Theory of Supper-Quantization of Quantized Field and Its Applications

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A new treatment of quantum field theory is proposed by applying the theory of super-quantization of quantized field. For examples of its application, it is shown that the symmetrical treatment of the four potentials of electromagnetic field is justified without introduction of the indefinite metric for the scalar photon, and the physical interpretation of the regulator theory of Bose field may be given.

§ 1. Introduction

The theory of super-quantization of quantized field has been proposed by the author. In this paper, we shall present the theory in a little more refined form. A main idea of the theory consists in a super-quantization of the state vectors of a quantized field by which a quantized field is treated as something like an elementary particle having infinite degrees of freedom. What relation between the ordinary theory and the present theory may be expected? This relation may be explained in a following simple model. Let us imagine a quantum field as a motion picture projected on a screen in which we see an assembly of elementary particles of the field in motion. We know that an original film of a motion picture consists of an assembly of the separate cuts of film. In contrast with the ordinary quantized field likened to a projected picture on a screen, we may imagine that the super-quantized field corresponds to the separate cuts of the film and a motion of elementary particles of the field will be given by a sequence of cuts of the film in this model.

Both of Fermi and Bose super-quantization might be applied to the state vectors of a quantized field obeying the Fermi statistics or the Bose statistics in an ordinary sense. A faithful correspondence between the ordinary field theory and the super-quantized field theory, however, seems to be obtained only in the case of Fermi quantization of the field state vectors.

It will be seen that the theory of super-quantization is redundant for the Fermi field and the positive energy Bose field, but an application of the theory to a negative energy Bose field will offer a justification for an introduction of a negative energy Bose field into the quantum field theory.

We shall propose a new treatment of quantum electrodynamics in which the symmetrical treatment of the four potentials of electromagnetic field is justified by applying this theory, but without introduction of the indefinite metric for the scalar potential used by several authors.2)
An application to the regulator theory of Bose field also is treated. The well-known difficulties of negative energy Bose field accompanied by the theory of regulator will be solved. We see that the anti-field of the negative energy Bose field can be defined and it is expected to become a observable positive energy field just like the positron corresponding to the negative energy electron. In this point, we hope that the theory may play some positive roles in the future theory.

§ 2. Theory of super-quantization of field

1. Introductory remarks

For a sake of simplicity, we take a real scalar Bose field. The field equation and the commutation relation is given in usual manner by

\[ \Box \varphi - x^2 \varphi = 0 \]  

and

\[ [\varphi(x), \varphi(x')] = i\Delta (x - x') \]  

respectively. The notations have usual meanings. We assume that the field is enclosed in a box of unit volume with periodic boundary conditions, and expand the \( \varphi \) as follows:

\[ \varphi(x) = \sum_k (2k_0)^{-1} \{ C(k) \exp[i(k \cdot r - k_0 t)] + C^*(k) \exp[-i(k \cdot r - k_0 t)] \}. \]  

For the commutation relations of the \( C(k) \) and \( C^*(k) \), we obtain

\[ [C(k), C^*(k)] = \delta_{k,k'}, \]  

while the other commutators \( [C(k), C(k')], [C^*(k), C^*(k')] \) vanish. The Lagrangian function and the Hamiltonian density is

\[ \mathcal{L} = \frac{1}{2} \left( \left( \frac{\partial \varphi}{\partial x^\mu} \right)^2 + x^2 \varphi^2 \right) \]  

and

\[ \mathcal{H} = \frac{1}{2} \left( \left( \frac{\partial \varphi}{\partial t} \right)^2 + \left( \frac{\partial \varphi}{\partial x_i} \right)^2 + x^2 \varphi^2 \right) \]  

respectively, where \( i = 1, 2, 3 \) and \( \mu = 1, 2, 3, 4 \).

We obtain for the Hamiltonian omitting the zero point energy

\[ \mathcal{H} = \int H dv = \sum_k k_0 C^*(k) C(k). \]  

The quantity:

\[ C^*(k) C(k) = n(k) \]  

which gives the number of the Bose particles with the momentum \( k \), has the eigen values...
The constant state vector $\phi$ of the field in the Heisenberg picture is represented as follows:

$$\phi = \sum_{n(k_1), n(k_2), \ldots} a(n(k_1), n(k_2), \ldots) \phi_{n(k_1), n(k_2), \ldots} (C^*(k_1), C^*(k_2), \ldots), \quad (9)$$

where the $\phi_{n(k_1), n(k_2), \ldots}$ is the eigenfunction for a state containing $n(k_1), n(k_2), \ldots$ particles with the momentum $k_1, k_2, \ldots$ respectively. The explicit form of the $\phi$ and its Hermite conjugate $\phi^*$ are given by

$$\phi_{n(k)} (C^*(k)) = (n(k_1)!, n(k_2)!, \ldots)^{-\frac{1}{2}} C^*(k)^{n(k_1)} C^*(k_2)^{n(k_2)} \ldots, \quad (10)$$

and

$$\phi^*_{n(k)} (C(k)) = (n(k_1)!, n(k_2)!, \ldots)^{-\frac{1}{2}} C(k)^{n(k_1)} C(k_2)^{n(k_2)} \ldots, \quad (11)$$

where for a brevity, the abbreviated notations

$$n(k) = (n(k_1), n(k_2), \ldots)$$

and

$$C^*(k) = (C^*(k_1), C^*(k_2), \ldots)$$

are used. The eigenfunction $\phi$'s are normalized and orthogonal, for

$$\langle \phi_{m(k)}, \phi_{n(k)} \rangle = \delta_{m(k_1), n(k_1)} \delta_{m(k_2), n(k_2)} \ldots \quad (12)$$

The operators $C(k)$ and $C^*(k)$ have the usual meaning of an absorption and an emission operator for a particle with a momentum $k$.

2. Super-quantization

Here, we introduce the super-quantization of the quantized field which is of an analogy to Fock's second quantization method. The $q$-number state vector $\psi$ is defined by replacing the amplitude $a(n(k))$ with the quantized amplitude $a(n(k))$ in the equation (9).

The quantized amplitude $a$ may be defined by the commutation relation

$$[a(m(k_1), m(k_2), \ldots), a^*(n(k_1), n(k_2), \ldots)] = \delta_{m(k_1), n(k_2)} \delta_{m(k_1), n(k_2)} \ldots \quad (13)$$

and other commutators vanish, or for the exclusion quantization,

$$\{a(m(k_1), m(k_2), \ldots), a^*(n(k_1), n(k_2), \ldots) \} = \delta_{m(k_1), n(k_2)} \delta_{m(k_1), n(k_2)} \ldots \quad (14)$$

and other commutators

$$\{a(m(k)), a(n(k))\}, \{a^*(m(k)), a^*(n(k))\}$$

vanish.

From the reason mentioned in the introduction, we consider only the case of the exclusion commutation relation which is defined by (14).

Then, the quantized state vectors are written as


\[ \phi(C^*(k)) = \sum_{n(k_1), n(k_2), \ldots} \alpha(n(k_1), n(k_2), \ldots) \phi_{n(k_1), n(k_2), \ldots}(C^*(k_1), C^*(k_2), \ldots), \]

and its Hermitian conjugate
\[ \phi^*(C^*(k)) = \sum_{n(k_1), n(k_2), \ldots} \alpha^*(n(k_1), n(k_2), \ldots) \phi^*_{n(k_1), n(k_2), \ldots}(C^*(k_1), C^*(k_2), \ldots). \] (15')

From the relations (14) and (12), the commutation relations
\[ \{ \phi(C^*(k_1), C^*(k_2), \ldots), \phi^*(C^*(k_1)', C^*(k_2)', \ldots) \} = \delta(C^*(k_1) - C^*(k_1)') \delta(C^*(k_2) - C^*(k_2)') \ldots \] (16)

are obtained. This relations are time-independent and relativistic invariant, for the above mentioned definitions of \( \phi \).

In general, the super-quantized field operator \( \mathbf{O} \) is defined from the corresponding ordinary field operator \( \mathbf{O} \) by
\[ \mathbf{O} = (\phi, \mathbf{O}\phi) = \sum_{m(k)} \sum_{n(k)} \alpha^*(m(k)) (m(k) | \phi | n(k)) \alpha(n(k)), \] (17)

where the abbreviation as mentioned before
\[ m(k) \equiv (m(k_1), m(k_2), \ldots) \text{ and } n(k) \equiv (n(k_1), n(k_2), \ldots) \]

are used, and the \((m(k) | \phi | n(k))\) is a matrix element of the ordinary field operator given by
\[ (m(k) | \phi | n(k)) = (\phi_{m(k_1), m(k_2), \ldots} \mathbf{O} \phi_{n(k_1), n(k_2), \ldots}). \] (18)

For examples, the super-quantized operators corresponding to the absorption and emission operators \( C(k_i) \) and \( C^*(k_i) \) may be defined as follows:
\[ C(k_i) = \sum_{n(k_1), n(k_2), \ldots} n(k_i)^{1/2} \alpha^*(n(k_1), n(k_2), \ldots, n(k_i) - 1, \ldots) \times \alpha(n(k_1), n(k_2), \ldots, n(k_i), \ldots), \] (19)

and
\[ C^*(k_i) = \sum_{n(k_1), n(k_2), \ldots} (n(k_i) + 1)^{1/2} \alpha^*(n(k_1), n(k_2), \ldots, n(k_i) + 1, \ldots) \times \alpha(n(k_1), n(k_2), \ldots, n(k_i), \ldots). \] (18')

From (19) and the commutation relation (14), we have
\[ [C(k_i), C^*(k_j)] = \delta_{k_i, k_j} \sum_{n(k)} \alpha^*(n(k)) \alpha(n(k)), \] (20)

and
\[ \text{Other commutators}\ = \ 0. \]

The following relation, therefore, is easily obtained,
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\[ [\varphi(x), \varphi(x')] = i\hbar (x - x') (\varphi, \varphi), \]

where

\[ \varphi(x) = (\varphi, \varphi(x)\varphi) \]

and

\[ (\varphi, \varphi) = \sum_{n(k)} a^*(n(k)) a(n(k)). \]

The operator (23) has a simple meaning such that it gives an operator of total numbers of the fields in the assembly of the identical fields and each \( a^*(n(k_1), n(k_2), \ldots) \times a(n(k_1), n(k_2), \ldots) \) gives the numbers of the field to which the given distribution of particles \( n(k_1), n(k_2), \ldots \) belongs. According to the exclusion quantization (14), the eigenvalue of the operator \( a^* (n(k)) a(n(k)) \) is zero or one. In other words, there does not exist more than one field which has the same distribution of the particles at the same time in the assembly.

The total Hamiltonian operator of the field assembly is given by

\[ \mathcal{H} = (\varphi, \mathcal{H}_\varphi) = \sum_{n(k_1), n(k_2), \ldots} (k_0(k_1)n(k_1) + k_0(k_2)n(k_2) + \cdots) \mathcal{N}(n(k_1), n(k_2), \ldots) \]

and

\[ \mathcal{N}(n(k_1), n(k_2), \ldots) = a^*(n(k_1), n(k_2), \ldots) a(n(k_1), n(k_2), \ldots) \]

whose eigenvalue is zero or one.

The operator \( C(k_i) \) and \( C^*(k_i) \) are the corresponding one of an absorption and an emission of the particle associated with the momentum \( k_i \), because the \( C(k_i) \) gives a transition such that one field having the particle distribution \( n(k_1), n(k_2), \ldots, n(k_i) \) is destroyed and another field having the particle distribution \( n(k_1), \ldots, n(k_i) - 1, \ldots \) is created, then the numbers of particles with the momentum \( k_i \) belonging to the field decrease by one, and for the \( C^*(k_i) \), vice versa.

We denote an eigen state vector of the super-quantized field by

\[ \psi[N'(n'(k_1), n'(k_2), \ldots), N''(n''(k_1), n''(k_2), \ldots) \ldots] \]

which belongs to an eigenvalue of \( \mathcal{N}(n(k)) \) having \( N' \) for number of the field of particles distribution \( n'(k_1), n'(k_2), \ldots \) and \( N'' \) for numbers of the field of particles distribution \( n''(k_1), n''(k_2), \ldots \) etc.

For this eigenstate, we have

\[ \mathcal{H}\psi[N', N'', \ldots] = \{(k_0(k_1)n'(k_1) + k_0(k_2)n'(k_2) + \cdots)N' + (k_0(k_1)n''(k_1) + k_0(k_2)n''(k_2) + \cdots)N'' + \cdots \} \psi[N', N'', \ldots] \]

where

\[ N', N'', \ldots = 0 \text{ or } 1. \]
We shall easily see that the ordinary description will be given in terms of the super-quantized field description. If the special assembly of fields contains only one field, i.e., the number $N'$ of the field which has a definite particle distribution $n'(k_1), n'(k_2), \ldots$ is one, all of other number $N''$s are zero, this special assembly of fields is originally equivalent to the ordinary quantized field.

3. Interaction with other field

Even if we introduce an interaction of the field $\varphi$ with any other field denoted by $A(x)$, whose special properties are not needed here, we can see that the above equivalence between the super-quantized theory and the ordinary theory is conserved.

We denote the interaction energy density

$$H'(x) = g \varphi(x) A(x).$$

If the corresponding super-quantized operator is defined as follows:

$$\mathbf{H}'(x) = g \varphi(x) A(x)$$

and

$$\varphi(x) = (\phi, \varphi(x) \phi),$$

the numbers of the field $\varphi$ will not be changed by this interaction, because the operator $\varphi$ has a bilinear form of the field absorption and emission operator $a, a^*$, then in any change of the field assembly state, the total numbers of the field $\varphi$ are conserved, i.e.,

$$\sum N'(n'(k_1), n'(k_2), \ldots) = \text{const.}$$

We start from the initial condition at which we have only one field, then the above mentioned equivalence will be easily proved.

In the interaction representation of the super-quantized theory of the field $\varphi$, the equation of motion may be given by

$$i \frac{\delta \Psi[\sigma]}{\delta \sigma(x)} = \mathbf{H}'(x) \Psi[\sigma],$$

where $\mathbf{H}'$ is defined by (26). The state vector $\Psi[\sigma]$ in the equation (27) does not refer to the particle assembly state of the field $\varphi$, but the $\varphi$ field assembly state.

In general, the total numbers of the fields in the state $\Psi[\sigma]$ are defined by

$$\sum_{n'(k)} N(n'(k)) \Psi[\sigma] = \sum_{n'(k)} N'(n'(k)) \Psi[\sigma],$$

then if the initial condition

$$\sum_{n'(k)} N'(n'(k)) = 1$$

is given, we can prove that

$$[C(k_1), C^*(k_2)] \Psi[\sigma] = \delta_{k_1, k_2} \Psi[\sigma],$$

and

$$[\varphi(x), \varphi(x')] \Psi[\sigma] = i \delta(x-x') \Psi[\sigma].$$
We see, therefore, that in this case, the super-quantized operator \( \varphi(x) \) has a meaning as equivalent as that of the ordinary operator \( \varphi'(x) \) and the state vector \( \Psi[\sigma] \) describing a change of the special assembly state which includes only one field just corresponds to the ordinary state vector \( \varphi[\sigma] \) describing a change of the state of the field in terms of the particle state.

The expectation value of any operator \( \mathbf{O} \) on a space-like surface will be defined as

\[
\langle \mathbf{O} \rangle = \langle \Psi[\sigma], (\varphi, 0\varphi) \Psi[\sigma] \rangle. \tag{30}
\]

This expectation value will have a same value as the corresponding one in Tomonaga-Schwinger’s theory, if the field assembly state \( \Psi[\sigma] \)'s are restricted to one defined by (28) and (29). For an example, we can prove

\[
\langle [\varphi(x), \varphi(x')] \rangle = i\delta(x-x') \langle (\varphi, \varphi) \rangle = i\delta(x-x').
\]

Following to the equation of motion (37), the change of a state is described in manners of a destruction of one field and a subsequent creation of another field. Therefore, we see that an ordinary motion of particle assembly will be described in terms of the transitions between the states of such a special field assembly to which only one field always belongs.

It must be noted that the vacuum in the super-quantized theory is not a zero field state, but a state such that there exists only one vacuum field defined in a sense of the ordinary theory, namely for the vacuum state, we have

\[
N' = 1 \quad \text{only for } n'(k_1) = n'(k_2) = \cdots = 0,
\]

all other \( N'' = 0 \).

§ 3. Application to quantum electrodynamics

1. Introductory remarks

We follow to Gupta’s representation of quantum electrodynamics. The electromagnetic potentials are represented by

\[
A_\mu = e^{(0)}_\mu A^{(0)} + e^{(1)}_\mu A^{(1)} + e^{(2)}_\mu A^{(2)} + e^{(3)}_\mu A^{(3)} + e^{(\sigma)}_\mu A^{(\sigma)}, \tag{3}
\]

where all \( A^{(\sigma)} \)'s are scalar and Hermitian, but it should be noted that we take also \( A^{(0)} \) to be Hermitian in contrast to Gupta’s one of anti-Hermitian \( A^{(0)} \), and \( e^{(1)}_\mu, e^{(2)}_\mu, e^{(3)}_\mu, e^{(\sigma)}_\mu \) are orthogonal unit four vectors, \( e^{(i)}_\mu, i = 1, 2, 3 \), being space-like and \( e^{(0)}_\mu \) time-like. Following to Gupta, the relations

\[
e^{(1)}_\mu e^{(1)}_\nu + e^{(2)}_\mu e^{(2)}_\nu + e^{(3)}_\mu e^{(3)}_\nu - e^{(0)}_\mu e^{(0)}_\nu = \delta_{\mu\nu}
\]

are satisfied. The commutation relations are given by

\[
[A^{(i)}(x), A^{(0)}(x')] = iD(x-x'), \quad i = 1, 2, 3,
\]

\[
[A^{(0)}(x), A^{(0)}(x')] = -iD(x-x'). \tag{32}
\]
Each $A^{(a)}$ is expanded as same as (3) respectively,
\[ A^{(a)} = \sum_k \sqrt{\frac{2\pi}{k}} \left\{ C_a(k) \exp \left[ i (k \cdot r - \omega t) \right] + C_a^*(k) \exp \left[ -i (k \cdot r - \omega t) \right] \right\}, \tag{33} \]
where $a = 1, 2, 3, 0$ and $C_a^*$'s are hermite-conjugate to $C_a$'s.

From (32) and (33), the following commutation relations are obtained
\[ [C_a(k), C_a^*(k')] = \delta_{k,k'}, \]
and
\[ [C_0(k), C_0^*(k')] = -\delta_{k,k'}. \tag{34} \]

As the usual manner, the Lagrangian density is taken to be
\[ L = -\frac{1}{8\pi} \left\{ \left( \frac{\partial A_{\mu}}{\partial x_{\mu}} \right)^2 + \frac{1}{2} \left( \frac{\partial A_{\mu}}{\partial x_{\nu}} - \frac{\partial A_{\nu}}{\partial x_{\mu}} \right)^2 \right\}, \]
then, we get the Hamiltonian, omitting the zero-point energy,
\[ \tilde{H} = \sum_k \hbar \left( C_a^*(k) C_a(k) + C_a(k) C_a^*(k) + C_0(k) C_0(k) - C_0(k) C_0^*(k) \right). \tag{35} \]

We see that the radiation field consists of four scalar fields.

2. Super-quantization of electromagnetic field

Following to § 2, we introduce the quantized state vectors for each $A^{(a)}$:
\[ \phi_i(C_i^*(k)) = \sum_{n_i(k_1), n_i(k_2), \ldots} \alpha_i(n_i(k_1), n_i(k_2), \ldots) \phi_{n_i(k_1), n_i(k_2), \ldots} (C_i^*(k_1), C_i^*(k_2), \ldots), \tag{36} \]
and
\[ \phi_0(C_0(k)) = \sum_{n_0(k_1), n_0(k_2), \ldots} \alpha_0(n_0(k_1), n_0(k_2), \ldots) \phi_{n_0(k_1), n_0(k_2), \ldots} (C_0(k_1), C_0(k_2), \ldots), \tag{36'} \]
where $\phi_{n_i(k)}(C_i^*(k))$ and $\phi_{n_0(k)}(C_0(k))$ is the eigenfunction for the state containing $n_a(k_1), n_a(k_2), \ldots$-scalar photons respectively.

The quantized amplitudes are defined by
\[ \{ \alpha_a(m_a(k)), \alpha_a^*(n_a(k)) \} = \delta_{m_a, n_a} \delta_{m_a(k), n_a(k)}. \tag{37} \]

The super-quantized quantities of each $a$-field are taken to be
\[ O_a = (\phi_a, \phi_a^*). \]

For examples, we have
\[ C_i(k_2) = \sum_{n_i(k_1), n_i(k_2), \ldots} n_i(k_2)^{\frac{1}{2}} \alpha_i^*(n_i(k_1), n_i(k_2), \ldots, n_i(k_2) - 1, \ldots) \]
\[ \times \alpha_i(n_i(k_2), n_i(k_2), \ldots, n_i(k_2), \ldots), \]
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\[ C_i^*(k_i) = \sum_{n_i(k_i), n_i(k_i)'} (n_i(k_i) + 1)^{\frac{1}{2}} \alpha_i^*(n_i(k_i), n_i(k_i'), \ldots, n_i(k_i) + 1, \ldots) \]
\[ \times \alpha_i(n_i(k_i), n_i(k_i'), \ldots, n_i(k_i) + 1, \ldots), \quad (38) \]

for \( i = 1, 2, 3, \) and

\[ C_0(k_i) = \sum_{n_0(k_i), n_0(k_i)'} (n_0(k_i) + 1)^{\frac{1}{2}} \alpha_0^*(n_0(k_i), n_0(k_i), \ldots, n_0(k_i) + 1, \ldots) \]
\[ \times \alpha_0(n_0(k_i), n_0(k_i), \ldots, n_0(k_i) + 1, \ldots), \]
\[ C_0^*(k_i) = \sum_{n_0(k_i), n_0(k_i)'} n_0(k_i)^{\frac{1}{2}} \alpha_0^*(n_0(k_i), n_0(k_i), \ldots, n_0(k_i) - 1, \ldots) \]
\[ \times \alpha_0(n_0(k_i), n_0(k_i), \ldots, n_0(k_i), \ldots). \quad (39) \]

The modified supplementary conditions given by Gupta are

\[ \frac{\partial A^+_\mu}{\partial x_\mu} \phi = 0, \quad \text{or} \quad \phi^* \frac{\partial A^-_\mu}{\partial x_\mu} = 0 \]

where \( A^+_\mu \) is a positive frequency part of \( A_\mu \). In the super-quantized theory, we assume that the supplementary conditions are given by

\[ \frac{\partial A^+_\mu}{\partial x_\mu} \Phi = 0 \quad (40) \]

or

\[ \Phi^* \frac{\partial A^-_\mu}{\partial x_\mu} = 0 \quad (40') \]

where

\[ A^\pm_\mu = e^{(1)}_\mu A^{\pm(1)} + e^{(2)}_\mu A^{\pm(2)} + e^{(3)}_\mu A^{\pm(3)} + e^{(0)}_\mu A^{\pm(0)}, \quad A^{(a)} = (\phi_a, A^{(a)} \phi_a). \quad (41) \]

If we choose \( e^{(a)}_\mu \) to satisfy the following condition for a plane wave component

\[ k_{\mu} e^{(1)}_\mu = k_{\mu} e^{(2)}_\mu = 0, \quad k_{\mu} e^{(3)}_\mu = -k_{\mu} e^{(0)}_\mu, \quad (42) \]

the supplementary conditions (40), (40') will become

\[ \{ C_0(k') - C_0(k) \} \Phi = 0 \quad (43) \]

and

\[ \Phi^* \{ C_0^*(k') - C_0^*(k) \} = 0. \quad (43') \]

The compatibility of these conditions are easily proved, because we have

\[ [C_0(k') - C_0(k'), C_0(k) - C_0(k)] = 0. \]

3. Vacuum definition and modified supplementary conditions

The usual symmetrical treatment of four potentials of electromagnetic field leads the vacuum definition.
For each plane wave component, it becomes

\[ A_\mu \mathcal{F}_\text{vac} = 0, \quad \mu = 1, 2, 3, 4, \]

or

\[ A^{(a)} \mathcal{F}_\text{vac} = 0. \]  \hspace{1cm} (44)

A physical meaning of this condition is as follows: A vacuum is such a state that there exists only one \( i \)-th field \( (i = 1, 2, 3) \) which has the zero-particle distribution, i.e.,

\[ N_i(n_i(k_1) = 0, n_i(k_2) = 0, \ldots) = 1, \quad \text{all other } N_i(n_i(k_1), n_i(k_2), \ldots) = 0, \]

where \( N_i(n_i(k_1), n_i(k_2), \ldots) \) is number of the \( i \)-th field having a particle distribution \( n_i(k_1), n_i(k_2), \ldots \) in the super-quantized system. But we must take a special care for the 0-th field. From the commutation relation (32) and the Hamiltonian (35), we see that the 0-th field is a negative energy field and the emission and absorption character of \( C_0(k) \) and \( C_0^*(k) \) is interchanged comparing with the corresponding one of other fields. This was one of the well-known difficulties of the negative energy Bose field in the ordinary theory. But, in the present theory we consider a quantized field assembly which obeys Fermi statistics, we will see that the vacuum condition

\[ C_0(k) \mathcal{F}_\text{vac} = 0 \]  \hspace{1cm} (46)

can be satisfied, if the vacuum for the 0-th field is defined as follows; the vacuum consists of infinite numbers of the 0-th field which occupy all of the states having any particle distribution except the zero particle distribution one, i.e.,

\[ N_0(n_0(k_1) = 0, n_0(k_2) = 0, \ldots) = 0, \quad \text{all other } N_0(n_0(k_1), n_0(k_2), \ldots) = 1. \]

This situation is very similar to the negative energy electron sea in the electron vacuum. The above mentioned vacuum definition for the 0-th field satisfies the condition (46).

We see that the supplementary conditions are satisfied for the simultaneous vacuum state of the 3rd and the 0-th field. We, therefore, replace the supplementary conditions by the stronger one:

\[ C_3(k) \mathcal{F} = 0, \quad C_0(k) \mathcal{F} = 0. \]  \hspace{1cm} (47)

This means that the pure radiation field is defined as the simultaneous vacuum state of the 3rd and the 0-th field.

4. Commutation relations and expectation values

From the equations (37) and (31), we obtain for the commutation relations of the super-quantized four potentials

\[ [A_\mu(x), A_\nu(x')] = i [\epsilon_\mu^{(1)} \epsilon_\nu^{(1)}(\Phi_1, \Phi_1) + \epsilon_\mu^{(2)} \epsilon_\nu^{(2)}(\Phi_2, \Phi_2) + \epsilon_\mu^{(3)} \epsilon_\nu^{(3)}(\Phi_3, \Phi_3) - \epsilon_\mu^{(0)} \epsilon_\nu^{(0)}(\Phi_0, \Phi_0)] D(x-x'). \]  \hspace{1cm} (48)

For the real radiation field, we must put the coditions:
(\phi_o, \phi_1) = (\phi_o, \phi_2) = (\phi_o, \phi_0) = \mathcal{T}.

(49)

This means that number of the \(i\)-th field is one. Although the eigen-value of \((\phi_o, \phi_0)\) is really infinite as we see from the vacuum definition of the 0-th field, we may normalize this number of the 0-th field as one, namely we define

\(\langle \phi_o, \phi_0 \rangle = \mathcal{T}.\)  

(50)

We then obtain

\[ \langle [A_{\mu}(x), A_{\nu}(x')] \rangle = i \delta_{\mu \nu} D(x-x'). \]  

(51)

This expectation value of the commutation relation has the same value as that of the ordinary theory. The expectation values of other quadratic form can be computed and they also become to be as same as the corresponding one in the ordinary theory.

For an example, we obtain

\[ \langle \{ A_{\mu}(x), A_{\nu}(x') \} \rangle_{\text{vacuum}} = D^{(0)}(x-x'). \]  

(52)

These results also certify an equivalence of the super-quantized theory to the ordinary quantum electrodynamics. It should be noted that the supplementary conditions are given in the physically simple form of (47) without any inconsistency.

5. Interaction with electrons

The theory of interaction with electrons is given in a straightforward generalization of the ordinary one. In the present theory, the interaction operator of electromagnetic field with electrons is given by

\[ H'(x) = -j_{\mu}(x) A_{\mu}(x) \]  

(53)

where \(j_{\mu}\) represents the usual operator of electron charge and current, and the super-quantized potentials \(A_{\mu}\) defined in (2). The interaction representation gives the equation of motion:

\[ i \frac{\delta [\mathcal{T} \sigma]}{\delta \sigma(x)} = H'(x) \mathcal{T} \sigma \]  

(54)

with the supplementary condition

\[ \left[ \frac{\partial A_{\mu}^+(x)}{\partial x_{\mu}} - \int \sigma D^+(x-x') j_{\mu}(x') d\sigma_{\mu} \right] \mathcal{T} \sigma = 0, \]  

(55)

in which the used notations are the usual one.

As a physical illustration of the theory, we shall consider the calculation of the self-energy of a free electron at rest by means of usual second order perturbational method. In the zero-order approximation, the superquantized electromagnetic field is in the vacuum state as mentioned above. The second order interactions will take place through the following processes:

A. First, the \(A^{(0)}\) field being in its own zero-particle state is absorbed, subsequently the \(A^{(0)}\)-field having the particle distribution: \(n_i(k_i) = 0, \ldots, n_i(k_i) = 1, \)
$n_i(k_{i+1}) = 0$, \ldots, is created and the electron takes the recoil $(-k_i)$; then this created field is absorbed, subsequently the original vacuum field is again created and the electron comes to rest.

These processes correspond to the ordinary emission and absorption processes of a transverse and a longitudinal photon with a momentum $k_i$ respectively.

B. First, one of the $A^{(0)}$-field occupying the state of the particle distribution:

\begin{align*}
(n_0(k_1) = 0, & n_0(k_2) = 0, \ldots, n^0(k_i) = 1, n^0(k_{i+1}) = 0, \ldots),
\end{align*}

originally in the vacuum is absorbed, subsequently the zero-particle distribution $A^{(0)}$ field which was absent in the vacuum state is created, and the electron takes the recoil $k_i$; then the latter field is absorbed, the original field is again created and the electron comes to rest. This corresponds to the absorption and emission process of negative energy scalar photon. From our definition of the vacuum state of $A^{(0)}$-field, however, the process can be interpreted as the emission and absorption process of an anti-scalar photon with positive energy. It will be seen that all other processes cannot contribute to the second-order perturbational calculation, especially for $A^{(0)}$-field, all processes except B are forbidden according to Pauli’s exclusion principle.

In this calculation, we can obtain the well-known self-energy expression of an electron without the ambiguities remarked by several authors.\textsuperscript{8)

\section*{§ 4. Regulator theory of Bose field}

According to Pais and Uhlenbeck,\textsuperscript{6) the generalized Bose field theory—for an example, the generalized quantum electrodynamics—may be said to be a realization of the regulator type of theory proposed by Pauli and Villars,\textsuperscript{8) but it was shown that the theory is physically inconsistent. This inconsistency of the theory, however, are considered to be only connected with the negative energy Bose field. We could solve this type of difficulty by means of the super-quantization theory and the appropriate vacuum definition for the negative energy Bose field as shown in the preceding section. We shall show that the same procedure will be applicable also to the generalized field theory and then, the regulator theory becomes to be physically an acceptable scheme. We consider the generalized field theory (scalar and neutral). The generalized scalar potential is defined as follows:

\begin{align*}
\varphi = \sum_{k=1}^{N} \zeta_k \varphi^{(k)}
\end{align*}

which obeys the field equation

\begin{align*}
\prod_{k=1}^{N} \left( 1 - \frac{\Box}{\mu_k^2} \right) \varphi = 0,
\end{align*}

where $\Box$ is the D’Alembertian and $\mu_k$ is the mass of the $k$-th associated field $\varphi^{(k)}$. The associated field $\varphi^{(k)}$ satisfies the equation

\begin{align*}
(\Box - \mu_k^2) \varphi^{(k)} = 0,
\end{align*}

and the commutation relation
where $\Delta_k$ is the four dimensional $\delta$-function involving the mass $\mu_k$. The generalized field should be regularized, i.e., it satisfies the regularized commutation relation

$$[\varphi(x), \varphi(x')] = i\Delta_R(x-x')$$

(60)

where $\Delta_R$ is the regularized $\Delta$-function.

This requirement will be fulfilled by some regularization condition for $\zeta_k$. For example, we may put

$$\sum_{k=1}^N \zeta_k \mu_k^{2n} = 0, \quad n = 0, 1, \ldots, m \leq N,$$

(61)

Other types of condition also are possible.

It is clear that this condition cannot be satisfied by positive values of $\zeta_k$'s for real masses of $\mu_k$'s. We have, therefore, some positive $\zeta_k$'s and other negative $\zeta_k$'s (we assume all of $\zeta_k$ to be real).

For a negative $\zeta_k$, the commutation relation changes its sign comparing with the ordinary commutation relation. This means that the field $\varphi^{(i)}$ having a negative $\zeta$ is a negative energy Bose field. Thus, the generalized field $\varphi$ is considered to be a mixed field of positive and negative energy fields. The case is very similar to the electromagnetic field which was treated in the preceding section. The super-quantization of the generalized field can be performed in the parallel manner to that. We introduce the quantized state vectors $\varphi_k$ for each $\varphi^{(k)}$ which is defined as (15) and (15'). We denote the super-quantized quantities by gothic letters as before. We have

$$\varphi^{(k)} = (\varphi, \varphi^{(k)} \varphi)$$

(62)

and

$$\varphi = \sum_k \zeta_k \varphi^{(k)}.$$

(63)

For vacuum definition, we put the conditions:

$$\varphi^{+(k)} \mathcal{F}_{+0+} = 0,$$

(64)

where $\varphi^{+(k)}$ is a positive frequency part of $\varphi^{(k)}$. We have seen that these conditions are consistently satisfied. As before, number of the positive energy field is one and infinite numbers of the negative energy field is normalized to one, we obtain

$$\langle \mathcal{F}, [\varphi(x), \varphi(x')] \rangle \mathcal{F} = i\Delta_R(x-x')$$

(65)

which corresponds to the ordinary regularized commutation relation. If there is an interaction with other field, the sufficiently regularized $\Delta_R$ will give all finite results which arise from the interaction fluctuation of the field $\varphi$.

If we have not the supplementary condition:

$$\varphi^{+(k)} \mathcal{F} = 0,$$

(66)
in this case, the field $\varphi^{(k)}$ will become to be observable as an assembly of Bose particles with mass $\mu_k$. However, this situation gives no physical inconsistency even for the negative energy field.

The vacuum for a negative energy field is defined as such state that only the zero-particle state is unoccupied and all other states are occupied. Then, if it happens that the zero-particle state is occupied and one of the states occupied in the vacuum comes to be vacant, we observe a field in which some particle distribution with positive energy corresponding to an anti-particle of the original negative energy one is realized. We may, therefore, expect a mass spectrum of Bose particles associated to a regularized field. It may be imagined to have any relation between this mass spectrum and that of the observed mesons in cosmic rays.

§ 5. Conclusions

The super-quantization theory affords a physically consistent reinterpretation of negative energy Bose field which serves to give a physical scheme for the symmetrical treatment of electromagnetic potentials and the theory of regulator of Bose field.

It is regretful that the present theory can offer no clue to solve the difficulties associated to the regulator theory of Fermi field. This problem remains to study in future.

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References

3) V. Fock, ZS f. Phys. 75 (1932), 622.
4) S. N. Gupta, loc. cit.
7) F. Bopp, ZS f. Naturforschung 1 (1945), 53.
8) W. Pauli and F. Villars, Rev. Mod. Phys. 21 (1949), 434.