Harmonic Expansions of Fields and Fermion Mass Problem in Generalized Kaluza-Klein Theory

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Harmonic expansions of fields including scalar and spinor fields are investigated in the generalized Kaluza-Klein Theory. Special attention is paid to the difference between two types of extra-coordinate space, i.e., the group manifold type and the coset space type. The Kaluza-Klein transformation in the vielbein formalism is reasonably introduced as the special coordinate transformation combined with (local) vielbein rotation. The invariant structure of the 4-dimensional fermion mass operator in each case of the above two types of extra-coordinate space is clarified. New geometrical constraints on the spinor field are proposed, in parallel with the Killing vector condition on the metric field, which give a possibility of deriving the chiral invariant gauge theory and protect fermions from acquiring huge masses of the order of the Planck mass in the dimensional reduction.

§ 1. Introduction

Recently special attention has been paid to the generalized Kaluza-Klein (K-K) theory as the unified theory of gravity, gauge and Higgs fields. It is occasionally anticipated that it possibly explains the origin of generations of quarks and leptons as the result of the dimensional reduction of spinor field, which is described covariantly on the multi-dimensional K-K space. Viewed from such a realistic standpoint, however, there arises the serious problem concerning the fermion mass, i.e., how to protect fermions from acquiring huge masses in the dimensional reduction which is expected to occur at energy scale of the order of the Planck mass $\sim 10^{19}$ GeV.

In studying the generalized K-K theory including the above fermion mass problem, it is quite important to notice that there exists a diversity in describing the extra-coordinate space according to different dimensional reduction schemes, for instance, ordinary reduction or generalized reduction leading to Abelian or non-Abelian gauge theories, respectively. Further, in deriving the latter non-Abelian gauge theory, there exist two viewpoints essentially different in applying the so-called Killing vector condition: The first one is of the principal fibre bundle type (let us call it PFB-type hereafter) which imposes the Killing vector condition directly on the $(4+n)$-dimensional metric tensor $\gamma_{AB}$. In this case, as is well-known, the $(4+n)$-dimensional Einstein equation must be modified on account of the above Killing vector condition, which is most naively to be taken into account by applying Lagrange's method of undetermined multipliers to the $(4+n)$-dimensional Einstein-Hilbert action. According to this procedure, as was shown in our previous papers and will be briefly summarized in § 2, one finds out that after the dimensional reduction the $n$-dimensional extra-coordinate space becomes a group-manifold of the symmetry group $G$ generated by the assumed $n$ Killing vector fields, and the extra-coordinate dependence of all the “$4$-dimensional” fields derived from $\gamma_{AB}$, i.e., gravitational fields $g_{\mu\nu}$, gauge potential fields $A_{\mu}^{a}$, and Higgs-type scalar fields $g_{\phi}$ ($n$-dimensional metric fields) becomes essentially fixed and given in terms of the (adjoint) representation matrix of the symmetry group. The $(4+n)$-dimensional scalar curvature $R^{(4+n)}$ becomes
then extra-coordinate independent and leads us to the familiar 4-dimensional generally covariant gravity and Yang-Mills Lagrangian under the possible condensation (let us use this kind of terms hereafter in anticipation of quantization) of scalar fields of the form $\langle g_{ab}(x) \rangle = x^2 \delta_{ab}$ (we assume here semisimple and compact symmetry group with Killing metric tensor $g_{ab}(\propto \delta_{ab})$ with gauge coupling constant $x^{-1}$.

The second one is of the homogeneous space type (let us call it HGS-type hereafter), which quite differently from the first one, claims that the $(4+n)$-dimensional Einstein equation ultimately holds although the “ground mode” of $n$-dimensional metric fields $g_{ab}$ is assumed to possess $(n+p)$ Killing vector fields. In this case, the $n$-dimensional extracoordinate space becomes a homogeneous space or a coset space like $G/H$ associated with the $(n+p)$-dimensional symmetry group $G$ generated by the above Killing vectors and its certain subgroup (isotropy group) $H$ with dimension $p$. Contrary to the first one, the extra-coordinate dependence of the $4$-dimensional fields including gravitational fields is not fixed, but they are to be subject to harmonic expansions in terms of complete eigenfunctions on the homogeneous space, as will be shown in § 3. Recently, Salam and Strathdee\textsuperscript{5} gave a general procedure to this problem from the point of view of nonlinear realization. In the present paper, we shall give more concrete consideration to it, avoiding any subjective “ansatz” on the possible form or coordinate dependence of vielbeine or metric fields imposed so far by each author, except for the assumption of the Killing vector condition on the ground mode mentioned above.

It is important for building realistic models of quarks and leptons to investigate the behaviour of the spinor field on K-K space. In the case of HGS-type K-K space, it is reasonable to consider that the extra-coordinate dependence of the spinor field is free just like that of the metric fields, so there arise an infinite number of higher excitation modes associated with harmonic expansions. The mass spectrum of these spinor modes are directly calculated as eigenvalues of the mass operator, which is, as will be shown explicitly in § 3, derived from the original $(4+n)$-dimensional covariant spinor field equation in the dimensional reduction. As was pointed out by many authors, however, one can hardly expect in general the existence of massless modes\textsuperscript{58–10} as the candidates for quarks and leptons or preons.

Contrarily, in the case of PFB type K-K space, we have an interesting possibility that the spinor field is also subject to some geometrical constraints in parallel with the Killing vector condition on the metric field, as will be proposed in § 3. The extra-coordinate dependence of the spinor field is then fixed, and one finds that this approach enables us to derive a realistic chiral-invariant gauge theory which protects spinor fields from acquiring huge masses in the dimensional reduction and yields ample freedom of generations of quarks and leptons.

The present paper is organized as follows. In the next section, § 2, we consider the K-K space of PFB (principal fibre bundle) type and HGS (homogeneous space) type in the vielbein formalism. The K-K transformation as the gauge transformation is reasonably introduced in close connection with harmonic expansions of fields. Section 3 is devoted to study of the fermion mass problem on the basis of the knowledge of the invariant structure of the mass operator. In Appendix A, group-theoretical consideration of harmonic expansions of fields on homogeneous space is given in connection with those on the group manifold.
§ 2. Harmonic expansions of fields on K-K space

In order to relate \((4+n)\)-dimensional covariant space \(\{Z^A\} \quad (A=1, 2, \cdots, 4+n)\) (with the metric tensor field \(\gamma_{AB}(Z)\)) to 4-dimensional space-time \(\{x^\mu\}\), we first introduce the following \textit{orthogonal} vielbein system \((\eta_\bar{\mu}, \xi_\bar{a})\):

\[
\begin{align*}
\eta_{\bar{\mu}} &\equiv \eta_{\bar{\mu}}(Z) \frac{\partial}{\partial Z^{\bar{\mu}}} , \quad (\bar{\mu}=1, 2, 3, 4) \\
\xi_{\bar{a}} &\equiv \xi_{\bar{a}}(Z) \frac{\partial}{\partial Z^{\bar{a}}} , \quad (\bar{a}=1, 2, \cdots, n)
\end{align*}
\]

(2.1)*)

\(\xi_{\bar{a}}\)'s are in the \textit{vertical} direction in the sense that

\[
\xi_{\bar{a}} \frac{\partial x^\mu}{\partial Z^{\bar{a}}} = 0 ,
\]

(2.2)

where \(x^\mu = \pi^\mu(Z)\) are 4 scalar functions giving a mapping from \(\{Z^A\}\)-space onto \(\{x^\mu\}\)-space. Let us now define the special coordinate system, where one has

\[
\begin{align*}
x^\mu &\equiv \pi^\mu(Z) = Z^{\bar{\mu}+\bar{a}} \\
\theta^a &\equiv Z^{a+4+n}
\end{align*}
\]

(2.3)

with \(\mu=1, 2, 3, 4\) and \(a=1, 2, \cdots, n\). In this coordinate system, vielbein fields become

\[
\eta^{\mu}_{\bar{\mu}} \equiv \frac{\partial \pi^\mu}{\partial Z^{\bar{\mu}}} (\equiv \pi^\mu_{\bar{\mu}}) = \begin{pmatrix} \frac{\partial x^\mu}{\partial z^{\bar{\mu}}} \\ \vdots \\ \frac{\partial x^\mu}{\partial z^{\bar{a}}} \end{pmatrix}
\]

(2.4)

\[
\begin{align*}
\xi_{\bar{a}} &\equiv \begin{pmatrix} \xi_{\bar{a}}^{\mu}(x, \theta) \\ \vdots \\ \xi_{\bar{a}}^{\mu}(x, \theta) \end{pmatrix} \\
\eta_{\bar{a}} &\equiv \begin{pmatrix} h_{\bar{a}}^{\mu}(x, \theta) \\ \vdots \\ -h_{\bar{a}}^{\mu}(x, \theta) A_\mu(x, \theta) \end{pmatrix}
\end{align*}
\]

(2.5)

(2.6)

by dividing \((4+n)\) vector components as \(A=(\nu, b)\), where \((\xi_{\bar{a}}^{\mu}, \eta_{\bar{a}}^{\mu})\) is the orthogonal vielbein system inverse to \((\xi_{\bar{a}}^{\mu}, \eta_{\bar{a}}^{\mu})\). \(\xi_{\bar{a}}^{\mu}(h_{\bar{a}}^{\nu})\)'s and their inverted \(\xi_{\bar{a}}^{\nu}(h_{\bar{a}}^{\nu})\)'s denote \((4+n)\)-dimensional orthogonal vielbein systems. The metric tensors \(\gamma_{AB}\) and \(\gamma^{AB}\) can then be written as

\[
\begin{align*}
\gamma^{AB} &\equiv \eta_{\bar{a}}^{\mu} \eta_{\bar{b}}^{\nu} + \xi_{\bar{a}}^{\mu} \xi_{\bar{b}}^{\nu} \\
&= \begin{pmatrix} g^{\nu\sigma} & -g^{\nu\sigma} A_\sigma^b \\ -g^{\nu\sigma} A_\sigma^b & g^{ab} + g^{\alpha\sigma} A_\sigma^b A_{\alpha}^b \end{pmatrix}
\end{align*}
\]

(2.7)

\[
\gamma_{AB} = \eta_{\bar{a}}^{\mu} \eta_{\bar{b}}^{\nu} + \xi_{\bar{a}}^{\mu} \xi_{\bar{b}}^{\nu}
\]

\[
\begin{pmatrix} g^{\nu\sigma} + g_{\alpha\sigma} A_\sigma^a A_{\alpha}^a & g_{ac} A_\sigma^c \\ g_{bc} A_\sigma^c & g_{ab} \end{pmatrix}
\]

(2.8)

*) A summation over doubled indices is always implied hereafter.
with
\[ g^{\mu \nu} = h^a_b h^b_\nu, \]  
\[ g^{ab} = \frac{\delta^a_{\mu} \delta^b_\nu}{\delta \theta^a / \delta Z^b} , \]  
and their inverted \( g_{\mu \nu} \) and \( g_{ab} \).

2.1. **PFB-type K-K space**

Here we assume that there exist locally linear-independent \( n \) vertical Killing vector fields, i.e.,
\[ \mathcal{L}_{\xi_a} \gamma^{ab} = 0, \text{ or } \xi_{a \lambda, b} + \xi_{ab, \lambda} = 0 \]  
with
\[ \xi_a^a \pi_a = 0, \quad (a = 1, 2, \cdots, n) \]  
As is well known,
\[ \xi_a = \xi_a^A \frac{\partial}{\partial Z^a} , \quad (a = 1, 2, \cdots, n) \]  
as a whole constitute a Lie Algebra \( \mathcal{G} \):
\[ [\xi_a, \xi_b] = f_{ab}^c \xi_c \]  
with structure constants \( f_{ab}^c \). On account of (2.12), one has in the special coordinate system
\[ \xi_a^A = \begin{pmatrix} 0 \\ \cdots \\ \xi_a^b (\theta) \end{pmatrix} \]  
and
\[ [\xi_a^\phi, \xi_b^\gamma] = f_{ab}^c \xi_c^\gamma \]  
with
\[ \xi_a^\phi = \xi_a^b (\theta) \frac{\partial}{\partial \theta^b} . \]  
This result implies that the \( n \)-dimensional extra-coordinate space \( \theta^a \) becomes a group manifold of group \( G \) associated with the Lie Algebra \( \mathcal{G} \) with dimension \( n \).

The Killing vector condition (2.11) gives rise to the relations:
\[ \xi_a^\phi g^{\mu \nu} = 0, \]  
\[ \xi_a^\phi g^{\beta \gamma} = -f_{ab}^\phi g^{\beta \gamma} - f_{ad}^\phi g^{\beta \delta} \]  
\[ \xi_a^\phi A_{\mu}^\beta = -f_{a \mu}^\phi A_{\mu}^\beta , \]  
where \( g^{\alpha \beta} \) and \( A_{\mu}^\alpha \) are defined through

\[ ^* \text{Details of this case can be seen in Ref. 4).} \]
\[ ^{**} \mathcal{L}_{\xi_a} \text{denotes the Lie derivative with respect to } \xi_a. \]
\[ g^{ab} = \xi^{a\alpha}_s g^{\alpha\beta}_s \xi^b_s , \]
\[ A^a_s = A^a_s \xi^{a\alpha}_s . \]

(2.19)

Equations in (2.18) essentially determine the extra-coordinate dependence of \( g^{\mu\nu} \), \( g^{as} \) and \( A^a_s \) as follows,
\[ g^{\mu\nu}(x, \theta) = g^{\mu\nu}(x) , \]
\[ g^{as}(x, \theta) = g^{as}(x) d_s^a(\theta) d_s^a(\theta) , \]
\[ A^a_s(x, \theta) = A^a_s(x) d_s^a(\theta) , \]

(2.20)

with the generalized rotation matrix or the adjoint representation matrix \( d_s^a(\theta) \) of group \( G \), which satisfies (see (A·9) in the Appendix)
\[ \xi^{a\alpha}_s d_s^a(\theta) = - f_{ab}^c d_s^c(\theta) . \]

(2.21)

In this way, we finally obtain the expressions
\[ \eta^{a\alpha}_s = \begin{pmatrix} h^{a\alpha}_s(x) \\ \cdots \\ - h^{a\alpha}_s(x) A^a_s(x) \xi^{a\alpha}_s(\theta) \end{pmatrix} , \]
\[ \xi^{a\alpha}_s = \begin{pmatrix} 0 \\ \cdots \\ \xi^{a\alpha}_s(x) C^a_s(\theta) \end{pmatrix} , \]

(2.22, 2.23)

where
\[ g^{\mu\nu}(x) = h^{a\alpha}_s(x) h^{a\alpha}_s(x) , \]
\[ g^{as}(x) = \xi^{a\alpha}_s(x) \xi^{a\alpha}_s(x) . \]

(2.24)

\( \xi^{a\alpha}_s(\theta) \) in Eqs. (2.22) and (2.23) are related to \( \xi^{a\alpha}_s(\theta) \) by
\[ \xi^{a\alpha}_s(\theta) = d_s^a(\theta) \xi^{a\alpha}_s(\theta) , \]

(2.25)

and \( \xi^{a\alpha}_s = \xi^{a\alpha}_s(\theta)(\partial/\partial \theta^a) \) satisfy the following relations:
\[ [\xi^{a\alpha}_s, \xi^{b\beta}_s] = - f_{ab}^c \xi^{c\beta}_s , \]
\[ [\xi^{a\alpha}_s, \xi^{b\beta}_s] = 0 . \]

(2.26, 2.27)

One finds out from these relations that \( \xi^{a\alpha}_s \)'s and \( \xi^{a\alpha}_s \)'s constitute the so-called left- and right-invariant Lie Algebras, respectively (see Appendix A).

Next let us consider the K-K transformation among special coordinate systems, which leaves Eqs. (2.4) and (2.15) invariant,
\[ x^\mu \rightarrow x'^\mu = x^\mu , \]
\[ \theta^a \rightarrow \theta'^a = \theta^a + C^a(x) \xi^{a\alpha}_s(\theta) \]

(2.28)

with infinitesimal function \( C^a(x) \). Noticing the \((4+n)\)-vector character of \( \eta^{a\alpha}_s \) and \( \xi^{a\alpha}_s \) in Eqs. (2.22) and (2.23), one finds out that the transformation (2.28) induces the following gauge transformation
\[ \delta g_{\mu\nu}(x) = 0, \quad (2.29) \]
\[ \delta A_\sigma(x) = - C^\sigma_\rho(x) - f^\rho_{\alpha\beta} A_\alpha(x) C^\sigma_\beta(x), \quad (2.30) \]
\[ \delta \xi^\alpha_\sigma(x) = - f^\sigma_{\alpha\beta} \xi^\alpha_\beta(x) C^\sigma_\chi(x), \quad (2.31) \]
\[ \delta g^{\alpha\sigma}(x) = - \{ f^\sigma_{\alpha\beta} g^{\beta\sigma}(x) + f^\sigma_{\beta\gamma} g^{\alpha\gamma}(x) \} C^\sigma_\chi(x). \quad (2.32) \]

The invariant action of the system becomes
\[ \int d^4x d^n\theta (\det^\circ \tilde{\xi}^\sigma_\alpha(\theta))^{-1} (\det h_\sigma^\alpha(x))^{-1} (\det \tilde{\xi}^\sigma_\alpha(x))^{-1} [R^{(4+n)} + B^{\alpha\beta}(\xi_{2\alpha} + \xi_{3\alpha} + \xi_{4\alpha})], \quad (2.33) \]

taking into account the Killing vector condition explicitly in terms of Lagrange's method of undetermined multipliers. The \( (4+n) \)-dimensional scalar curvature \( R^{(4+n)} \) becomes then extra-coordinate independent and gives rise to the usual 4-dimensional gravity and Yang-Mills Lagrangian coupled to scalar fields \( g_{\alpha\beta}(x) \) under the possible condensation of \( \langle g_{\alpha\beta}(x) \rangle = x^3 \delta_{\alpha\beta} \quad (2.34) \)

or
\[ \langle \xi^\alpha_\sigma(x) \rangle = x^{-1} \delta^\alpha_\sigma \quad (2.35) \]

with gauge coupling constant \( x^{-1} \) (see Ref. 4).

It is important here to note that the condensation of \( g_{\alpha\beta} \) in (2.34) is gauge invariant, but that of \( \xi^\alpha_\sigma \) in (2.35) is not, i.e., the relation (2.35) is not invariant under the transformation (2.28). However, this difficulty is removed if we modify (2.28) by adding the \( n \)-dimensional vielbein rotation
\[ \xi^\alpha_\sigma \rightarrow \xi^{\prime\alpha}_\sigma = \xi^\alpha_\sigma - C^\alpha(x) f^\sigma_{\alpha\beta} \xi^\beta_\chi. \quad (2.36) \]

Under this extended K-K transformation, (2.31) is now replaced by
\[ \delta \xi^\alpha_\sigma(x) = \{ f^\sigma_{\alpha\beta} \xi^\beta_\chi(x) - f^\sigma_{\beta\chi} \xi^\alpha_\beta(x) \} C^\sigma_\chi(x), \quad (2.31)' \]

which leaves the condensation (2.35) invariant, because of the relation \( f^\sigma_{\alpha\beta} = - f^\sigma_{\beta\alpha} \). The above consideration becomes important to treat the spinor field in the vielbein formalism, as will be seen in § 3.

2.2. HGS-type K-K space

In this case, we start with the following assumption that the ground mode of the \( n \times n \) metric fields \( g^{\alpha\beta} \) defined by (2.10) in the special coordinate system possesses (globally) linear-independent \( (n+p) \) Killing vector fields
\[ \xi^\alpha_\sigma(\theta) = \frac{\partial}{\partial \theta^\alpha} \Rightarrow (a=1,2,\ldots,n+p \text{ and } \alpha=1,2,\ldots,n) \quad (2.37) \]

This assumption can be expressed as
\[ \langle g^{\alpha\beta}(x, \theta) \rangle = g^{\zeta^\alpha_\zeta^\beta}(\theta) = \xi^\alpha_\sigma(\theta) \xi^{\zeta\zeta_\sigma}(\theta) \quad (2.38) \]

with
\[ [\xi^\alpha_\zeta, \xi^\beta_\zeta] = f^\zeta_\sigma \xi^\alpha_\sigma. \quad (2.39) \]
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In fact, one can confirm that the assumption (2.38) with (2.39) gives rise to the Killing vector condition:

\[
\mathcal{L}_{\xi} g^{ab}(\theta) = 0,
\]

or

\[
\begin{align*}
\mathcal{L}_{\xi} g^{ab} &= \xi^{c} g_{ac} + \xi^{c} g_{bc}, \\
\xi^{c} g_{ac} &= -\xi^{c} g_{ac} - \xi^{c} g_{bc}.
\end{align*}
\] (2.40)

If we use the expression

\[
g^{\alpha\beta}(\theta) = \xi^{\alpha}(\theta) \xi^{\beta}(\theta)
\]

with

\[
\langle \xi^{\alpha}(x), \theta \rangle = \xi^{\alpha}(\theta),
\]

then Eq. (2.40) gives the relation

\[
[k^{\alpha}, k^{\beta}]^\delta g^{\gamma\delta} + [k^{\gamma}, k^{\delta}]^\alpha g^{\beta\gamma} = 0,
\] (2.43)

where

\[
[k^{\alpha}, k^{\beta}] = k^{\alpha} k^{\beta} - k^{\beta} k^{\alpha},
\] (2.44)

and \(\xi^{\alpha}(\theta)\)'s are the inverses of \(\xi^{\alpha}(\theta)\).

It should be noted that in the present case the Killing vector condition is irrelevant to fixing of extra-coordinate dependence of fields \(h^a\), \(A^a\), and \(\xi^{\alpha}\), and these fields are to be subject to harmonic expansions on extra-coordinate space. The \(n\)-dimensional extra-coordinate space, on which linearly independent \(n+p\) vector fields \(\xi^{\alpha}\) are introduced, by assumption, so as to satisfy the Lie Algebra \(G\) (2.39) with dimension \((n+p)\), becomes the homogeneous space, i.e., the parameter space of cosets \(G/H\) generated by group \(G\) (associated with \(G\)) and its suitable subgroup \(H\) with dimension \(p\). As will be shown in Appendix A, the \(n\)-dimensional homogeneous space \(\{\theta^a\} \sim G/H\) is related to the \((n+p)\) dimensional group manifold \(\{u^a\}\) of \(G\) by a (left- or right-invariant) mapping \(\theta^a = \pi^a(u^a)\). The complete basis functions of harmonic expansions on the group manifold is known to be provided by the series of representation matrices of \(G\), i.e.,

\[
D^{(\alpha \beta)}(u) = \langle A | g(u) | \bar{B} \rangle_{\alpha \beta},
\] (2.45)

where \(g(u)\) denotes any group element of \(G\), and \( \langle A \rangle_{\alpha \beta}, \ldots \), the set of basis states belonging to a representation \((\sigma)\) of \(G\). Then the basis functions on the homogeneous space are provided by

\[
D^{(\alpha \beta)}(u) = \langle A | g(u) | \bar{H} \rangle_{\alpha \beta},
\] (2.46)

where \(\bar{H}\) implies any singlet state with respect to subgroup \(H\) among the \((\sigma)\)-representation basis states, and they satisfy

\[
\xi^{\alpha} D^{(\alpha \beta)}(\theta) = \langle A | J_{\alpha} | A' \rangle_{\sigma} D^{(\alpha \beta)}(\theta),
\] (2.47)

with representation matrix \(\langle A | J_{\alpha} | A' \rangle_{\sigma}\) of the Lie Algebra \(G\).

Before entering into discussion on harmonic expansions of fields, let us introduce the
following extended K-K transformation, which in the present case preserves the following form assumed above

\[ \langle \xi^A(x, \theta) \rangle = \begin{pmatrix} 0 \\ \cdots \\ \xi^a_{\alpha}(\theta) \end{pmatrix} \]  

(2.48)

together with that of (2.4) characteristic of the special coordinate system, i.e., the special coordinate transformation

\[
\begin{align*}
    x^m &\rightarrow x'^m = x^m, \\
    \theta^a &\rightarrow \theta'^a = \theta^a + C^a(x) \xi^\alpha a(\theta),
\end{align*}
\]

(2.49)

combined with the local vielbein rotation:

\[
\xi^A_a(x, \theta) \rightarrow \xi'^A_a(x, \theta) = \xi^A_a(x, \theta) + C^a(x) \epsilon^\delta_b(\theta) \xi^\delta b(x, \theta).
\]

(2.50)

Here rotation coefficients \( \epsilon^\delta_b(\theta) \) are given by

\[
\epsilon^\delta_b(\theta) = [\xi^\alpha_a, \xi^\alpha_{\beta}]^b \xi^\delta \beta = -\epsilon^\delta_a(\theta),
\]

(2.51)

whose anti-symmetric property is provided by the result of Killing vector conditions, i.e., (2.43). Under this transformation, one can derive \(^6\)

\[
\begin{align*}
    \delta h^A_a(x, \theta)(= h^A_a(x, \theta') - h^A_a(x, \theta')) &= -C^a(x) \xi^\alpha a h^A_a(x, \theta), \\
    \delta A_{,a}^a(x, \theta) &= -C^a(x, \theta) + C^a(x)(\xi^a_{\alpha} A_{,\alpha}^a(x, \theta) + f_{ab}^a A_{,b}^a(x, \theta)), \\
    \delta \xi^A_a(x, \theta) &= -C^a(x)[[\xi^\alpha_{\beta}, \xi^\gamma_a] \xi^\delta \beta] - \epsilon^\delta_k(\theta) \xi^\delta b(x, \theta), \\
    \delta g_{ab}(x, \theta) &= -C^a(x)[\xi^a_{\alpha} g_{ab}(x, \theta) + g_{ab}(x, \theta) \xi^a_{\alpha} + g_{ab}(x, \theta) \xi^a_{\alpha}],
\end{align*}
\]

(2.52) - (2.55)

where \( A_{,a}^a(x, \theta) \)'s are defined through

\[
A_{,a}^a(x, \theta) = A_{,a}^a(x, \theta) \xi^\alpha a(\theta).
\]

(2.56)

One can confirm that the transformation (2.54) makes (2.48) invariant as anticipated, because two terms on the right-hand side of Eq. (2.54) cancel for the ground mode of \( \xi^a \).

Now let us study harmonic expansions of fields. As regards \( h^A_a(x, \theta) \) and \( A_{,a}^a(x, \theta) \), one can simply perform their expansions in conformity with transformations (2.52) and (2.53) as follows:

\[
\begin{align*}
    h^A_a(x, \theta) &= h^A_a(x) + \sum (\sigma) h_{\sigma a}^A(x) D^{(\sigma)} h(\theta), \\
    A_{,a}^a(x, \theta) &= A_{,a}^a(x) + \sum A_{ab}^a h(x) D^{(\sigma)} h(\theta).
\end{align*}
\]

(2.57) - (2.58)*

Inserting these expressions into Eqs. (2.52) and (2.53), one obtains the familiar gauge transformation for ground modes of gravity and gauge potential fields:

\(^*\) Possible new gauge symmetry \((\xi, \text{ see Appendix A})\) related to components \( \tilde{H} \) will be discussed elsewhere.
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\[ \delta h^a_{\beta}(x) = 0, \]
\[ \delta A^a_{\beta}(x) = -C^a_{\beta}(x) + f^a_{\gamma}A^a_{\gamma}(x)C^\gamma(x), \]

\[ \delta \begin{pmatrix} \varphi^a_l \cr \varphi^a_{\hat{a}} \end{pmatrix} = -\begin{pmatrix} \delta \varphi^a_l(x) \\ \delta \varphi^a_{\hat{a}}(x) \end{pmatrix}, \]
\[ \delta \begin{pmatrix} \varphi'^{\hat{a}}_l \cr \varphi^{\hat{a}}_{\hat{a}} \end{pmatrix} = -\begin{pmatrix} \delta \varphi'^{\hat{a}}_l(x) \\ \delta \varphi^{\hat{a}}_{\hat{a}}(x) \end{pmatrix} + f^a_{\gamma}A^a_{\gamma}(x). \]

As for scalar fields, let us first consider harmonic expansions of vielbein fields \( \xi_{\alpha}^a(x, \theta) \) subject to the transformation (2.54). In this case one finds it reasonable to apply harmonic expansions not to \( \xi_{\alpha}^a(x, \theta) \) directly but to \( \xi_{\alpha}^{ab}(x, \theta) \) defined through

\[ \xi_{\alpha}^{ab}(x, \theta) = \xi_{\alpha}^{ab}(x, \theta) \xi_{\alpha}^{c}(\theta), \]

which transforms as

\[ \delta \xi_{\alpha}^{ab}(x, \theta) = -C^a(x)(\Sigma^a_{\alpha} \xi_{\alpha}^{cd}(x, \theta)). \]

Here \( \Sigma^a_{\alpha} \)'s are given by

\[ \Sigma^a_{\alpha} = \xi_{\alpha}^{a} - \frac{i}{2} \xi_{\alpha}^{a}(\theta) T^a_{\hat{a}} \]

with \( SO(n) \) representation matrix

\[ (T^a_{\hat{a}})_{\hat{a} \hat{b}} = -i(\hat{a}_{\alpha} \hat{b}_{\alpha} + \hat{a}_{\alpha} \hat{b}_{\alpha} - (1 \leftrightarrow m)). \]

One can verify that they satisfy the Lie Algebra \( G \):

\[ [\Sigma^a_{\alpha}, \Sigma^b_{\beta}] = f^{abc}_{\gamma} \Sigma^c_{\gamma}. \]

Therefore, harmonic expansions of \( \xi_{\alpha}^{ab} \) can be given by the series of representation bases of \( \Sigma^a_{\alpha} \)'s:

\[ \xi_{\alpha}^{ab}(x, \theta) = \delta_{\alpha}^{ab} + \Sigma^a_{\alpha} \varphi^a_{\alpha}(x)U^a_{\alpha}(\theta), \]

\[ (\Sigma^a_{\alpha} \varphi^a_{\alpha}(x))^{(e)} = (\xi^a_{\alpha} \varphi^a_{\alpha}(x))^{(e)} \]

with \( (x) \)-representation matrix \( (\Sigma^a_{\alpha})_{\alpha' \alpha''} \), of \( G \). Then, Eq. (2.64) gives

\[ \delta \varphi^a_{\alpha}(x) = -C^a(x)(\Sigma^a_{\alpha} \varphi^a_{\alpha}(x)). \]

Next let us consider the \( n \times n \) metric (scalar) fields \( g^{ab}(x, \theta) \) or \( g_{ab}(x, \theta) \), which can be expressed in terms of \( \xi_{\alpha}^a(x, \theta) \) considered above through Eq. (2.10), but, compared with the latter fields are devoid of freedoms of vielbein rotations and known to be interpretable as the Nambu-Goldstone fields.\(^3\) Therefore, it is nontrivial to consider harmonic expansions of \( g_{ab} \) separately from those of \( \xi_{\alpha}^a \)'s, though the use of the latter becomes essential in considering the spinor field, as will be seen in § 3. Now let us rewrite Eq. (2.55) as

\[ \delta g_{ab}(x, \theta) = -C^a(x)(\Sigma^a_{\alpha} \varphi^a_{\alpha}(x, \theta)), \]

where

\[ \Sigma^a_{\alpha} = \xi_{\alpha}^a + \xi_{\alpha, \beta}^a(\theta)K_{\alpha \beta} \]

(2.72)
\[
(K_{1m})_{ab}^{cd} = \delta_{ic} \delta_{am} \delta_{bd} + \delta_{ia} \delta_{ac} \delta_{bm}.
\]

(2.73)

By noticing
\[
[K_{1m}, K_{p\nu}] = (\delta_{1\nu} \delta_{mp} \delta_{1\tau} - \delta_{mp} \delta_{1\nu} \delta_{1\tau}) K_{st},
\]

(2.74)

one can confirm that \( \Sigma_a \)'s also satisfy the Lie Algebra \( \mathcal{G} \):
\[
[\Sigma_a, \Sigma_b] = f_{ab}^{\;c} \Sigma_c.
\]

(2.75)

Consequently, harmonic expansions of \( g_{ab}(x, \theta) \) can be given by the series of representation bases of \( \Sigma_a \)'s:
\[
g_{ab}(x, \theta) = g_{ab}^{ss}(\theta) + \sum_{s} (\Sigma_a)^{s} V_{ab}^{s}(\theta),
\]

(2.76)

\[
(\Sigma_a)^{s} V_{ab}^{s}(\theta) = (\Sigma_a)^{s} V_{ab}^{s}(\theta)
\]

(2.77)

with \((\eta)\)-representation matrix \((\Sigma_a)^{s}\)M \. Then, the transformation (2.55) induces
\[
\delta \sigma_M(x) = - C^{a} (x) (\Sigma_a)^{s} M \cdot (\sigma_M(x)).
\]

(2.78)

The invariant action of the system can now be written as
\[
\int d^4 x d^\theta (\det (\epsilon^{ss})^{-1} (\det h_{ab}(x, \theta))(\det (\epsilon^{ss})^{-1} R^{(4+n)}(x, \theta)).
\]

(2.79)

This expression is invariant under the extended K-K transformation (2.49) combined with (2.50), so the effective 4-dimensional action which is obtained after harmonic expansions of fields given above and integrations over extra-coordinates \( \theta^{a} \), must be invariant under the gauge transformation of 4-dimensional fields, (2.59), (2.62) and (2.70) or (2.78). Consequently one can conclude that the ground modes of gravity and gauge fields, \( g_{ab}(x) \) and \( A_{a}(x) \), give rise to the usual gravity and Yang-Mills gauge theory coupled to an infinite number of higher excitation modes.\(^{(a)}\) In order to see, however, how these excitation modes, especially of gravity, are troublesome, we give in Appendix B the expression of \( R^{(4+n)}(x, \theta) \) before harmonic expansions of fields.

§ 3. Harmonic expansions of spinor field and Fermion mass problem

The spinor field \( \psi(Z) \) on the \((4+n)\)-dimensional covariant space can be described by adding to the scalar curvature \( R^{(4+n)}(Z) \) the following term
\[
\bar{\psi}(Z) \Gamma^{\hat{a}} \left( V_{\hat{a}} \partial \bar{Z}^{\hat{a}} + \frac{i}{2} S_{\hat{a}}^{\bar{b}}(Z) \Sigma^{\bar{b}} \right) \psi(Z)
\]

(3.1)

with
\[
\Sigma^{\bar{a}} = - \frac{i}{4} [\Gamma^{\hat{a}}, \Gamma^{\hat{b}}].
\]

Here \( \Gamma^{\hat{a}} \)'s constitute \((4+n)\)-dimensional Clifford Algebra.
Harmonic Expansions of Fields and Fermion Mass Problem

\[ \{ \Gamma^\bar{a}, \Gamma^\bar{b} \} = 2 \delta_{\bar{a} \bar{b}}, \]

and \( S_\bar{A}^\bar{c}(Z) \) are the Ricci rotation coefficients described in terms of orthogonal vielbein fields \( V^\mu_\bar{A}(Z) \) as

\[ S_\bar{A}^\bar{c} = -V^\bar{b}_\bar{e} V^\bar{b}_\bar{d} V^\sigma_\bar{A} \]

and guarantee the \((4+n)\)-dimensional local vielbein rotation invariance of (3·1). Vielbein fields \( V^\mu_\bar{A}(Z) \) in the K-K theory are given by \((\eta^\mu_\bar{A}, \zeta_\bar{A})\) studied in the preceding section in detail.

3.1. Harmonic expansions of spinor field

First let us investigate the spinor transformation under the extended K-K transformations, (2·28) combined with (2·36) for PFB-type and (2·49) combined with (2·50) for HGS-type.

For the PFB type K-K transformation, one has

\[ \delta \psi(x, \theta) = \psi' \psi(x, \theta') - \psi(x, \theta), \]

where

\[ \Pi_\sigma = \psi_\sigma + G_\sigma, \]

\[ G_\sigma = \frac{i}{2} f^{\bar{a} \bar{b}} \sigma_\bar{a} \]

with

\[ \sigma_\bar{a} = -\frac{i}{4} \{ \gamma^{\bar{a}}, \gamma^\bar{a} \}, \]

by putting \( I^\bar{a} = (\gamma^\bar{a}, \gamma^\bar{a}) \). One can confirm the following relations

\[ [\gamma^{\bar{a}}, G_\sigma] = -f^{\bar{a} \bar{b}} \gamma^{\bar{b}}, \]

\[ [G_\sigma, G_\tau] = -f^{\bar{a} \bar{b}} G_\tau, \]

\[ G_\sigma^2 = \frac{1}{8} f^{\bar{a} \bar{b}} f^{\bar{a} \bar{b}} \]

\[ [\Pi_\sigma, \Pi_\tau] = -f^{\bar{a} \bar{b}} \Pi_\tau. \]

Harmonic expansions of spinor field on PFB-type K-K space is thus performed by the series of representation bases of \( \Pi_\sigma \)'s,

\[ (\Pi_\sigma)_{\alpha \nu} U^{\nu}_{\gamma}(\theta) = (\Pi_\sigma)^{\alpha \nu} \] \( U^{\gamma}(\theta) \)

with \((\gamma)\)-representation matrix \((\Pi_\sigma)_{\alpha \nu} \) of \( \mathcal{G} \). \( U^{\nu}_{\gamma}(\theta) \)'s are actually constructed according to the usual addition theorem of angular momenta applied to \( \psi_\sigma \) and \( G_\sigma \) which are mutually commutable, i.e.,

\[ U^{\nu}_{\gamma}(\theta) = C^{\nu \gamma \sigma}_{\rho \sigma} D^{\rho \sigma}(\theta) U^{\sigma}, \]

with extended CG coefficients \( C^{\nu \gamma \sigma}_{\rho \sigma} \) connecting \((\sigma)\)-representation of \( \psi_\sigma \) (i.e., \( D^{\rho \sigma}(\theta) \),
see Eq. (A.2) in the Appendix) and \((\rho)\)-representation of \(G, \psi^{(\rho)}\),

\[
(G_{\rho})_{\kappa \lambda} \psi^{(\rho)}_{\kappa} = (G_{\rho})_{\lambda \kappa} \psi^{(\rho)}_{\kappa}
\]  \(3\cdot13\)

with \(n\)-dimensional-spinor index \(N (= 1, 2, \cdots, 2^{n/2})\). The harmonic expansion of spinor field is thus given by

\[
\psi(x, \theta) = \sum_{\kappa} \psi_{\kappa}^{(r)}(x) U^{(r)}_{\kappa \lambda}(\theta)
\]  \(3\cdot14\)

with the result

\[
\delta \psi_{\kappa}^{(r)}(x) = -C^a(x) \sum_{\kappa} \psi_{\alpha}^{(r)}(x).
\]  \(3\cdot15\)

Here suffices \(H (= L, R)\) represent the left- and right-handed components corresponding to the usual 4-dimensional Dirac spinor.

For the HGS type K-K transformation, one has

\[
\delta \psi(x, \theta) = -C^a(x) \Theta^a \psi(x, \theta)
\]  \(3\cdot16\)

with

\[
\Theta^a = \frac{i}{2} \left[ \gamma^a \Theta^a \right]
\]  \(3\cdot17\)

which can also be verified to satisfy the Lie Algebra

\[
[\Theta^a, \Theta^b] = f^{abc} \Theta^c.
\]  \(3\cdot18\)

The harmonic expansion of spinor field is thus given by the series of representation bases of \(\Theta^a\)’s:

\[
\psi_{\kappa}^{(r)}(x, \theta) = \sum_{\kappa} \psi_{\kappa}^{(r)}(x) U^{(r)}_{\kappa \lambda}(\theta),
\]  \(3\cdot19\)

\[
(\Theta^a)_{\kappa \lambda} U^{(a)}_{\lambda \kappa} = \sum_{\kappa} \psi_{\kappa}^{(r)}(x) U^{(r)}_{\kappa \lambda}(\theta)
\]  \(3\cdot20\)

with the result

\[
\delta \psi_{\kappa}^{(r)}(x) = -C^a(x) \Theta^a \psi_{\kappa}^{(r)}(x).
\]  \(3\cdot21\)

3.2. Fermion mass problem

Now let us turn our attention to the fermion mass problem. The 4-dimensional mass operator is obtained if we use in \((3\cdot1)\) \(\xi_8^a\) and \(\eta_8^a\) replaced, respectively, by \(\xi_8^a\) given in §2 and

\[
\langle \eta_8^a(x, \theta) \rangle = \left( \begin{array}{c} h_8^a(x, \theta) \\ \cdots \\ -h_8^a(x, \theta) A^a(x, \theta) \end{array} \right) = \left( \begin{array}{c} I_8 \delta_{ab} \\ \cdots \\ 0 \end{array} \right)
\]  \((3\cdot22)^a\)

with \(I_8\) identified with the Planck length \(\sim 10^{-32}\) cm\((\sim (10^{16} \text{GeV})^{-1})\). Equation \((3\cdot1)\) becomes then, except for constant factor \(I_8\),

\(^{a)}\) We use the following dimensional convention: \([Z^a] = L, [\gamma_{\kappa\lambda}] = [\gamma^{\kappa\lambda}] = L^{-1}, [L^{\kappa\lambda\cdots}] = L^0\). Consequently, \([\eta_8^a] = [\xi_8^a] = L, [L^{\kappa\lambda\cdots}] = L^0\) and \([\xi_8^a] = L\) because of \([\xi_8^a] = L^0\).
\[ \psi(x, \theta) = \left( \gamma^\mu \frac{\partial}{\partial x^\mu} + M \right) \phi(x, \theta) \]  
(3.23)

with
\[ M^{\text{PFB}} = \frac{X}{l_p^2} \gamma^\mu \left( \xi_{\alpha=\bar{a}} + \frac{1}{2} G_{\alpha=\bar{a}} \right) \]  
for PFB-type, \(^{\text{PFB}} \)
(3.24)

and
\[ M^{\text{HGS}} = \frac{1}{l_p^2} \gamma^\mu \left( \xi_{\alpha=\bar{a}}(\theta) \frac{\partial}{\partial \theta^b} + i \frac{1}{2} \gamma_{\alpha=\bar{a}}(\theta) \gamma^b \right) \]  
for HGS-type. \(^{\text{HGS}} \)
(3.25)

\[ G_\alpha \] are given by (3.6) and \( S^{\text{HGS}}(\theta) \) are n-dimensional Ricci rotation coefficients constructed from the ground mode vielbein fields \( \xi_{\alpha=\bar{a}}(\theta) \),
\[ S^{\text{HGS}}(\theta) = -\frac{1}{2} \left( C^{\bar{b}}_{\bar{b} \alpha} + C^{\bar{a}}_{\bar{a} \alpha} - C^{\bar{b}}_{\bar{a} \alpha} \right) \]  
(3.26)

with
\[ C^{\bar{a}}_{\bar{b} \alpha} = \left( \xi_{\bar{b}}(\theta) \right) \left( \xi_{\alpha}(\theta) \right) \frac{\partial}{\partial \theta^\alpha} \]  
(3.27)

As anticipated from the conditions used in deriving the above mass operator, one can see the following invariant structure of \( M \) under the (global) K-K transformation:
\[ \begin{bmatrix} M^{\text{PFB}}, \Pi_\alpha \end{bmatrix} = 0, \]  
\[ \begin{bmatrix} M^{\text{HGS}}, \Xi_\alpha \end{bmatrix} = 0 \]  
(3.28)

with \( \Pi_\alpha \) and \( \Xi_\alpha \) given by Eqs. (3.5) and (3.17). Consequently, one finds that \( M^{\text{PFB}} \) and \( M^{\text{HGS}} \) become diagonal under the representation bases of \( \Pi_\alpha \)'s and \( \Xi_\alpha \)'s, respectively, i.e.,
\[ (M^{\text{PFB}})_{\lambda\eta} U^{\lambda_1 A}_N(\theta) = \gamma_\lambda m^{\eta_1} U^{\lambda_1 A}_N(\theta), \]  
(3.29)*\(^{\ddagger}\)

\[ (M^{\text{HGS}})_{\lambda\eta} U^{\lambda_1 K}_N(\theta) = \gamma_\lambda m^{\eta_1} U^{\lambda_1 K}_N(\theta). \]  
(3.30)

By using harmonic expansions of the spinor fields, (3.14) and (3.19), we finally obtain 4-dimensional spinor mass terms (except for a constant factor)
\[ \int d^8\theta [\det \xi(\theta)]^{-1} \sum \left( m^{(\sigma)} U^{(\gamma_1)K}_N(\theta) U^{(\gamma_1)\lambda_1 M}_N(\theta) \bar{\psi}_{\lambda_1}^{(\gamma_1)S}(x) \psi_{\lambda_1}^{(\gamma_1)S}(x) - (L \leftrightarrow R) \right) \]  
for PFB type,  
(3.31)

and
\[ \int d^8\theta [\det \xi(\theta)]^{-1} \sum \left( m^{(\sigma)} U^{(\gamma_1)K}_N(\theta) U^{(\gamma_1)\lambda_1 M}_N(\theta) \bar{\psi}_{\lambda_1}^{(\gamma_1)S}(x) \psi_{\lambda_1}^{(\gamma_1)S}(x) - (L \leftrightarrow R) \right) \]  
for HGS-type.  
(3.32)

Here it is very important to note that, so far as harmonic expansions of the left- and right-handed spinor fields extend unlimitedly over all possible representations of \( G \), then any \( \psi^{\lambda_1}(x) \) can find its counter part \( \psi_{\lambda_1}(x) \) with \( \gamma_L = \gamma_R \) and thus the integrals over \( \theta^a \) in

\(^{\ddagger}\) Note here \([M^{\text{PFB}}, \xi(\theta) = 0].\)

\(^{\ddagger\ddagger}\) Neglect of possible degeneracies of representations is not essential here, though we write explicitly those related to helicities, \( \gamma_L(\gamma_R, \gamma^a = 0). \)
(3·31) and (3·32) associated with them give necessarily non-vanishing values. This consideration, however, suggests a very interesting possibility that the original spinor field $\phi(x, \theta)$ obeys some geometrical (associated fibre bundle type) constraints, which lead to harmonic expansions different for left- and right-handed spinor components, and eventually the integrals over $\theta^a$ give rise to vanishing mass terms, irrespectively of mass-eigenvalues $(m^{(v)}$ or $m^{(a)}$). In fact, as was emphasized in Ref. 4, this possibility is very likely for the spinor field on the PFB-type K-K space, where the metric tensor fields $\gamma_{ab}$ obey a similar kind of constraint, i.e., the Killing vector condition. Indeed let us put, in parallel with the latter Killing vector condition (2·18), the following form of spinor constraints,

$$\xi^{(a)} N \phi(x, \theta) = (\xi^{(a)} N \phi(x, \theta)), \quad (3·33)$$

where $(\xi^{(a)} N \phi(x, \theta))$ denotes some suitable representation $(\rho)$ of Lie Algebra $\mathfrak{g}$ embedded in the $SO(n)$ spinor representation.

The K-K transformation in this case should be the original one (2·28) unaccompanied with vielbein rotations, which leaves the above spinor constraints invariant,

$$\delta \phi(x, \theta) = -C^a(x) \xi^a \phi(x, \theta). \quad (3·34)$$

The solution of (3·33) can be written as

$$\psi_{HN}(x, \theta) = \psi_{HM}(x) D^{(a)} N M(\theta) \quad (3·35)$$

with $D^{(a)} N M(\theta)$;

$$\begin{align*}
\xi^{(a)} N D^{(a)} N M(\theta) &= (\xi^{(a)} N M D^{(a)} N M(\theta), \\
\xi^{(a)} N D^{(a)} N M(\theta) &= (\xi^{(a)} N M D^{(a)} N M(\theta).
\end{align*} \quad (3·36)$$

So the transformation (3·34) gives

$$\delta \phi_{HN}(x) = -N \xi^{(a)} N \phi(x, \theta). \quad (3·37)$$

One can realize, for example, the chiral invariant $SO(10)$ gauge theory by assuming the condensation $\langle \xi^{(a)} N \phi(x) \xi^{(b)} N \phi(x) \rangle = x^2 \delta_{ab}$ (3·34) compatible with (2·28) and taking

$$\begin{align*}
\xi^{(a)} N &\xi^{(b)} N = -i \left( \gamma^a, \gamma^b \right) (1 + \gamma^{(a)}/2, \\
\xi^{(a)} N &\xi^{(b)} N = -i \left( \gamma^a, \gamma^b \right) (1 - \gamma^{(b)}/2, \quad (3·38)
\end{align*}$$

(a, b = 1, 2, ..., 10)

where $\gamma^{a}$s $(a = 1, 2, \cdots, 10)$ are any ten operators among $\gamma^{a}$s $(a = 1, 2, \cdots, \dim SO(10) = 45)$ and $\gamma^{(a)} = i \gamma^a \gamma^b \cdots \gamma^{(a)}$. Spinor constraints (3·33) can then be written as

$$\xi^{(a)} N \phi(x, \theta) = -i \left( \gamma^a, \gamma^b \right) (1 + \gamma^{(a)}/2 \phi(x, \theta) \quad (3·39)$$

with $\gamma^{(a)} = \gamma^a \gamma^{(a)}$. The massless property of spinor fields characteristic of the chiral
invariant gauge theory is still preserved, even at the condensation $\langle \xi_{\alpha}(x) \rangle = x^{-1} \delta_{\alpha\beta} (2.35)$ assumed in derivation of (3.1), by the vanishing overlapping integrals between the left- and right-handed spinor fields whose extra-coordinate dependence is fixed by (3.35) and (3.36):

$$x/l_p \int d^a \theta \left[ \text{det} \xi^{a}(\theta) \right]^{-1} D^{(a)*} \xi^{a}(\theta) D^{(a)} \xi^{a} \xi^{a} \phi_{\theta h}(x) \right] \phi_{\theta h}(x) + (L \leftrightarrow R).$$

(3.40)

§ 4. Discussion

In the present paper, we have obtained the general procedure for harmonic expansions of fields in the K-K theory in conformity with the K-K transformation as the gauge transformation. It should be noted that the K-K transformation defined in the vielbein formalism which is needed for description of the spinor field, becomes a coordinate transformation combined with (local) vielbein rotation. Harmonic expansions of fields, however, are not in general unique. Appendix C is devoted to further study of harmonic expansions of $A_{\alpha}^a(x, \theta)$ and $g^{ab}(x, \theta)$ different from those given in § 2.2.

We have proposed a spinor constraints model in the PFB-type K-K theory to get a chiral invariant gauge theory which protects spinor fields from acquiring huge masses of the order of the Planck mass in the dimensional reduction. It is important to note that the spinor constraints given by (3.33) actually play the role of "gauge fixing" with respect to local vielbein rotations of the invariant action, in parallel with the Killing vector condition which plays a similar role with respect to the local transformation, $\xi_{\alpha} \rightarrow \xi_{\alpha} + \xi_{\alpha}(z) \xi_{\alpha}$, as was pointed out in Ref. 4.

Finally let us remark on the possible origin of generations of quarks and leptons in the present scheme. For example, in the chiral $SO(10)$ spinor constraint model given by (3.39), one finds that the spinor indices (or freedoms) related to $\gamma_{\alpha}^i$ ($\alpha = 1, 2, \cdots, 45$) other than $\gamma_{\alpha}^a$ ($a = 1, 2, \cdots, 10$) are irrelevant to the $SO(10)$ gauge transformation, as seen from Eqs. (3.37) and (3.38), and thus possibly provide ample freedom of generations, although further investigations are needed for building realistic models of quarks and leptons.

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Appendix A

Let us consider a compact and semi-simple group $G$ with dimension $n + p$ and denote its parameter space by $\{ u^a \}$ ($a = 1, 2, \cdots, n + p$). The representation matrix of group element $g(u)$ is defined by
\[ D^{(\sigma A)}(u) = \langle A|g(u)|B\rangle_\sigma, \]  
\( (A.1) \)

where \( |A\rangle_\sigma, \cdots \) denote a set of basis vectors belonging to the representation \( \sigma \) of \( G \). The left- and right-invariant vector fields, \( \xi_\sigma^- \) and \( \xi_\sigma^+ \), are defined as

\[ \xi_\sigma^- D^{(\sigma A)}(u) = \xi_\sigma^+ D^{(\sigma A)}(u) = \langle A|g(u)|B\rangle_\sigma = \langle B'|g(u)J_\sigma|B\rangle_\sigma = \langle B'|D^{(\sigma A)} - \partial_{\partial u^\sigma} D^{(\sigma A)}(u) \rangle_\sigma, \]  
\( (A.2) \)

\[ \xi_\sigma^+ D^{(\sigma A)}(u) = \xi_\sigma^+ D^{(\sigma A)}(u) = \langle A|D^{(\sigma A)}(u)J_\sigma|B\rangle_\sigma = \langle A|J_\sigma|B\rangle_\sigma = \langle A|D^{(\sigma A)}(u)J_\sigma|B\rangle_\sigma, \]  
\( (A.3) \)

where \( J_\sigma \)s are generators of \( G \) and satisfy the Lie Algebra \( \mathfrak{g} (\equiv \text{Lie } G) \):

\[ \begin{align*}
gJ_\sigma g^{-1} &= \langle g\alpha|g\beta\rangle_\sigma_{ab} J_\sigma^b, \\
[J_\sigma, J_\tau] &= -f_\sigma^{ab} J_\tau^a,
\end{align*} \]

\( (A.4) \)

with structure constants \( -f_\sigma^{ab} \), i.e., the adjoint representation matrix of \( \mathfrak{g} \):

\[ f_\sigma^{ab} = -f_\sigma^{ba} = -f_\sigma^{ab} = \langle g\gamma|g\alpha\rangle_\sigma_{ab}. \]  
\( (A.5) \)

From the above definition,

\[ [\xi_\sigma^+, \xi_\sigma^-] = \pm f_\sigma^{ab} \xi_\tau^a, \]  
\( (A.6) \)

\[ [\xi_\sigma^+, \xi_\sigma^-] = 0. \]  
\( (A.7) \)

If we set

\[ d_\sigma^a(u) \equiv D^{(\sigma A)}(A^a = g^{-1}u), \]  
\( (A.8) \)

then

\[ \xi_\sigma^+ d_\sigma^a(u) = -f_\sigma^{ab} d_\sigma^b(u), \]  
\( (A.9) \)

\[ \xi_\sigma^- d_\sigma^a(u) = -f_\sigma^{ab} d_\sigma^b(u), \]  
\( (A.10) \)

and \( \xi_\sigma^ \)s are related with each other by

\[ \xi_\sigma^- = d_\sigma^a(u) \xi_\sigma^+. \]  
\( (A.11) \)

The left- and right- invariant metric tensor \( g^{ab} \) is defined by

\[ g^{ab}(u) = \xi_\alpha^a \xi_\beta^b = \xi_\beta^a \xi_\alpha^b, \]  
\( (A.12) \)

which satisfies \( \mathcal{L}_\xi g^{ab} = 0 \).

If \( G \) has subgroup \( H \) with dimension \( p \), the parameter space of cosets \( G/H \), which is denoted by \( \theta^a (a = 1, 2, \cdots, n) \), is related to the group manifold of \( G \) by a mapping

\[ \theta^a = \pi^a(u), \]  
\( (A.13) \)

which satisfies (in the case of right invariant mapping, for instance)

\[ \xi_\sigma^- \pi^a(u) = 0, \quad (a = 1, 2, \cdots, p) \]  
\( (A.14) \)

\( \xi_\sigma^ \)s and \( \xi_\sigma^- \)s introduced in § 2 correspond to \( \xi_\sigma^- \)s and \( \xi_\sigma^- \)s, respectively.
with \( \mathcal{E}_n \in \mathcal{H} \), the Algebra associated with \( H \) as the isotropy group. As is well known, the parameter space \( \{ \theta^a \} \) works as a homogeneous space under the action of \( G \), whose generators (vector fields) are defined in the following way

\[
\xi^a_\theta = \xi^a_\theta(\theta) \frac{\partial}{\partial \theta^a}
\]

(A·15)

with

\[
\xi^a_\theta(\theta) = \mathcal{E}_n^a \pi^a(u) = \mathcal{E}_n^a(u) \pi^a(u).
\]

(a = 1, 2, ..., \( n + p \) and a = 1, 2, ..., \( n \))

(A·16)

One can confirm that \( \xi^a_\theta \)'s defined above are actually functions of \( \theta^a \)'s by noticing the relation

\[
\mathcal{E}_n^a \xi^a_\theta = \mathcal{E}_n^a(\mathcal{E}_n^a \pi^a) = \mathcal{E}_n^a(\mathcal{E}_n^a \pi^a) = 0,
\]

(A·17)

and satisfy the Lie Algebra \( \mathcal{L} \),

\[
[\xi^a_\theta, \xi^b_\theta] = f^a_{bc} \xi^c_\theta.
\]

(A·18)

Further, (A·14) combined with (A·11) leads one to the relations

\[
d^a_{\theta^h}(u) \xi^a_\theta(\theta) = 0
\]

(A·19)

or

\[
d^a_{\theta^h}(u(\theta)) \xi^a_\theta(\theta) = 0 \quad (h = 1, 2, \ldots, p)
\]

(A·19')

with \( \pi^a(u(\theta)) = \theta^a \), which explicitly show the local linear dependence of \( \xi^a_\theta(\theta) (a = 1, 2, \ldots, n + p; a = 1, 2, \ldots, n) \) on homogeneous space.

Next let us consider harmonic expansions on the homogeneous space. One can construct their expansion bases from those on the group manifold, i.e., \( D^{a\sigma}(u) \)'s given in (A·1), in the following way :

\[
D^{a\sigma}(u) \equiv \langle A|g(u)|B = \tilde{H}_{\sigma},
\]

(A·20)

where \( \tilde{H}_{\sigma} \)'s denote singlet states with respect to subgroup \( H \) among \( \sigma \)-representation bases of \( G \), i.e.,

\[
J_h \tilde{H}_{\sigma} = 0, \quad (h = 1, 2, \ldots, p)
\]

(A·21)

and hence they are described in general by representations of possible Lie Algebra \( \mathcal{K} (\subset \mathcal{L}) \) generated by \( J_{\tilde{k}} \)

\[
[J_{\tilde{k}}, \mathcal{K}] = 0. \quad (\tilde{k} = 1, 2, \ldots, l)
\]

(A·22)

From the relation

\[
\mathcal{E}_n D^{a\sigma}(u) = \langle A|g(u)J_h|\tilde{H}_{\sigma} = 0,
\]

(A·23)

one can confirm that \( D^{a\sigma} \) is actually the function of \( \theta^a \)'s. Further, from (A·3), one can derive the relation
\[ \xi^D_{\alpha} D^{(\alpha)A}_{\beta} \theta = \langle A[j_{\alpha}]A^\beta \rangle D^{(\alpha)\beta}_{\alpha} \theta. \] (A·24)

Peter and Weyl's expansion theorem on compact homogeneous space tells us the following harmonic expansions of function \( \phi(\theta) \):

\[ \phi(\theta) = \sum \phi^{(\sigma)}_{\beta} D^{(\sigma)\beta}_{\alpha} \theta. \] (A·25)

**Appendix B**

We shall give here the expression of scalar curvature \( R^{(x+n)}(x, \theta) \) in the HGS-type K-K theory (as for the one corresponding to PFB-type, see Ref. 4), for instance) before harmonic expansions of fields:

\[
R^{(x+n)}(x, \theta) = R^{(x)}(\zeta^+_{\bar{a}}) + R^{(n)}(\zeta^+_{\bar{a}})
\]

\[
- \frac{1}{4} (\xi^{\rho}_{\alpha} \xi^{\sigma}_{\beta}) (\xi^{\rho}_{\alpha} \xi^{\sigma}_{\beta}) g_{\rho \sigma} g^{\nu \rho} g^{\delta \sigma}
\]

\[
+ \frac{1}{2} (D_{\xi^{\rho}_{\alpha}} \xi^{\sigma}_{\beta}) (D_{\xi^{\rho}_{\alpha}} \xi^{\sigma}_{\beta}) g^{\rho \sigma} g^{\nu \rho} - 2(h^{\rho \sigma}_{\alpha \beta} D_{\xi^{\rho}_{\alpha}} \xi^{\sigma}_{\beta})
\]

\[
+ (S_{\beta}^{\gamma} S_{\gamma^* \alpha} + S_{\beta}^{\gamma} S_{\gamma^* \alpha}) A_{\gamma^*}^\alpha
\]

\[
+ (S_{\beta}^{\gamma} S_{\gamma^* \alpha} - S_{\beta}^{\gamma} S_{\gamma^* \alpha}) A_{\gamma^*}^\alpha A_{\gamma^*}^\alpha
\]

\[
- 2 h^{\rho \sigma}_{\gamma^* \rho} (D_{\xi^{\rho}_{\alpha}} \xi^{\sigma}_{\beta}) S_{\gamma^* \alpha}^{\rho \sigma}
\]

\[
+ h^{\rho \sigma}_{\gamma^* \rho} h^{\rho \sigma}_{\gamma^* \rho} S_{\gamma^* \alpha}^{\rho \sigma} \xi^{\gamma^* \rho} \xi^{\gamma^* \rho}
\]

\[
- S_{\gamma^* \alpha}^{\rho \sigma} \xi^{\gamma^* \rho} \xi^{\gamma^* \rho}
\]

\[
- 2 (S_{\gamma^* \alpha} A_{\gamma^*}^\alpha) \bar{a} + 2 (h^{\rho \sigma}_{\gamma^* \rho} A_{\gamma^*}^\alpha) \bar{a}.
\] (B·1)

In the above expression, the following abbreviations are used:

\[
S_{\alpha}^{\beta} = - \frac{1}{2} (C_{\alpha}^{\rho} + C_{\beta}^{\rho} - C_{\rho}^{\rho})
\] (B·2)

with

\[
C_{\beta}^{\rho} = (h^{\rho}_{\bar{a}} h^{\rho}_{\bar{a}} - h^{\rho}_{\bar{a}} h^{\rho}_{\bar{a}}) h^{\rho}_{\bar{a}},
\] (B·3)

\[
S_{\gamma^* \alpha}^{\rho \sigma} = \frac{1}{2} (d_{\gamma^* \rho}^{\sigma} + d_{\gamma^* \sigma}^{\rho} - d_{\gamma^* \rho}^{\rho})
\] (B·4)

with

\[
d_{\gamma^* \rho}^{\sigma} = (h^{\rho}_{\bar{a}} \xi^{\sigma}_{\alpha} h^{\rho}_{\bar{a}} - h^{\rho}_{\bar{a}} \xi^{\sigma}_{\alpha} h^{\rho}_{\bar{a}}) h^{\rho}_{\bar{a}}
\] (B·5)

and
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\[ S_\mu^{\nu\overline{\nu}} = -\frac{1}{2} \left( \xi_{\alpha} h_{\alpha,\rho} h_{\rho}^{\overline{\nu}} + \xi_{\overline{\alpha}} h_{\overline{\alpha},\rho} h_{\rho}^{\nu} \right). \]  

(B.6)

Further,

\[ D_{\alpha}^{\overline{\overline{\nu}}} = \xi_{\alpha}^{\overline{\overline{\nu}}} + A_{\alpha}^{\overline{\overline{\nu}}} \left( \xi_{\overline{\alpha}}^{\overline{\overline{\nu}}} - \xi_{\alpha}^{\overline{\overline{\nu}}} \right) \xi_{\overline{\alpha}} \xi_{\overline{\alpha}}, \]  

(B.7)

and

\[ D_{\overline{\alpha}}^{\overline{\nu}} = A_{\overline{\alpha}}^{\overline{\nu}} - f^{\overline{\beta}}_{\overline{\gamma}} A_{\beta}^{\overline{\nu}} A_{\gamma}^{\overline{\nu}} - A_{\alpha}^{\overline{\nu}} \xi_{\alpha} \xi_{\alpha}, \]  

(B.8)

with

\[ G_{\alpha}^{\nu} \equiv G_{\alpha}^{\nu} h_{\alpha}^{\nu} + S_{\alpha}^{\nu} G_{\overline{\alpha}}, \]  

(B.9)

\[ F_{\alpha}^{\nu} = F_{\overline{\alpha}}^{\nu} + S_{\alpha}^{\nu} F_{\overline{\alpha}}. \]  

(B.10)

It should be remarked that troublesome terms surrounded by the dotted line in (B.1) come from the extra-coordinate dependence of \( h_{\alpha}^{\nu}(x, \theta) \), i.e., the excitation modes of gravity.

Appendix C

Here let us consider harmonic expansions of \( A_{\alpha}^{\mu}(x, \theta) \) and \( g^{\alpha\beta}(x, \theta) \) different from those given in \$2.2\). With respect to \( A_{\alpha}^{\mu}(x, \theta) \), component fields \( A_{\alpha}^{\mu}(x) \)'s defined in (2.58) belong to the direct product representations of \( G \) with "intrinsic" components \( a \) and "orbital" components \( B \), as seen in (2.62). Another expansion is given, instead of (2.58), as follows. Let us set

\[ A_{\alpha}^{\mu}(x, \theta) = A_{\alpha}^{\mu}(x) \xi_{\alpha}^{\theta}(\theta) + B_{\alpha}^{\mu}(x, \theta), \]  

(C.1)

and insert it into Eq. (2.53) by taking into account Eq. (2.60), then

\[ \delta B_{\alpha}^{\mu}(x, \theta) = -C^{\mu}(x)(\Omega_{\alpha})_{a}^{\mu} B_{a}(x, \theta) \]  

(C.2)

with

\[ (\Omega_{\alpha})_{a}^{\mu} = \xi_{\alpha}^{\mu} \delta_{\alpha a} - \xi_{\alpha}^{\mu} \delta_{a \alpha}(\theta). \]  

(C.3)

On account of the relations

\[ [\Omega_{\alpha}, \Omega_{\beta}] = f_{\alpha \beta \gamma} \Omega_{\gamma}, \]  

(C.4)

harmonic expansions of \( B_{\alpha}^{\mu}(x, \theta) \) are given in terms of the series of representation bases of \( \Omega_{\alpha} \)'s

\[ B_{\alpha}^{\mu}(x, \theta) = \sum B_{\alpha}^{\mu}(x) V_{\alpha}^{\mu}(\theta), \]  

(C.5)

\[ (\Omega_{\alpha})_{a}^{\mu} V_{a}^{\mu}(\theta) = (\Omega_{\alpha})_{a}^{\nu} V_{a}^{\mu}(\theta). \]  

(C.6)
with \((\sigma\)-representation matrix \((Q_{\sigma})_{MM'}\) of \(G\). Then, Eq. (C.2) gives

\[
\delta^{(\mu)}_{\sigma} B_{\mu}^{M'}(x) = -C^{a}(x)(Q_{\sigma})_{MM'}(Q_{a})^{(\mu)}_{\sigma}(x).
\]  

(C.7)

With respect to scalar fields \(g^{ab}(x, \theta)\), let us here, instead of Eq. (2.76), apply harmonic expansions to \(g^{ab}(x, \theta)\) defined through

\[
g^{ab}(x, \theta) = \xi^{\alpha}_{a} g^{ab}(x, \theta) \xi^{\beta}_{b}.
\]  

(C.8)

Equation (2.55) gives

\[
\delta g^{ab}(x, \theta) = -C^{c}(x)(\xi^{\alpha}_{a} g^{ab} + f^{\alpha}_{a} g^{ab} + f^{\beta}_{b} g^{ab}).
\]  

(C.9)

First let us expand \(g^{ab}(x, \theta)\) simply in terms of the representation bases of \(\xi^{\alpha}_{a}\)’s, i.e., \(D^{\sigma}\rho \phi_{\sigma}(\theta)\),

\[
g^{ab}(x, \theta) = \sum_{\sigma}^{(\sigma)} \phi^{(\sigma)}_{\rho}(x)(D^{\sigma})_{\rho \phi}(\theta).
\]  

(C.10)

Then Eq. (C.9) gives

\[
\delta^{(\sigma)}_{\rho} \phi^{(\sigma)}_{\rho}(x) = -C^{c}(x)(\Gamma^{(\sigma)}_{\rho})^{(\sigma)}_{\rho} g^{ab}(x, \theta)
\]  

(C.11)

the direct product representations of \(G\). On the other hand, if one rewrites Eq. (C.9) as

\[
\delta g^{ab}(x, \theta) = -C^{c}(x)(\Gamma^{(\sigma)}_{\rho})^{(\sigma)}_{\rho} g^{ab}(x, \theta)
\]  

(C.12)

with

\[
(\Gamma^{(\sigma)}_{\rho})^{(\sigma)}_{\rho} = \xi^{\alpha}_{a} \delta_{\alpha \beta} \delta_{ab} + f^{\alpha}_{a} \delta_{ab} + f^{\beta}_{b} \delta_{ab},
\]  

(C.13)

and notices the relation

\[
[\Gamma_{\alpha}, \Gamma_{\beta}] = f^{\delta}_{\alpha \beta} \Gamma_{\delta},
\]  

(C.14)

one can give harmonic expansions of \(g^{ab}(x, \theta)\) in terms of the series of representation bases of \(\Gamma_{\alpha}\)’s,

\[
g^{ab}(x, \theta) = \sum_{\alpha}^{(\sigma)} g^{ab}(x, \theta) V^{(\sigma)}_{\alpha}(\theta),
\]  

(C.15)

\[
(\Gamma^{(\sigma)}_{\rho})^{(\sigma)}_{\rho} V^{(\sigma)}_{\alpha}(\theta) = \Gamma^{(\sigma)}_{\rho} M^{(\sigma)} \rho \phi^{(\sigma)}_{\rho}(\theta)
\]  

(C.16)

with the result

\[
\delta^{(\sigma)}_{\rho} \phi^{(\sigma)}_{\rho}(x) = -C^{c}(x)(\Gamma^{(\sigma)}_{\rho})^{(\sigma)}_{\rho} \phi^{(\sigma)}_{\rho}(x).
\]  

(C.17)

Finally it should be noted that on account of the local linear dependence of \(\xi^{\alpha}_{a}(\theta)\) on homogeneous space, as explicitly shown in (A.19), there exists ambiguity in definition of \(A^{a}(x, \theta)\) and \(g^{ab}(x, \theta)\) considered in the preceding arguments, i.e., \(A^{a}(x, \theta) = \xi^{\alpha}_{a} A^{a}(x, \theta)\) and \(g^{ab}(x, \theta) = \xi^{\alpha}_{a} g^{ab}(x, \theta) \xi^{\beta}_{b}\), which generates transformations

\[
\delta A^{a}(x, \theta) = F^{a}_{b}(x, \theta) d^{a}_{b}(u(\theta)),
\]  

(C.18)

\[
\delta g^{ab}(x, \theta) = f^{ab}(x, \theta) d^{a}_{b}(u(\theta)) + g^{ab}(x, \theta) d^{a}_{b}(u(\theta))
\]  

(C.19)
with arbitrary functions $F^a_\mu$, $f^{ab}$ and $g^{aa}$, and thus gives rise to the corresponding transformations of their component fields after harmonic expansions.

References

7) A. Salam and J. Strathdee, Ann. of Phys. 141 (1982), 316.