THE RANGE OF EXISTENCE OF STONELEY WAVES IN AN INTERNAL STRATUM. II. ANTISYMMETRIC VIBRATIONS

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(Communicated by E. R. Lapwood)

(Received 1957 February 20)

Summary

We derive the frequency equation and the condition of existence of Stoneley type waves with antisymmetric vibrations which can be propagated along the interfaces between an internal stratum and two adjacent halfspaces of identical elastic properties, all perfectly elastic, homogeneous and isotropic. The ranges of existence of such waves are next obtained by numerical computation and the results are presented both in tabular form and graphically.

For large values of the frequency, the frequency equation for these waves, like the one for waves with symmetric vibrations discussed in an earlier paper *, reduces to the velocity equation of Stoneley waves propagated along the interface between two halfspaces.

For a low-velocity stratum, the results are similar to those for waves with symmetric vibrations. It is found that, for a pair of media for which waves of some frequency can exist, as the frequency of the waves or the thickness of the stratum is decreased, there is a cut-off value of either below which such waves cannot be propagated.

For a high-velocity stratum, the results are in sharp contrast with those for waves with symmetric vibrations. Stoneley waves of all frequencies can exist for some pairs of materials. As the frequency of the waves or the thickness of the stratum is decreased, that is, as the ratio of the wave-length to the thickness of the stratum is increased, the regions of existence expand until a certain limiting region of existence is obtained when this ratio tends to infinity.

Another difference between waves with symmetric and antisymmetric vibrations is that whereas for the former the phase velocity could not be less than the smaller of the two Rayleigh wave velocities, for the latter there is no theoretical lower limit.

1. Introduction.—The range of existence of Stoneley type waves with symmetric vibrations propagated along the interfaces between an internal stratum and two adjacent halfspaces of identical elastic properties has been discussed in I. When we have a point source of compressional waves in the middle of such a stratum, then, besides the various reflected and refracted body waves, boundary waves of Stoneley type with symmetric vibrations of the stratum only can be generated. If, however, the source is not situated at the middle of the stratum, then boundary waves with antisymmetric vibrations can also be generated. We obtain here the range of existence of these boundary waves with antisymmetric vibrations. The frequency equation of these waves and the condition for their existence are obtained in the same way as for waves with symmetric vibrations and we shall, therefore, give only the outlines of this work. We follow the notation used in I.

* Chopra, S. D. (1957), M.N., Geophys. Suppl., 7, 256. This will be referred to as I throughout the present work.
2. Derivation of the frequency equation.—Fig. 1 shows the stratum $M$, the halfspaces $M_1, M_2$, and the cylindrical polar coordinates $r, \theta, z$. $\rho, \lambda, \mu$ are the density and Lamé's constants and $\alpha, \beta$ the dilatational and distortional wave velocities for the stratum. The same symbols with the suffix 1 denote the corresponding quantities for the halfspaces $M_1, M_2$ of identical elastic properties. $2h$ is the thickness of the stratum.

Since we are interested only in the $P$ and $SV$ types of boundary waves, the displacement in the transverse direction is taken as zero. There is symmetry about $Oz$ and therefore all the quantities are independent of $\theta$, so that $\partial/\partial \theta = 0$. The displacements $U, W$ in the $r, z$ directions respectively can be derived, as in I (1), from a dilatational potential $\phi$ and a distortional potential $\psi$, which satisfy the wave equations $I (3)$ and $I (4)$ respectively.

The method of separation of variables gives suitable values of the potentials $\phi, \psi$, etc. for the antisymmetric type of vibrations as

$$\phi = P J_0(r\xi) \sinh \alpha x e^{i\omega t},$$
$$\psi = Q \frac{d}{dr} (J_0(r\xi)) \cosh \beta x e^{i\omega t},$$
$$\phi_1 = R J_0(r\xi) e^{-i\alpha x} e^{i\omega t},$$
$$\psi_1 = S \frac{d}{dr} (J_0(r\xi)) e^{-i\beta x} e^{i\omega t},$$
$$\phi_2 = -R J_0(r\xi) e^{i\alpha x} e^{i\omega t},$$
$$\psi_2 = S \frac{d}{dr} (J_0(r\xi)) e^{i\beta x} e^{i\omega t},$$

where

$$a = \left( \xi^2 - \frac{\omega^2}{\alpha^2} \right)^{1/2}, \quad b = \left( \xi^2 - \frac{\omega^2}{\beta^2} \right)^{1/2}, \quad a_1 = \left( \xi^2 - \frac{\omega^2}{\alpha_1^2} \right)^{1/2}, \quad b_1 = \left( \xi^2 - \frac{\omega^2}{\beta_1^2} \right)^{1/2}.$$
and $P, Q, R, S$ are functions of $\zeta$ to be determined from the boundary conditions. $a, b, a_1, b_1$ are defined to have their real parts positive. $\zeta$ is a parameter of the dimensions of $1/r$.

The boundary conditions, the same at $z = h$ and $z = -h$, viz. the continuity of displacements and stresses, lead to four homogeneous equations in $P, Q, R, S,$ and the condition of their consistency gives the frequency equation as

$$\begin{vmatrix} -\sinh ah & b \sinh bh & 1 & b_1 \\ a \cosh ah & -\zeta^2 \cosh bh & a_1 & \zeta^2 \\ -\mu \left( \zeta^2 - \frac{\omega^2}{2\beta^2} \right) \sinh ah & \mu \zeta^2 b \sinh bh & \mu_1 \left( \zeta^2 - \frac{\omega^2}{2\beta_1^2} \right) & \mu_1 \zeta^2 b_1 \\ \mu a \cosh ah & -\mu \left( \zeta^2 - \frac{\omega^2}{2\beta^2} \right) \cosh bh & \mu_1 a_1 & \mu_1 \left( \zeta^2 - \frac{\omega^2}{2\beta_1^2} \right) \end{vmatrix} = 0. \quad (5)$$

Putting $\zeta = \omega \xi / \beta$, so that $\xi$ is a dimensionless quantity, dividing by $\mu_1^2 / \beta^6$ and rearranging according to powers of $\mu / \mu_1$, the equation takes the form

$$(\xi^2 - C_1 D_1) \left[ (2\xi^2 - 1)^2 \tanh HC - 4\xi^2 CD \tanh HD \right] \theta^2 - 2[ (2\xi^2 - \gamma - 2C_1 D_1) \{ \xi^2 (2\xi^2 - 1) \tanh HC - 2\xi^2 CD \tanh HD \} \\
+ \frac{1}{2} \gamma CD_1 + \frac{1}{2} \gamma C_1 D \tanh HC \tanh HD \theta \\
+ \{ (2\xi^2 - \gamma)^2 - 4\xi^2 C_1 D_1 \} \{ \xi^2 \tanh HC - CD \tanh HD \} = 0. \quad (6)$$

where

$$\theta = \frac{\mu}{\mu_1}, \quad \gamma = \beta^2 / \beta_1^2, \quad H = \omega h / \beta, \quad C = (\xi^2 - \beta^2 / \beta_1^2)^{1/2}, \quad D = (\xi^2 - 1)^{1/2}, \quad C_1 = (\xi^2 - \beta^2 / \beta^2)^{1/2}, \quad D_1 = (\xi^2 - \gamma)^{1/2}. \quad (7)$$

When the frequency $\omega \to \infty$ and therefore $H \to \infty$, the hyperbolic tangents tend to unity. In this case, equation (6), like the frequency equation for waves with symmetric vibrations, becomes the wave-velocity equation for Stoneley waves at the interface between two halfspaces.

At large distances from the origin where the waves approximate to plane waves, $\omega / \zeta$, that is, $\beta / \xi$ represents their phase velocity. Since Stoneley waves*, by definition, have their phase velocity less than the distortional wave velocities of both media, the condition for the existence of Stoneley type waves with antisymmetric vibrations of the stratum is that equation (6) shall have a root in the range $\xi^2 > \max (1, \gamma)$. In this range $C, D, C_1, D_1$ are all real and positive.

Equation (6) is satisfied for $C = 0$, i.e. $\xi^2 = \beta^2 / \beta_1^2$. Since $\beta^2 / \beta_1^2 < 1$, this root is of no interest to us. Moreover, when $C = 0$, the equations for $P, Q, R, S$ lead, in general, to an identically zero solution.

3. Discussion of the frequency equation.—Equation (6) may be written in the form

$$F'(\xi^2) = L' \theta^2 - 2M' \theta + N' = 0, \quad (8)$$

where

$$L' = (\xi^2 - C_1 D_1) \left[ (2\xi^2 - 1)^2 \tanh HC - 4\xi^2 CD \tanh HD \right], \quad M' = (2\xi^2 - \gamma - 2C_1 D_1) \left[ \xi^2 (2\xi^2 - 1) \tanh HC - 2\xi^2 CD \tanh HD \right] \\
+ \frac{1}{2} \gamma CD_1 + \frac{1}{2} \gamma C_1 D \tanh HC \tanh HD, \quad N' = \{ (2\xi^2 - \gamma)^2 - 4\xi^2 C_1 D_1 \} \{ \xi^2 \tanh HC - CD \tanh HD \}. \quad (9)$$

* We are not concerned here with the so-called normal modes of vibration of the stratum which can be explained as being due to the constructive interference of multiple reflected body waves travelling up and down the stratum at critical angles. These exist if $\beta < \beta_1$ and have their phase velocities lying between $\beta$ and $\beta_1$. 

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The range of existence of Stoneley waves in an internal stratum

We shall limit ourselves, as in I, to the Poisson case, \( \lambda = \mu, \lambda_1 = \mu_1 \), so that \( \alpha^2 = 3\beta^2 \), \( \alpha_1^2 = 3\beta_1^2 \) and \( C = (\xi^2 - \frac{1}{2})^{1/2} \), \( C_1 = (\xi^2 - \frac{1}{2}\gamma)^{1/2} \), while \( D, D_1 \) remain unchanged.

A discussion of the signs of \( L', M', N' \), as in I, leads to the following conclusions:

(i) The sign of \( L' \) is determined by its second factor, the first being positive for \( \xi^2 > \max (1, \gamma) \). \( L' \) is positive for \( 1 < \xi^2 < 1.183 \) for all \( \mu \). For any fixed \( \mu \), \( L' \) changes sign from positive to negative for some value of \( \xi^2 > 1.183 \) and thereafter remains negative. The value of \( \xi^2 \) at which this change of sign takes place can be made as large as we please by taking \( \mu \) sufficiently small, that is, by making either the frequency of the waves or the thickness of the stratum sufficiently small.

(ii) \( M' \) is a sum of three terms, each of which is positive for \( \xi^2 > \max (1, \gamma) \) and all \( H \). Hence \( M' \) is positive for \( \xi^2 > \max (1, \gamma) \) and all \( H \).

(iii) The first factor of \( N' \) is positive for \( \xi^2 < 1.183 \) and negative for \( \xi^2 > 1.183 \), while the second factor is positive for all \( \xi^2 \geq 1 \), \( \xi^2 = \max (1, \gamma) \) and negative for \( \xi^2 > 1.183 \).

Remembering that \( \theta = \mu / \mu_1 \) is positive, it follows that, for any fixed \( \mu \), \( F'(\xi^2) \) is negative for all sufficiently large \( \xi^2 \) and may change sign from positive to negative in the range \( \xi^2 > \max (1, \gamma) \). Hence in the range \( \xi^2 > \max (1, \gamma) \) and for all \( \mu \), \( N' \) is positive for \( \xi^2 < 1.183 \gamma \) and negative for \( \xi^2 > 1.183 \gamma \).

4. Case I: \( \gamma < 1 \), i.e. \( \beta^2 < \beta_1^2 \), low velocity stratum.—The condition for the existence of Stoneley waves is obtained by requiring that \( F'(\xi^2) \) be positive at \( \xi^2 = 1 \). This gives

\[
\theta^2 [\{1 - (1 - 3/2\gamma)\}^{1/2}(1 - \gamma)^{1/2}] \tanh K \\
- 2\theta^2 \{2 - \gamma - 2(1 - 3/2\gamma)^{1/2}(1 - \gamma)^{1/2} \} \tanh K + 6^{1/2} \gamma (1 - \gamma)^{1/2} \\
+ ((2 - \gamma)^2 - 4(1 - 3/2\gamma)^{1/2}(1 - \gamma)^{1/2}) \tanh K > 0,
\]

(10)

where, as before, \( K = \sqrt{8} \mu \).

Numerical calculations are made for the same values of \( K \) and \( \gamma \) as in I and the results are shown in the first half of the attached table. The case \( K = 5 \) is illustrated separately in Fig. 2, where the shaded areas to the left of the line \( \gamma = 1 \) show the regions in which \( \gamma, \theta \) must lie. The boundary curves of the regions of admissible values of \( \gamma, \theta \) for different values of \( K \) are shown in Fig. 3 to the left of the line \( \gamma = 1 \). For any given \( K \), these regions lie between the corresponding boundary curve and the line \( \gamma = 1 \).

The calculations show the following features in common with the case of symmetric vibrations.

(i) There is no lower region of existence for \( \gamma < 0.8453 \). This shows that if, as is usual, a velocity ratio less than unity is associated with a ratio of rigidities less than unity or, what comes to the same thing in the present case, with a density ratio less than unity, then Stoneley waves cannot exist unless \( \gamma = \beta^2 / \beta_1^2 > 0.8453 \), that is, unless the distortional wave velocity in the stratum is greater than the velocity of simple Rayleigh waves in the outer medium.

(ii) The regions of existence shrink, though at first very slowly, as \( K \) decreases from a very large value. If \( K \) is taken so small that \( \tan K \) may be replaced by \( K \), then (10) becomes

\[
K[\{1 - (1 - 3/2\gamma)^{1/2}(1 - \gamma)^{1/2}\} \theta^2 - 2\{2 - \gamma - 2(1 - 3/2\gamma)^{1/2}(1 - \gamma)^{1/2}\} \theta \\
+ (2 - \gamma)^2 - 4(1 - 3/2\gamma)^{1/2}(1 - \gamma)^{1/2}] - \sqrt{8} \theta \gamma (1 - \gamma)^{1/2} > 0,
\]

(11)
### Table I

Admissible values of $\theta = \mu/\mu_1$ for different values of $\gamma = \beta^2/\beta_2^2$ and $K = \sqrt{\frac{\omega}{\omega_0}} \alpha / \beta$.  

<table>
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<th>$\gamma$</th>
<th>$K \to \infty$</th>
<th>$K=5$</th>
<th>$K=2.64$</th>
<th>$K=1.034$</th>
<th>$K=0.1$</th>
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<td></td>
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<td>$\theta &lt; \theta &gt;$</td>
<td>$\theta &lt; \theta &gt;$</td>
<td>$\theta &lt; \theta &gt;$</td>
<td>$\theta &lt; \theta &gt;$</td>
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<tr>
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<td>6.044</td>
</tr>
<tr>
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<td>0.343</td>
<td>0.343</td>
<td>$\infty$</td>
<td>0.343</td>
</tr>
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</table>

**Fig. 2.** Shaded areas are the regions in which $\gamma = \beta^2/\beta_2^2$ and $\theta = \mu/\mu_1$ must lie for the existence of Stoneley waves with antisymmetric vibrations in the case $K = \sqrt{\frac{\omega}{\omega_0}} \alpha / \beta = 5$.  

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which cannot be satisfied for any given $\gamma$ and $\theta$ if $K = \sqrt{\frac{\omega h}{\beta}}$ be taken sufficiently small. Hence, for a pair of materials for which waves of some frequency can exist, as the frequency of the waves or the thickness of the stratum is decreased, there is a cut-off value of either, below which such waves cannot be propagated.

One point of difference from the case of symmetric vibrations may be noticed. The upper region of admissible values of $\gamma$, $\theta$, though decreasing in extent with decreasing $K$, never disappears entirely and its boundary curve starts at the point $\gamma = 1$, $\theta = 1$ for all values of $K$.

5. Case II: $\gamma > 1$, i.e. $\beta^2 > \beta_1^2$, high-velocity stratum.—The condition for the existence of Stoneley waves, obtained by putting $\xi^2 = \gamma$ in $F(\xi^2)$, is now

$$
\theta^2[2(2\gamma - 1) \tanh \{H(\gamma - \frac{1}{2})^{1/2}\} - 4\gamma(\gamma - \frac{1}{2})^{1/2}(\gamma - 1)^{1/2} \tanh \{H(\gamma - 1)^{1/2}\}]$$

$$-2\theta[\gamma(2\gamma - 1) \tanh \{H(\gamma - \frac{1}{2})^{1/2}\} - 2\gamma(\gamma - \frac{1}{2})^{1/2}(\gamma - 1)^{1/2} \tanh \{H(\gamma - 1)^{1/2}\}$$

$$+6(\gamma - 1)^{1/2}(\gamma - 1)^{1/2} \tanh \{H(\gamma - \frac{1}{2})^{1/2}\} \tanh \{H(\gamma - 1)^{1/2}\}]$$

$$+\gamma^2 \tanh \{H(\gamma - \frac{1}{2})^{1/2}\} - \gamma(\gamma - \frac{1}{2})^{1/2}(\gamma - 1)^{1/2} \tanh \{H(\gamma - 1)^{1/2}\} > 0. \quad (12)$$

The results of numerical calculations for this case are given in the second half of the above table. As for a low-velocity stratum, the case $K = 5$ is illustrated separately in Fig. 2 where the shaded areas to the right of the line $\gamma = 1$ show the regions in which $\gamma$, $\theta$ must lie. The boundary curves of the regions of admissible
values of $\gamma$, $\theta$ for different values of $K$ are shown in Fig. 3 to the right of the line $\gamma = 1$. For any given $K$, these regions lie between the corresponding boundary curve and the line $\gamma = 1$.

There are some marked differences from the case of symmetric vibrations.

(i) As $K$, and therefore $H$, decreases from a large value, the regions of existence increase in size until, as $K \rightarrow 0$, a certain limiting region is obtained. This is in sharp contrast with the case of symmetric vibrations where these regions become smaller as $K$ decreases. See Fig. 4 for this contrast in the case $K = 0.1$.

(ii) The boundary curves of the lower regions of existence all have the asymptote $\theta = 0.343$ as is seen by making $\gamma \rightarrow \infty$ in (12) which yields $0 < \theta < 0.343$. This is the same as for symmetric vibrations except that this asymptote is now approached from above by all the boundary curves.

$\theta = \mu^2/\mu$, valid in the region where $\gamma$ is not too large, and therefore certainly near the point $(1, 1)$. This curve is a hyperbola through $(1, 1)$ with the asymptotes $\theta = \frac{1}{2}$ and $\theta = 2\gamma - \frac{1}{2}$. The values of $\theta$ corresponding to $\gamma = 1, 1.01, 1.04$ in the case $K = 0.1$, i.e. $H = 0.12245$, in Table 1, are in close agreement with (13).
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The coefficient of \( \theta^2 \) in (12) equated to zero gives

\[
(2y-1)^2 \tanh \{H(y-1)^{1/2}\} - 4\sqrt{y-1} \tanh \{H(y-1)^{1/2}\} = 0.
\]  

When \( H \to \infty \), this yields \( y = 1.183 \). As \( H \) decreases, the value of \( y \) increases until, when \( H \to 0 \), \( y \to \infty \). Thus for \( H \to \infty \), the boundary curve for the upper region of existence has the asymptote \( y = 1.183 \). As \( H \) decreases, this asymptote moves to the right, receding to an infinite distance as \( H \to 0 \).

6. We have seen in 1 that the velocity of Stoneley waves with symmetric vibrations cannot be smaller than the velocity of simple Rayleigh waves for the lower velocity medium. For antisymmetric vibrations, there is no lower limit to this velocity on our assumptions. Going back to (8), we observe that \( F'(\xi^2) = 0 \) may be satisfied for as large a value of \( \xi^2 \) as we like provided \( \theta = \mu/\mu_1 \) has a suitable value and \( H = \omega h/\beta \) is sufficiently small. For however large \( \xi^2 \) may be, we can make \( L' \) positive by choosing \( H \) sufficiently small. Then \( L'\theta^2 \) is positive and \(-2M'\theta + N'\) is negative, and the two can cancel each other if \( \theta \) has a suitable value. Since the phase velocity of the waves is \( \beta/\xi \), it follows that there is no theoretical lower limit to this velocity.

In the preceding argument, the larger the chosen value of \( \xi^2 \), the smaller must \( H \) be to make \( L' \) positive. In fact, if \( \xi^2 \gg 1 \), it is sufficient that \( \xi H \) be so small that \( \tanh HC = \tanh \{H(\xi^2 - 1)^{1/2}\} \) and \( \tanh HD = \tanh \{H(\xi^2 - 1)^{1/2}\} \) may be replaced by their arguments. Then the second factor of \( L' \), which determines its sign,

\[
L' = H(\xi^2 - 1)^{1/2}(2\xi^2 - 1)^{1/2} - 4\xi^2(\xi^2 - 1) = H(\xi^2 - 1)^{1/2} > 0.
\]  

A small value of \( H \) means that the frequency \( \omega \) is small and consequently the wave length is large, or that the thickness \( 2h \) of the stratum is small. Hence a small value of \( H \) implies that the ratio of the wave length to the thickness of the stratum is large. The above argument shows that this ratio is large for waves which have very small velocities. This argument is of necessity of a rough kind because the relation between the frequency and the wave length of the waves, which is only another form of (5), is not a simple one. It may be added that this is not the only case where infinitely slow waves are possible. Lamb (1917, Proc. Roy. Soc. A, 93, 114–128) finds that Rayleigh-type waves with antisymmetric vibrations which can be propagated in an infinite elastic plate can have indefinitely small speeds.

7. Conclusions.—The main results of the preceding discussion are as follows:

(a) For a low-velocity stratum, the results are similar to those for waves with symmetric vibrations. It is found that: (i) If, as is usual for most rocks in the Earth, a velocity ratio less than unity is associated with a density ratio less than unity, then such waves cannot exist unless the shear wave velocity for the stratum is greater than the velocity of simple Rayleigh waves for the outer medium. (ii) For a pair of media for which waves of some frequency can exist, as the frequency of the waves or the thickness of the stratum is decreased, there is a cut-off value of either below which such waves cannot be propagated.

(b) For a high-velocity stratum, the results are in sharp contrast with those for waves with symmetric vibrations. Waves with antisymmetric vibrations of all frequencies can exist for some pairs of materials. As the frequency of the waves or the thickness of the stratum is decreased, that is, as the ratio of the wave length to the thickness of the stratum increases, the regions of existence expand until, when this ratio tends to infinity, a certain limiting region of existence is obtained.
To illustrate this contrast, the regions of existence of waves with symmetric and antisymmetric vibrations in the case $K = 0.1$ have been shown in Fig. 4(a) and 4(b) respectively.

(c) There is no theoretical lower limit to the velocity of Stoneley waves with antisymmetric vibrations.

8. Acknowledgments.—The author thanks Dr E. R. Lapwood for suggesting this investigation and for his encouragement during the course of the work. He is also grateful to the Government of India and the Panjab University for a scholarship.

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1957 February.