Orthogonal Functions in the Complex Domain

Giiti IWATA

Physical Institute, University of Tokyo

(Received February 15, 1952)

Eigenfunctions associated to an operator in the complex domain constitute a system of orthogonal functions along the path of integration in the complex plane. There are orthogonal functions free from operators, examples of which are given. At the end of this paper are collected some transformation functions constructed with orthogonal functions.

§ 1. Orthogonal functions in the complex plane

The relations

\[ (y_m, y_n) = 0, \quad (\varepsilon_m, \varepsilon_n) = 0, \quad (y_m, \varepsilon_n) = \delta_{mn} \]  

that are satisfied by the eigenfunctions associated to an operator in the complex domain remind us of the relations that are satisfied by a system of basis vectors in the euclidian space of even dimensions. Instead of the usual form of the innerproduct

\[ (q, p) = \sum q^\mu p^\nu, \quad \mu = 1, 2, \cdots, n, \quad \nu = 1, 2, \cdots, n' \]

of the two vectors \((q^\mu), (p^\nu)\) in the 2n-dimensional euclidian space, here will be used the innerproduct of the form

\[ (x, y) = \sum (x^r y^r + x'^r y'^r) = g_{\rho\sigma} x^\rho y^\sigma, \]

where the vectors \((x^r), (y^\rho)\) may be regarded as those obtained by the change of variables

\[ x^r = (q^r + iq'^r)/\sqrt{2}, \quad x'^r = (q^r - iq'^r)/\sqrt{2}, \]

\[ y^\rho = (p^\rho + ip'^\rho)/\sqrt{2}, \quad y'^\rho = (p^\rho - ip'^\rho)/\sqrt{2} \]

from the vectors \((q^\rho), (p^\rho)\).

There exists then a system of basis vectors \(e_\rho\) that satisfies the orthogonality relations

\[ (e_\rho, e_\sigma) = 0, \quad (e_\rho, e_\sigma') = 0, \quad (e_\rho, e_\sigma') = \delta_\rho\sigma' \]

or

\[ (e_\rho, e_\sigma) = g_{\rho\sigma}. \]

The relations (1) and (3) are identical except that while the innerproduct of two vectors is defined by (2) the innerproduct of two functions is defined to be the complex integral of the product of the two functions divided by \(2\pi i\) and taken along a certain closed curve \(C\) in the complex plane, viz.
$(f, g) = \frac{1}{2\pi i} \int_C f(z)g(z)dz$.

Hence the eigenfunctions associated to the operators may be called orthogonal functions along the curve $C$.

There are infinitely many systems of orthogonal functions on the same curve. For simplicity, let $C$ be the unit circle with its center at the origin, then

$$e_r = z^r, \quad e_{-r} = z^{-r-1}, \quad r = 0, 1, 2, \ldots$$

constitute a system of orthogonal functions.

Let $f_p$ be another system of orthogonal functions on the same curve $C$, there must be then the orthogonality relations

$$(f_p, f_q) = g_{pq}.$$  

Since $f_p$ is a linear combination of $e_p$, there is a linear transformation $A$ such that

$$f_p = e_p A_p,$$

the sign of summation with respect to $A$ being omitted. The orthogonality relations require the linear transformation to be orthogonal, viz.

$$g_{pq} = g_{qp} A^* p A^* q.$$  \hspace{1cm} (4)

With regard to the orthogonal transformations the suffices may be raised or lowered with the aid of the fundamental tensors $g_{pq}$ and $g^{pq}$ which is inverse to it.

§ 2. Decomposable systems of orthogonal functions

If $f_r$ are regular inside the curve $C$, $f_r$ regular outside $C$, the linear transformation $A$ decomposes into the direct sum of a non-singular matrix and its transposed inverse matrix, since $A$ must be of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where the elements of $a$ and $b$ are labeled with the indices $r, r'$ respectively, and that the orthogonality relation (4) requires that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (a^* b^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (a^* b^*)$$

(*: transposition) or

$$a^* b = 1$$

i.e.

$$b = (a^{-1})^* = a^{-*}.$$  

For simplicity, $(e_r, f_r)$ shall be denoted by $e, f, (e_r, f_r)$ by $e', f'$. The relations between $e, e'$ and $f, f'$ will be then represented by

$$f = e a, \quad f' = e' a^{-*}.$$  

Hence

$$f(z)f^*(z') = e(z) a a^{-1} e'(z') = e(z) e'^* (z') = (z|z') = 1/(z' - z).$$
Orthogonal Functions in the Complex Domain

In case \( f \) are regular inside \( C \) while \( f' \) are not regular inside as well as outside \( C \), the linear transformation \( A \) takes the form

\[
A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}
\]

so that the orthogonality relations require

\[
b = a^{-*}, \quad ac^* + ea^* = 0
\]

or \((ac^*)^* = -ac^*\).

One gets then

\[
f(z)f'^*(z') = e(z)\alpha(e(z')c + e'(z')b)^*
\]

\[
= e(z)\alpha(e^*e^*(z') + b^*e^*(z'))
\]

\[
= e(z)\alpha ac^* e^*(z') + e(z)e'^*(z').
\]

Since \( ac^* \) is skew symmetric, the first term in the right member will vanish when \( z' = z \).

Here is an example. The Bessel's equation

\[
\frac{d}{dz}z^{-\frac{1}{2}} \frac{dy}{dz} + z^{-1}y = 0
\]

has

eigenvalues \( l_n = n(n + 1), \quad n = 0, 1, 2, \cdots \),

eigenfunctions \( f_n(z) = (z|n) = z^{-\frac{1}{2}} f_{n, \frac{1}{2}}(z), \)

\( f_n'(z) = (n|z) = (-)^n(n + \frac{1}{2})z^{-\frac{1}{2}} f_{n-\frac{1}{2}}(z). \)

The orthogonality conditions are satisfied, but one gets

\[
(z|n)(n|z') = \frac{\cos(z'-z)}{z'-z}
\]

which is one of the addition theorems of Bessel functions. One sees

\[
\left[ \frac{\cos(z'-z)}{z'-z} - \frac{1}{z'-z} \right]_{z'=z} = 0.
\]

In the worst case when there exist no basis vectors regular inside or outside the curve \( C \), the sum \( f(z)f'^*(z') \) takes no definite form other than the \( \delta \)-function.

When \( f(z)f'^*(z') \) has a definite form, Laurent's expansion of any function will be readily given.

§ 3. The change of variable

When the correspondence between the variables \( z \) and \( w = f(z) \) is required to be one-to-one, \( f(z) \) must be of the form \((oz + \beta)(yw + \delta)^{-1}\).
The transformation rule for eigenfunctions \((w|n), (n|w)\) shall be given here for the transformation \(w \rightarrow z\).

Let
\[
(s|n) = A(z)(w(z)|n), \\
(n|s) = B(z)(n|w(s)),
\]
then the orthogonality conditions for \((s|n), (n|s)\) require that

1) \((s|n)(n|s') = A(z)B(z')(w(z)|n)(n|w(s')) = A(z)B(z')(w(z') - w(z)^{-1}) = A(z)B(z')(rz + \alpha)(rz' + \alpha)(\alpha^2 - \beta^2)^{-1}(z' - z)^{-1},
\]

\[
\therefore A(z) = B(z) = (\alpha^2 - \beta^2)^{1/2}(rz + \alpha)^{-1}\frac{dz}{dw}. \tag{5}
\]

2) \((n|s)(s|n') = B(z)A(z)(n|w(z))(s|w(z)) = B(z)A(z)(n|w(z))(w(z)|n') = B(z)A(z)(w(z)|n')(\alpha^2 - \beta^2)^{-1}\frac{dz}{dw},
\]

\[
\therefore A(z)B(z) = \frac{dz}{dw}.
\]

This condition is already satisfied by (5).

Hence one gets
\[
(s|n) = (w(z)|n)\sqrt{\frac{dw}{dz}}, \\
(n|s) = (n|w(z))\sqrt{\frac{dw}{dz}}.
\]

§ 4. The equation \(f[(az + \beta)(rz + \delta)^{-1}] = l f(z)\)

As an application of the change of variable, we take the equation
\[
f\left(\frac{az + \beta}{rz + \delta}\right) = l f(z).
\]

The transformation \(z \rightarrow (az + \beta)(rz + \delta)^{-1}\) leaves fixed the two points \(z_1, z_2,\)
\[
z_1 = \frac{a - \delta + \sqrt{D}}{2\gamma}, \quad z_2 = \frac{a - \delta - \sqrt{D}}{2\gamma}, \quad D = (a - \delta)^2 + 4\beta\gamma,
\]
which are the two roots of the equation \(z = (az + \beta)(rz + \delta)^{-1}\).

The two cases are distinguished.

1) \(D \neq 0\). If \((z - z_1)(z - z_2)^{-1}\) is denoted by \(w\), the transformation will be represented in terms of \(w\) as
\[
w \rightarrow kw
\]
where \(k\) is the ratio of the two characteristic roots of the transformation, viz.
Orthogonal Functions in the Complex Domain

\[ k = \frac{m - \sqrt{D}}{m + \sqrt{D}} \]

For simplicity it is assumed that \( k^m \neq 1 \) (\( m = 1, 2, \ldots \)). If we put

\[ f(z) = g(zv) \]

the equation will be changed into the equation

\[ g(kw) = lg(w) \]

which has

- eigenvalues \( l_n = k^n \), \( n = 0, 1, 2, \ldots \),
- eigenfunctions \( (w|n) = w^n \), \( (n|w) = w^{-n-1} \),

or

\[
(z|n) = \left( \frac{z_1 - z_2}{z - z_2} \right)^{1/2} \left( \frac{z - z_1}{z - z_2} \right)^n, \quad (n|z) = \left( \frac{z_1 - z_2}{z - z_2} \right)^{1/2} \left( \frac{z - z_1}{z - z_2} \right)^{-n-1}.
\]

2) \( D = 0 \). If \( c \) denotes \( (a + \delta)/2r \), \( w \) does \( c(z - z_1)^{-1} \), the transformation will be changed into the transformation

\[ w \rightarrow w + 1. \]

So the equation for \( g(w) = f(z) \) will be

\[ g(w + 1) = lg(w). \]

A continuous spectrum appears in this case in which are made three assumptions;

1) the path of integration is the imaginary axis or a straight line parallel to the imaginary axis,
2) \( (w|a) \rightarrow 0 \) as \( w \rightarrow -\infty \),
3) \( (a|w) \rightarrow 0 \) as \( w \rightarrow \infty \).

One gets then

- eigenvalues \( l = e^a \), \( a > 0 \),
- eigenfunctions \( (w|a) = e^{aw} \), \( (a|w) = e^{-aw} \),

the orthogonality relations:

\[
(a|w) (w|a') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-aw + a'w} dw = \delta(a - a') = (a|a'),
\]

\[
(w|a) (a|w') = \int_{0}^{\infty} e^{aw - aw'} da = 1/(w' - w) = (w|w'),
\]

Real part of \( w' - w > 0 \).

Returning to the old variable we have

\[ (z|a) = \exp \left( \frac{ac}{z - z_1} \right) (z_1)^{\nu} (z - z_1)^{-1}, \]
An easy generalization of the preceding functional equation may be the equation
\[ f(u_1 z + \beta_1) + f(u_2 z + \beta_2) + \cdots + f(u_m z + \beta_m) = l f(z) \]  
(6)
to which corresponds the operator \( L \) with its matrix element
\[ (z \mid L \mid z') = \frac{1}{z' - u_1 z - \beta_1} + \cdots + \frac{1}{z' - u_m z - \beta_m}. \]

So the equation adjoint to (6) will be
\[ \frac{1}{z} g(z - \beta_1) + \cdots + \frac{1}{z} g(z - \beta_m) = l g(z). \]

Eigenvalues and eigenfunctions are as follows:

- **eigenvalues**: \( l = u_1^n + u_2^n + \cdots + u_m^n \), \( n = 0, 1, 2, \ldots, \)
- **eigenfunctions**:
  - \( (z \mid n) \): polynomial of degree \( n \) in \( z \),
  - \( (n \mid z) \): ascending power series in \( z^{-1} \) starting at \( z^{-n-1} \).

A special case is of a little interest. The equation
\[ f \left( \frac{z}{m} \right) + f \left( \frac{z+1}{m} \right) + \cdots + f \left( \frac{z+m-1}{m} \right) = l f(z) \]
has
- **eigenvalues**: \( l_n = m^{1-n} \), \( n = 0, 1, 2, \cdots, \)
- **eigenfunctions**: \( (z \mid n) = B_n(z) \), Bernoulli’s polynomial,
\[ (n \mid z) = \frac{1}{n!} \left( -\frac{d}{dz} \right)^n \log \frac{z}{z-1}, \]
the path of integration being taken to be a circle of radius \( > 1 \) with its center at the origin. The result
\[ B_n \left( \frac{z}{m} \right) + B_n \left( \frac{z+1}{m} \right) + \cdots + B_n \left( \frac{z+m-1}{m} \right) = m^{1-n} B_n(z) \]
is known as the multiplication theorem for Bernoulli’s polynomials.\(^2\)

§ 5. Free orthogonal functions

So far the orthogonal functions have been considered as those associated to their operators. However, as was seen in § 2, a system of orthogonal functions can be generated with the aid of a non-singular matrix and transposed matrix inverse to it. If we can expand \( (z \mid z') \) in the series
\[ \frac{1}{z' - z} = f_0(z) g_0(z') + f_1(z) g_1(z') + \cdots + f_n(z) g_n(z') + \cdots \]
Orthogonal Functions in the Complex Domain

where \( f_n(z) \) are regular in a domain of the complex plane confined by a closed curve while \( g_n(z) \) are regular in the other domain of the complex plane, \( f_n(z) \), \( g_n(z) \) constitute a system of orthogonal functions, viz.

\[
f_n(z) = (z|n), \quad g_n(z) = (n|z)
\]

Here are a few examples.

1) Appell’s series.\(^3\) P. Appell expanded \((z' - z)^{-1}\) in the series

\[
\frac{1}{z' - z} = \frac{Q_0(z)}{P_1(z')} + \frac{Q_1(z)}{P_2(z')} + \cdots + \frac{Q_n(z)}{P_{n+1}(z')} + \cdots
\]

where \( P_n(z) \) is given polynomial of degree \( n \) having all its zeros inside the unit circle. \( Q_n(z) \) is a polynomial of degree \( n \) uniquely determined by the orthogonality conditions.

2) Neumann’s series.\(^4\)

\( (z|n) = [J_n(\sqrt{z})]^2 \)

\( J_n(z) \) : Bessel function,

\( (n|z) = \text{Neumann’s polynomial} \ Q_n(\sqrt{z}) \) multiplied by \( \epsilon_n \),

\( (0|z) = \frac{1}{z}, \quad (1|z) = \frac{2}{z} + \frac{4}{z^3} \quad (2|z) = \frac{2}{z} + \frac{16}{z^3} + \frac{64}{z^5} \cdots \)

3) Sonine’s series.\(^5\) Let \( \psi(w) \) be any function of \( w \) regular in the neighbourhood of \( w = 0 \); and, if \( \psi(w) = x \), let \( w = \psi(x) \) so that \( \psi \) is the function inverse to \( \phi \). We have then

\[
(z|n) = \frac{n!}{2\pi i} \int_0^{z^*} e^{tx} \frac{d\psi}{w^{n+1}} \quad (n|z) = \frac{1}{n!} \int_0^{z^*} e^{-tx} w^n dx,
\]

e.g.

\( w = \psi(x) = x + x^2 \), \( x = \psi(w) = \frac{1}{2} (\sqrt{1 + 4w} - 1) \)

\( = w - w^3 + \frac{2}{3}w^5 - \frac{5}{9}w^7 + \cdots \)

\( (z|0) = 1, \quad (0|z) = \frac{1}{z} \)

\( (z|1) = z, \quad (1|z) = \frac{1}{z^3} + \frac{2}{z^3} \)

\( (z|2) = z^2 - 2z, \quad (2|z) = \frac{1}{z^3} + \frac{6}{z^4} + \frac{60}{z^5} \)

\( (z|3) = z^3 - 6z^2 + 12z, \quad (3|z) = \frac{1}{z^4} + \frac{12}{z^5} + \frac{60}{z^6} + \frac{120}{z^7} \cdots \)

The orthogonality conditions are satisfied.

4) General interpolation series.\(^6\)

\[
\frac{1}{x - z} = \frac{1}{x - a_1} + \frac{z - a_1}{(x - a_1)(z - a_2)} + \frac{(z - a_1)(z - a_2)}{(x - a_1)(x - a_2)} + \cdots
\]
\((x|n) = (x-a_1) \cdots (x-a_n),\)
\((n|x) = 1/(x-a_1) \cdots (x-a_{n+1}).\)

The path of integration shall encircle all the points \(a_1, a_2, a_3, \ldots\) in the positive sense.

There are three special cases.

4') Newton's interpolation series.
\[
\frac{1}{x-z} = \frac{1}{x} + \frac{z}{x(x-1)} + \frac{z(z-1)}{x(x-1)(x-2)} + \cdots.
\]

4'') Everett's interpolation series.
\[
\frac{1}{x-z} = \frac{\zeta}{x} + \frac{z}{x(\zeta+1)(\zeta-1)} + \frac{z(z-1)}{x(\zeta+1)(\zeta-1)(x-1)} + \cdots,
\]
\((\zeta = 1-z, \, \xi = 1-x).\)

4''' Stirling's interpolation series.
\[
\frac{1}{x-z} = \frac{1}{x} + \frac{z}{x^2-1} + \frac{z^2}{x^2-1} + \frac{z(z-1)}{x^2-1(x^2-2)} + \frac{z^2(z-1)+z(z-1)}{x^2-1(x^2-2)} + \cdots.
\]

The orthogonality relations are easily verified.

§ 6. Some transformation functions

When there are two decomposable systems of orthogonal functions \(e_r, f_r, e, f\) being regular inside the curve \(C\), \(e', f'\) regular outside the curve \(C\), we can denote \(e(z)f^*(l)\) by \((x|l), f(l)e^*(z)\) by \((l|x)\). In place of the conditions for \((x|l), (l|x)\) in the preceding paper\(^7\), \((x|y)\) is conditioned to be regular qua function of \(x\) inside the path of integration, regular qua function of \(y\) outside the path of integration. One sees then
\[
e(z)f^*(l) = e_r(z)f^*(l) = (x|l),
\]
\[
f(l)e^*(z) = f_r(l)e^*(z) = (l|x),
\]
by virtue of the relations
\[
(x|l)(l|x') = e(z)f^*(l)f(l) e^*(z') = e(z)e^*(z') = (x|z'),
\]
\[
(l|x)(x|x') = f(l)e^*(z)e(z)f^*(l') = f(l)f^*(l') = (l|l'),
\]
\[
(e^*(z)e(z) = f^*(l)f(l) = 1).
\]

The fundamental system of orthogonal functions is the system
\[
e_r(z) = z^r, \quad e_r(z) = z^r(z) = z^{-r-1}
\]
which is modified to the system
\[
f_r(z) = z^r/e_r, \quad f_r(z) = z^r(z) = e(z)^{-r-1}.
\]

We have then
Orthogonal Functions in the Complex Domain

\[ (z|l) = \sum \frac{c_r z^r}{l^{r+1}}, \quad r = 0, 1, 2, \ldots, \]

\[ (l|z) = \sum \frac{l^r}{c_r z^{r+1}}. \]

Here are some examples.

1) \[ (z|l) = \frac{1}{l} + \frac{z}{l^2} + \frac{z^2}{l^3} + \cdots = \frac{1}{l-z}, \]
\[ (l|z) = \frac{1}{z} + \frac{l}{z^2} + \frac{l^2}{z^3} + \cdots = \frac{1}{z-l}. \]

2) \[ (z|l) = \frac{1}{l} + \frac{2z}{l^2} + \frac{3z^2}{l^3} + \cdots = \frac{l}{(l-z)^2}, \]
\[ (l|z) = \frac{1}{z} + \frac{l}{z^2} + \frac{l^2}{z^3} + \cdots = \frac{1}{l} \log \frac{z}{z-l}. \]

3) \[ (z|l) = \frac{1}{l} + \frac{3z}{l^2} + \frac{3 \cdot 5 z^2}{2 \cdot 4 l^3} + \cdots = \frac{l^{1/2}}{(l-z)^{3/2}} \]
\[ (l|z) = \frac{1}{z} + \frac{2l}{3z^2} + \frac{2 \cdot 4 l^2}{3 \cdot 5 z^3} + \cdots = \sqrt{\frac{1}{l(z-l)}} \arcsin \sqrt{\frac{l}{z}}. \]

4) \[ (z|l) = \frac{1}{l} + \frac{3z}{l^2} + \frac{5z^2}{l^3} + \cdots = \frac{l+z}{(l-z)^2}, \]
\[ (l|z) = \frac{1}{z} + \frac{l}{3z^2} + \frac{l^2}{5z^3} + \cdots = \frac{1}{(l,z)^{1/2}} \arctan \sqrt{\frac{l}{z}}. \]

5) \[ (z|l) = \frac{1}{l} + \frac{a}{l^2} + \frac{a(a+1)}{1 \cdot 2 l^3} z^2 + \cdots = \frac{l^{a-1}}{(l-z)^a}, \]
\[ (l|z) = \frac{1}{z} + \frac{l}{a z^2} + \frac{1 \cdot 2 l^2}{a(a+1) z^3} + \cdots = \frac{1}{z} \Gamma \left(1, 1; a, \frac{l}{z} \right). \]

6) \[ (z|l) = \frac{1}{l} + \frac{ab}{1 \cdot c l^2} + \frac{a(a+1) b(b+1)}{1 \cdot 2 \cdot c(c+1) l^3} z^2 + \cdots = \frac{1}{l} \Gamma \left(a, b; c, \frac{z}{l} \right), \]
\[ (l|z) = \frac{1}{z} + \frac{1}{ab} \frac{l}{z^2} + \frac{1 \cdot 2 \cdot c(c+1)}{a(a+1) b(b+1) z^3} l^2 + \cdots = \frac{1}{z} \Gamma \left(1, 1; c; a, b, \frac{l}{z} \right). \]

7) \[ (z|l) = \frac{1}{l} - \frac{1}{l^2} + \frac{1}{2! l^3} - \cdots = \frac{1}{l} e^{-z/l}, \]
\[ (l|z) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{2! z^3} - \cdots = \frac{1}{l} \xi \left(\frac{l}{z} \right), \]
\[ \xi(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \cdots. \]
8) \( g[l] = \frac{1}{l} \frac{1}{1^{l+1} l^2} + \frac{1}{2^{l+1} l^3} \ldots = \frac{1}{l} J_0\left(2\sqrt{\frac{z}{l}}\right), \)
\( (l | z) = \frac{1}{z} - \frac{1}{2^{l+1} z^2} + \frac{2^{l+1} l^2}{2^{l+1} z^3} \ldots = \frac{1}{l} \beta(\frac{t}{z}), \)
\( \beta(x) = \frac{1}{x} - \frac{1}{2^{l+1} x^2} + \frac{2^{l+1} l^2}{2^{l+1} x^3} \ldots. \)

The cases pertaining to well-known polynomials:

9) \( g[l]\) = \( P_0(z) + \frac{3 P_1(z)}{l^2} + \frac{5 P_2(z)}{l^3} + \ldots = \frac{l^2-1}{(l^2-2lz+1)^{1/2}}, \)
\( (l | z) = Q_0(z) + l Q_1(z) + l^2 Q_2(z) + \ldots = \frac{1}{(l^2-2lz+1)^{1/2}} \text{ arc coth } \frac{z-l}{(l^2-2lz+1)^{1/2}}, \)

\( L_n(z) \), \( Q_n(z) \): Legendre's functions.

10) \( g[l]\) = \( L_0(z) + \frac{L_1(z)}{l} + \frac{L_2(z)}{l^2} + \ldots = e^{-z/(l-1)}, \)
\( (l | z) = M_0(z) + l M_1(z) + l^2 M_2(z) + \ldots = \frac{1}{l-1} z\left(\frac{l-1}{z}\right), \)

\( L_n(z) \): Laguerre's polynomial,

\( M_n(z) = (-)^n \sum n! \binom{n}{m} z^{-n-1}, \quad n = m, \quad m + 1, \ldots. \)

11) \( g[l]\) = \( H_n(z) + \frac{H_1(z)}{l} + \frac{H_2(z)}{l^2} + \ldots = \varepsilon(1-z), \)
\( (l | z) = K_0(z) + l K_1(z) + l^2 K_2(z) + \ldots = i\varepsilon(i z - i l), \)

\( H_n(z) \): Hermite's polynomial,

\( K_n(z) = \frac{1}{2^n n!} + \frac{(n+1)(n+2)}{2^{n+2}} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot 2^{n+5}} + \ldots, \)
\( \varepsilon(x) = \int_0^x \exp \frac{x^2 - t^2}{2} dt = \frac{1}{x} - \frac{1}{x^3} \cdot 1 \cdot 3 \cdot 5 + \ldots. \)

12) \( g[l]\) = \( B_0(z) + \frac{B_1(z)}{l} + \frac{B_2(z)}{l^2} + \ldots = \int_0^z t e^{(z-t)^{1/2}} dt, \)
\( (l | z) = C_0(z) + l C_1(z) + l^2 C_2(z) + \ldots = \log \frac{z-l}{z-1-l}, \)

\( B_n(z) \), Bernoulli's polynomial,

\( C_n(z) = \frac{1}{n!} \left( - \frac{d}{dz} \right)^n \log \frac{z}{z-1}. \)
The relation between (7) and (10) is obviously observed. The change of variables will engender various systems of orthogonal functions.

Postscriptum To fill up the vacant space two remarks may be inserted here.
1) The path of integration considered so far is a closed curve without any node or a straight line extending to infinity at most. If the path of integration encircles each of the singular points of a differential equation a number of times in any sense, there appears the need of Riemann surfaces with topological considerations. The treatment of the case shall be reserved for a later occasion elsewhere.
2) Many examples of the system of orthogonal functions are known in the real domain as well as in the complex domain. We don't know any system of symplectic functions in the proper sense. A system of functions \((f_1, f_2, \ldots, f_n, f_2, f_1, \ldots)\) may be called symplectic when it satisfies the conditions

\[
[f_{r+1}, f_r] = 0, \quad [f_{r+1}, f_{r+1}] = 0, \quad [f_{r+1}, f_{r+1}] = -[f_{r+1}, f_r] = \delta_{r+1},
\]

where \([f, g]\) is defined as

\[
[f, g] = \int_C F(f, g) ds,
\]

\(F(f, g)\) being a function bilinear and alternate in \(f, g\), \(C\) an appropriate curve.
References

1) G. Iwata, Prog. Theor. Phys. 6 (1951), 220.
2) L. M. Milne-Thomson, Calculus of Finite Difference (1933), 141.
4) G. Watson, Bessel Functions (1922) 290.
5) " " , 280.
7) G. Iwata, Prog. Theor. Phys. 6 (1951), 526.