stress and the calculated value appears to be extremely close to those given in [3].

References

1 Chatterjee, S. N., "The Stress Field in the Neighborhood of a Branched Crack in an Infinite Elastic Sheet," *International Journal of Solids and Structures*, Vol. 11, 1975, pp. 521–538.

2 Theocaris, P. S., and Ioakimidis, N., "The Symmetrically Branched Crack in an Infinite Elastic Medium," *ZAMP*, Vol. 27, 1976, pp. 801–814.

3 Westmann, R. A., "Pressurized Star Crack," *Journal of Mathematics and Physics*, Vol. 43, 1964, pp. 191–198.

Author's Closure

The author wishes to thank Dr. S. N. Chatterjee for his interesting and correct remarks on the paper, as well as for the verification of the efficiency of the numerical technique used in the paper in the problem of a star-shaped crack with three equal branches. As regards the questions posed by the discusser, he wishes to mention that:

1 The Plemelj formulas can be used at corner points, taking on these points the forms mentioned in [4, pp. 31–32]. Nevertheless, since these formulas are valid only on a corner point and not in its neighborhood, they are not useful for the estimation of the behavior of the solution of a singular integral equation near a corner point. This can be achieved easily by taking into account the singular integral equation itself and applying to it the function-theoretic technique, extensively used in [5]. This is valid even in the case when no stress singularity exists at the point of branching, as mentioned in the paper for the derivation of conditions (23).

2 The ratio K_{IIA}/K_{IA} of the value of the mode II stress-intensity factor K_{IIA} at the main crack tip A to the value of the mode I stress-intensity factor K_{IA} at the same crack tip was seen both theoretically and experimentally to take very small values, less than 0.1 in all cases considered in the paper, where the main crack was assumed much longer than its branches. In Fig. 1, a typical case of a branched crack together with the caustics formed at its tips is shown. The stress field is normal to the main crack and the symmetry of the caustic at the main crack tip reveals that the ratio K_{IIA}/K_{IA} is insignificant. Unfortunately, the specimens tested have not been loaded up to fracture to see the mode in which they failed.

Finally, the author wishes to mention that the numerical technique used in this paper, as well as in reference [2] of the discussion, for a crack with three branches, has also been applied to the problem of a star-shaped crack with n branches emanating from the same apex, considered for the first time in reference [3] of the discussion. Some numerical results, confirming further the last remark of the discusser, have been published in [6 and 7, pp. 296–304].

References

4 Gakhov, F. D., *Boundary Value Problems*, Pergamon Press and Addison-Wesley, Oxford, 1966.

5 Erdogan, F., "Complex Function Technique," *Continuum Mechanics of Single-Substance Materials* (*Continuum Physics*, Vol. 2, ed., Eringen, A. C.), Academic Press, New York, 1975, Part III, Chapter 3, pp. 523–603.

6 Theocaris, P. S., and Ioakimidis, N. I., "A New Method for the Solution of the Problem of a Star-Shaped Crack in an Infinite Isotropic Elastic Medium," *Scientific Papers of the Faculty of Civil Engineering of the National Technical University of Athens*, Vol. 1, 1977, pp. 1–13.

7 Ioakimidis, N. I., "General Methods for the Solution of Crack Problems in the Theory of Plane Elasticity," doctoral thesis at the National Technical University of Athens, Athens, 1976 (available by University Microfilms, Order No. 76-21,056).

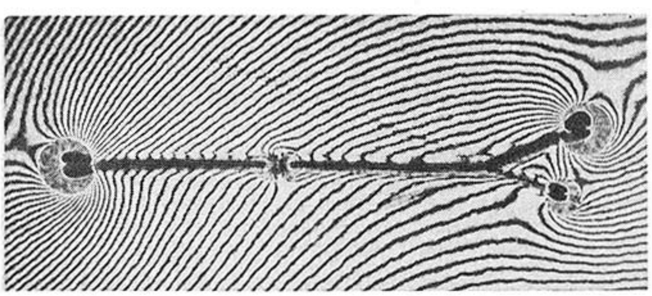

Fig. 1 Caustics formed at the tips of a typical branched crack under tension at infinity of direction normal to the main crack

A High-Order Theory of Plate Deformation[1]
Part 1: Homogeneous Plates

F. I. Niordson.[2] The engineering approach to improve the Kirchhoff theory of thin plates by including certain effects like deformation due to transverse shear, etc., has so far not lead to a consistent theory and the paper under discussion is no exception.

The authors' terminology for "high-order" or "higher-order" theories is misleading since the thickness of the plate is not properly accounted for. The important parameter in this case is the thickness h, or rather h/L, and there is no guaranty that third-order polynomials in z for the displacements will yield the correct terms of relative order $(h/L)^2$, as in fact they do not. The improvement may therefore be illusive.

The correct solution to the problem considered by the authors was actually given in 1972 by Brod [1] and to the corresponding problem for vibrating plates by Niordson [2] in 1977.

The asymptotic expansion of the midplane displacement for the example considered by the authors is (see reference [1])

$$w^0 = \frac{q_0 L^4}{D\pi^4}\left[1 + \frac{8-3\nu}{40\,(1-\nu)}\left(\frac{\pi h}{L}\right)^2 - \frac{3(227-157\nu)}{5\cdot 8!\,(1-\nu)}\left(\frac{\pi h}{L}\right)^4 + \ldots\right]\sin\frac{\pi x}{L}$$

and neither equation (30) in the paper under discussion, which includes terms up to and including order $(h/L)^4$, nor equation (25) including terms of order $(h/L)^6$ has even the second-order terms $(h/L)^2$ correctly accounted for.

Finally it should be stressed that the boundary conditions with regards to the *stresses* must be fulfilled to insure that the order of approximation for the inner region holds good for the whole finite plate.

References

1 Brod, K., Herleitung der Plattengleichung der klassischen Elastizitätstheorie durch systematische Entwicklung nach einem Dickenparameter, *Diplomarbeit*, Göttingen, 1972.

2 Niordson, F. I., "An Asymptotic Theory for Vibrating Plates," DCAMM Report No. 129, Nov. 1977.

Authors' Closure

Professor Niordson finds fault with the high-order plate theory, and indeed with all approximate plate theories of higher order than that of the classical treatment. He apparently prefers the direct application of asymptotic methods in the context of three-dimensional elasticity. We certainly have no objection to the latter approach. However, we do not believe it precludes the utility of the former approach. In fact, from his point of view, it would seem to be inconsistent

[1] By K. H. Lo, R. M. Christensen, and E. M. Wu and published in the December, 1977, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 44, No. 4, pp. 663–668.

[2] Technical University of Denmark, 2800 Lyngby, Denmark.

to even admit the classical level approach; after all, it just represents the result of the first terms in a trial expansion.

As evidence of the fault of the high-order theory, he cites the fact that the high-order theory solution for a particular problem, when arranged in an asymptotic expansion in (h/L), gives a result which is not term-by-term identical with that of the exact solution. This is a rather strange criterion. If there were term-by-term equality, the approximate theory would provide the exact result, which of course is impossible. The value of any approximate theory is best assessed by comparison of its complete prediction, obtained by summing terms, against that of an exact solution. Such comparisons were given in several figures in the papers. Professor Niordson chooses to ignore these results.

There remain many questions concerning the accuracy and utility of high-order plate theories. We hope others will be interested in pursuing these lines of investigation.

Stability of a Cluster of Flexible Cylinders in Bounded Axial Flow[1]

Authors' Closure

The authors are grateful to Mr. Leko for raising the points that he did in his discussion.[2]

The first point is concerned with the convergence of the series solutions given by equations (8) and (9) of the paper. These forms of the solution for $\phi_j^i(r_i, \theta_i)$ are obtained from equations (6) and (7) by using series expansions of the coordinate transformation

$$(r_je^{i\theta_j})^{\pm n} = (r_ie^{i\theta_i} - R_{ij}e^{i\psi_{ij}})^{\pm n};$$

convergence depends on whether the series expansion for this term converges with n. To simplify matters, let us write this simply as

$$(z - z_0)^{\pm n},$$

where z and z_0 are complex variables.

Now $(z - z_0)^n$, to be used in the first two terms of equation (6), may easily be expanded in a binomial series which converges for all z; however, the convergence of $(z - z_0)^{-n}$ requires further attention. Let us write

$$(z - z_0)^{-n} = \frac{1}{z_0^n\left(\dfrac{z}{z_0} - 1\right)^n}.$$

Now, writing $((z/z_0) - 1)^{-n}$ as a Taylor series about $z/z_0 = 0$, one obtains

$$\frac{1}{\left(\dfrac{z}{z_0} - 1\right)^n} = (-1)^n \sum_{m=0}^{\infty} \frac{(n + m - 1)!}{m!(n - 1)!}\left(\frac{z}{z_0}\right)^m,$$

which clearly converges for $|z/z_0| < 1$, or $r_i < R_{ij}$. This leads to equation (8), which is used with boundary condition (*ii*), i.e., with equation (11). To obtain equation (9), which must have a form convergent for $r_i > R_{ij}$—necessary for boundary condition (*i*), equation (10)—one proceeds to write

$$(z - z_0)^{-n} = \frac{1}{z^n\left(1 - \dfrac{z_0}{z}\right)^n}$$

and expands about $z_0/z = 0$, obtaining

$$\frac{1}{\left(1 - \dfrac{z_0}{z}\right)^n} = \sum_{m=0}^{\infty} \frac{(n - m - 1)!}{m!(n - 1)!}\left(\frac{z_0}{z}\right)^m$$

which is clearly convergent for $|z_0/z| < 1$, or $R_{ij} < r_i$, as required.

The second point raised by Mr. Leko arises, we believe, from lack of precision on our part in writing that section, which we shall try to correct here (indeed, as we have done in a subsequent paper[3]). If the total velocity potential is expressed by

$$\phi = \sum_{j=1}^{k} \phi_j,$$

each of the ϕ_j should strictly be viewed as the velocity potential due to the *presence* of cylinder j, rather than due to its motion. In that sense, each of the ϕ_j may be viewed as the velocity potential due to a source-sink doublet representing the cross section of a cylinder in an otherwise unperturbed arbitrary medium. The *form* of this velocity potential, in polar coordinates, is known and is that given by equation (6). After adding the ϕ_j together for all the cylinders, *then the total velocity potential* is "shaped" properly by application of the boundary conditions. This is clearly valid since the velocity potential used in the end, ϕ, satisifes all boundary conditions.

On the other hand, there would be no sense in applying boundary conditions due to the presence of other cylinders on each of the ϕ_j. In any case, this is mathematically impossible because the system would then become overdetermined (too many boundary conditions, for each ϕ_j, for the number of available free constants).

The difficulty in this matter clearly arises from the ambiguous definition of ϕ_j in the paper, and the authors are very grateful to the discusser for giving them this opportunity to clarify it.

---

[1] By M. P. Paidoussis, and S. Suss, published in the September, 1977, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 44, pp. 401–408.

[2] D. Leko's Discussion was published in the June, 1978, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 45, p. 455.

[3] Paidoussis, M. P., Suss, S., and Pustejovsky, M. "Free Vibration of Clusters of Cylinders in Liquid-Filled Channels," *Journal of Sound and Vibration*, Vol. 55, 1977, pp. 443–459.