropriate integration contours.

Equation (77) may be expanded as:

$$R_{yx}(\tau) = 2\sum_{m=1}^{\infty} h_{m0}\cdot g_{m0}\cos m\omega_0\tau + \sum_{k=1}^{\infty} h_{0k}\cdot g_{0k}\frac{R_{nn}{}^k(\tau)}{k!} + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\epsilon_m h_{mk}\cdot g_{mk}\cos m\omega_0\tau\,\frac{R_{nn}{}^k(\tau)}{k!} \tag{81}$$

Neglecting higher-order and cross-product terms,[4] the above equation reduces to:

$$R_{yx}(\tau) \approx 2h_{10}\cos\omega_0\tau + h_{01}R_{nn}(\tau) \tag{82}$$

Obviously, the respective equivalent gains K_{sn} and K_{ns} corresponding to the sinusoidal and random components are:

$$K_{sn} = 2h_{10}/A; \quad K_{ns} = h_{01} \tag{83}$$

K_{sn} is identical to the "equivalent admittance" given by equation (24) in this paper. It can be readily shown that K_{ns} approaches Booton's equivalent gain [7, 8] for Gaussian random inputs as the sinusoidal amplitude A diminishes. Explicit forms of K_{sn} and K_{ns} for various nonlinearities have been tabulated in [14].

In discussing the stability of closed-loop systems, the authors of this paper assumed that a noise with known variance is injected at the input of the nonlinear element and the low-pass characteristics of the plant is sufficient to filter out both the noise and other components with frequencies higher than the limit-cycle frequency. These assumptions permit considerable simplification of analysis. In stabilization of control systems by intentionally injecting an external random noise as proposed by Professor Oldenburger, the most convenient injection point (from analysis standpoint) is at the input of the nonlinear element. However, this point may not be accessible in many physical systems. Also, random noise may enter the system as an undesired disturbance. In these situations, the analysis may proceed with the aid of composite equivalent gains given by equation (83).

Consider a specific system configuration shown in Fig. 12. $D(t)$ is a Gaussian random disturbance. The variance of the random portion of the input to the nonlinear element is approximately:[5]

$$R_{nn}(0) \approx \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{G_1(j\omega)G_3(j\omega)H(j\omega)}{1 + K_{ns}G_1(j\omega)G_2(j\omega)G_3(j\omega)H(j\omega)}\right|^2 S_D(\omega)d\omega \tag{84}$$

$S_D(\omega)$ is the disturbance power spectral density.

The characteristic equation of the system is:

$$K_{sn}G_1(j\omega)G_2(j\omega)G_3(j\omega)H(j\omega) + 1 = 0 \tag{85}$$

K_{ns} and K_{sn} are related to $R_{nn}(0)$ and A by equation (83).

[2] International Business Machines Corp., San Jose, Calif.

[3] Numbers in brackets from 7 to 15 designate References at end of this Discussion.

[4] The validity of this approximation depends upon the form of the nonlinearity and the behavior of $R_{nn}(\tau)$.

[5] Restrictions must be imposed on the forms of $G_1(s)$, $G_2(s)$, $G_3(s)$, and $H(s)$ such that the random portion of $x(t)$ is nearly Gaussian and the approximations made in deriving equations (82) are valid.

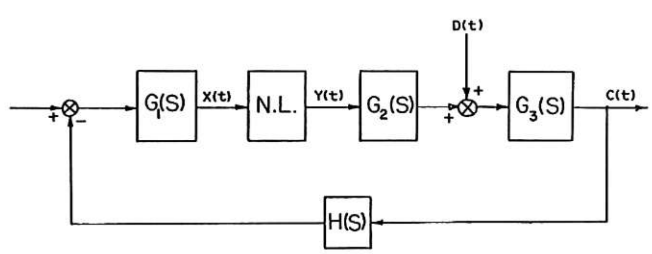

Fig. 12

They can be expressed conveniently as functions of a ratio $A/\sqrt{2}\ R_{nn}(0)$. To determine the limit-cycle amplitude, it is necessary to consider equations (83), (84), and (85) simultaneously. The analysis may be performed graphically [14].

Because of rather stringent assumptions, the application of statistical linearization method is limited to the analysis of a small class of nonlinear systems subjected to Gaussian random inputs. Formidable difficulties will be encountered when either the input statistics or the system falls outside the class considered here.

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Authors' Closure

The authors wish to thank Mr. P. K. C. Wang for his illuminating discussion which has materially increased the value of the paper. The derivation for K_{sn} in equation (83) is similar to the derivation of the equivalent admittance in the paper. However, the K_{ns} term in Equation (83), which the discusser points out is Booton's equivalent gain, is a special case of the more general definition of equivalent gain proposed in reference [1].

The method suggested by Mr. Wang for treating the noise source at a point different from the input to the nonlinearity is an obvious but useful extension.

The authors agree with Mr. Wang regarding the limitations of statistical linearization.