On the Polyakov Model for a Closed String

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A discussion is given of Polyakov's bosonic string model in terms of a doubly-connected domain mapped to simple surfaces in space of three dimensions. The minimal surface generated by a closed string is transformed to a catenoid, and it is shown, for instance, how the Liouville equation is associated with it. The problem of the critical dimension is also discussed by an estimation of the measure of the Pomeron propagator.

§ 1. Introduction

In 1981, Polyakov\(^1\) revised the dual model by introducing a fundamental change in our idea of quantisation of dual strings: In the new model the quantum fluctuation around the background metric of the two-dimensional surface spanned by the string is incorporated, and eventually this procedure has proved to solve the longstanding trouble of unworldly dimension inherent in the old dual models. Polyakov's bosonic (fermionic) model is so constructed as to be reducible to the old one when the space-time dimension.\(^D\) is 26 (10), which is indeed the critical dimension of the Veneziano (Neveu-Schwarz-Ramond) model. Detailed verifications of Polyakov's several expressions, especially factors including dimension, have been given since then in several contexts, for example, for the bosonic case in Refs. 2) and 3). As regards the problem of dimension, Marnelius' recent viewpoint should also be noticed.\(^4\)

The Lagrangean for the bosonic case with which Polyakov starts is that proposed by Brink, Di Vecchia and Howe\(^5\) of the form

$$\mathcal{L}_0 = \frac{1}{2} g^{ab} \partial_a x^{(m)} \partial_b x^{(n)}, \quad (1\cdot1)$$

where \(a,b=1,2\). The metric is mostly taken in the orthogonal gauge

$$g_{11} = g_{22} = e^\varphi, \quad g_{12} = 0, \quad (1\cdot2)$$

which itself is the same gauge as given by Virasoro in the old model. It is however the auxiliary field \(\varphi\) that features the new model. For example, the partition function is written in terms of a functional integral over \(\varphi\) such that \(\varphi\) which will extremise the action is governed by the Liouville equation.

It should be said however that the gauge \((1\cdot2)\) looks somewhat singular at first sight since, if we use \((1\cdot2)\) in \((1\cdot1)\) before everything, \(\mathcal{L}_0\) appears to turn out independent of \(\varphi\). In this respect, Marnelius in the paper cited above starts by counting \(\varphi\) in from the outset, using the Lagrangean of the form

$$\mathcal{L} = \mathcal{L}_0 + C \mathcal{L}_1, \quad (1\cdot3)$$

where \(\mathcal{L}_0\) is from \((1\cdot1)\), \(\mathcal{L}_1\) is one related to the Liouville action density and \(C\) is a
multiplier. Marnelius then refers to the method of quantisation in the light-cone frame as used in Ref. 6) to find in the bosonic case that $C$ in (1-3) should be
\begin{equation}
C = \frac{(26-D)}{48\pi}.
\end{equation}

This implies that $\mathcal{L}$ is reduced to $\mathcal{L}_0$ if $D=26$.

The present paper is concerned with some possible interpretations of the Liouville field $\varphi$ in the bosonic case when the model is applied to the motion of a closed string. Since it is claimed that the model should hold for any dimensions, we simply choose a space of three dimensions, and it will be shown how the Liouville equation is associated with the minimal surface (catenoid) generated by a motion of the closed string (§2). The physical quantity we refer mainly to is a Pomeron propagator which we construct as a generalised version of the Feynman propagator of a point particle (§3). We then employ Marnelius' hypothesis in the propagator and see by an approximation how the dimension trouble is remedied by the presence of the Liouville field, whereby we require the Pomeron intercept to be independent of the dimension (§4). We then close with some remarks (§5).

§ 2. Liouville equation associated with the propagation of a closed string

Let the surface generated by the closed string be defined by
\begin{equation}
y^{(\mu)} = y^{(\mu)}(s, t),
\end{equation}
where $s$ and $t$ vary within a doubly-connected domain. By the vector $y^{(\mu)}(s, t)$ we mean the dual position to $x^{(\mu)}(s, t)$ in (1-1) (in the sense as explained below). In conformity with (1-1), we consider the Lagrangean
\begin{equation}
\mathcal{L}_0 = \frac{1}{2} \sqrt{g^{ab}} \partial_0 y^{(\mu)} \partial_0 y^{(\nu)},
\end{equation}
where we mean $\partial_1 = \partial_s$ and $\partial_2 = \partial_t$. Variations with respect to $y^{(\mu)}(s, t)$ and $g^{ab}(s, t)$ respectively yield two kinds of equations of motion
\begin{align}
\partial_a (\sqrt{g} g^{ab} \partial_b y^{(\mu)}) &= 0, \\
g_{ab} &= \partial_0 y^{(\mu)} \partial_0 y^{(\nu)}.
\end{align}

Equation (2-3b) just says that $g_{ab}(s, t)$ is the metric tensor in the extremum.

Next we choose the Virasoro gauge:
\begin{align}
\partial_0 y^{(\mu)} \partial_0 y^{(\nu)} &= 0, \\
(\partial_0 y^{(\mu)})^2 &= (\partial_0 y^{(\nu)})^2 = e^{\varphi(s, t)}.
\end{align}

The gauge implies that in the presence of (2-3b) the coordinate curves constitute an orthogonal net such that the linear element of the surface is
\begin{equation}
d\sigma^2 = e^{\varphi(s, t)} (ds^2 + dt^2).
\end{equation}

In this gauge, Eq. (2-3a) takes the form
\begin{equation}
(\partial_s^2 + \partial_t^2) y^{(\mu)} = 0.
\end{equation}
That is, in the extremum, $y^{(s)}(s,t)$ is harmonic in the $(s+it)$-plane.

To see some geometrical meanings of (2.4b) and (2.6), we shall have recourse to a simple model—an a doubly-connected surface in space of three dimensions. We begin by regarding $t$ out of the isometric coordinates $(s,t)$ as the propertime, and consider the case where a closed string at $t=0$ moves upward to make another loop at $t=\beta$ ($\beta$ is a parameter fixed at present).

Let $C^*$ denote the domain in $(s,t)$ defined by

$$C^* = \{ s, t \mid 0 < t < \beta \}/\sim,$$  \hspace{1cm} (2.7)

where $\sim$ implies the equivalence class defined by $(s, t) = (s+2l, t)$, $l$ being another parameter. This domain is obviously equivalent to an annulus bounded by a pair of circles having radii $\exp(-\beta/\ell)$ and 1 respectively. We then map $C^*$ into a Euclidean space whose coordinates are

$$y = (y^{(1)}, y^{(2)}, y^{(3)}).$$ \hspace{1cm} (2.8)

We make further simplifications: We first assume the map is a surface of revolution$^7$ whereby identifying $t$ with the third axis (apart from a scaling factor) such that

$$y = \left( \rho(t) \cos \frac{\pi}{l}s, \rho(t) \sin \frac{\pi}{l}s, \gamma t \right),$$ \hspace{1cm} (2.9)

$\gamma$ being the scaling. We secondly assume that the boundaries of the surface shall be the circles of the same radius $a_0$:

$$\rho(t) = a_0 \quad \text{at} \quad t=0 \quad \text{and} \quad \beta.$$ \hspace{1cm} (2.10)

The extremum problem is then reduced to a Plateau problem, and we are to look for a minimal surface out of possible surfaces which are maps of $C^*$.

Substitution of (2.9) into (2.6) simply gives

$$dt^2 = \left( \frac{\pi}{l} \right)^2 \rho(t).$$ \hspace{1cm} (2.11)

It follows on the other hand from (2.4b) that

$$\left( \frac{\pi}{l} \right)^2 \rho^2 = e^\varphi,$$ \hspace{1cm} (2.12a)

$$\left( d\varphi \right)^2 + \gamma^2 = e^\varphi.$$ \hspace{1cm} (2.12b)

A consistent solution to (2.11), subject to the boundary condition (2.10), is given by

$$\rho(t) = a_0 \sech \frac{\pi \beta}{2l} \cosh \frac{\pi}{l}\left( t - \frac{\beta}{2} \right).$$ \hspace{1cm} (2.13)

while the gauge restriction (2.12a,b) determines

$$a_0 = \frac{l}{\pi} \cosh \frac{\pi \beta}{2l}.$$ \hspace{1cm} (2.14)

Hence we have

$$\rho(t) = \frac{l}{\pi} \cosh \frac{\pi \beta}{2l} \left( t - \frac{\beta}{2} \right).$$ \hspace{1cm} (2.15)
and from (2·12a) it follows that
\[ e^\varphi = \gamma^2 \cosh^2 \frac{\beta}{2} \left( t - \frac{\beta}{2} \right). \]  

Equation (2·12b) holds consistently.

The auxiliary field \( \varphi \) here depends only on \( t \) because we started by assuming the surface is a surface of revolution. Conversely if \( \varphi \) depends only on \( t \), and the metric is given by (2·5), then the surface can be conformally transformed to a surface of revolution.\(^7\)

The present Plateau problem is a long established one, and the vector (2·9) with (2·15) really forms a catenoid which minimises the area\(^5\)
\[ A = \int_C \mathcal{L} d\sigma dt \geq \frac{1}{2} \int \left[ (\partial_s \varphi)^2 + (\partial_t \varphi)^2 \right] d\sigma dt. \]  

Equality in (2·17) holds when the gauge (2·4) is satisfied.

We are next to ask what equation governs the field \( \varphi(s, t) \). Using (2·12) and (2·13) together with (2·11), we immediately see that the equation is
\[ d_i^2 \varphi = 2(\gamma \pi / l)^2 e^{-\varphi(t)}. \]  

This is a one-dimensional Liouville/Toda equation. In general we can envisage that (2·18) takes the form
\[ (\partial_s^2 + \partial_t^2) \varphi(s, t) = 2(\gamma \pi / l)^2 e^{-\varphi(s, t)}. \]  

To see this, we first refer to the Codazzi equations, which, when the element is given by (2·5), are known to be\(^7\)
\[ \begin{align*}
\partial_s \varphi &= 2(x_1/x_2(x_2 - x_1)) \partial_s x_2, \\
\partial_t \varphi &= 2(x_2/x_1(x_1 - x_2)) \partial_t x_1.
\end{align*} \]  

Here \( x_1 \) and \( x_2 \) are the parameters (principal curvatures) which represent Gauss' total curvature \( K \) and the mean curvature \( H \) as
\[ K = x_1 x_2, \]  
\[ H = (1/2)(x_1 + x_2), \]  
respectively. When (2·6) holds, \( H = 0 \). Hence \( x_1 + x_2 = 0 \), so that
\[ K = -x_1^2. \]  

It follows then from (2·20) that
\[ \partial_a \varphi = -(1/2)K \partial_a K, \quad (a=1, 2) \]  

which infers
\[ K = \pm \mu^2 e^{-2\varphi} \]  

with \( \mu \) being an integration constant. On the other hand, if the metric is given by (2·5), the general formula for the total curvature \( K \) is simply represented as\(^7\)
\[ K = -e^{-\varphi}(\partial_s^2 + \partial_t^2) \varphi. \]
We see then Eqs. (2·26) and (2·25) combine to give (2·19) if we choose the minus sign in (2·25) (the case of negative curvature) and put

$$
\mu^2 = 2(\gamma \pi / l)^2.
$$

(2·27)

It is to be noted that in our case $K$ is not constant, but given by

$$
K = -(\pi / l \gamma)^2 \text{sech}^2 \left( \frac{t - \beta}{2} \right)
$$

(2·28)

as easily seen from (2·25) and (2·26).

General solutions to the two-dimensional Toda, as well as one-dimensional Toda molecule equations are detailed in Refs. 9) and 10) respectively by Farwell and the present author: The general solution to

$$
de^\varphi = e^{-2\psi}
$$

(2·29)

is of the form

$$
e^\psi = \frac{1}{2} \left[ \left( \xi + \frac{1}{\xi} \right) \cosh \lambda \tau + \left( \xi - \frac{1}{\xi} \right) \sinh \lambda \tau \right]
$$

(2·30)

where $\lambda$ is a constant of the motion. Set

$$
\tau = (\gamma \pi / l) t, \quad \Psi = 2\varphi \quad \text{and} \quad \lambda = 1 / \gamma.
$$

(2·31)

Then (2·29) goes over to (2·18), and by the choice

$$
\xi = \exp(-\pi \beta / 2l),
$$

(2·32)

we have the special solution (2·16). This solution (2·16) is obtainable if the initial conditions are given by

$$
e^\varphi\big|_{t=0} = \gamma^2 \cosh^2(\pi \beta / 2l),
$$

$$
d\varphi\big|_{t=0} = (\pi / 2l) \tanh(\pi \beta / 2l).
$$

(2·33)

§ 3. Possible propagators in the generalised model

We would like to clarify here, digressing somewhat from the particular model, what meanings the parameters $\beta, l$ and the dual position $y^{(\mu)}(s, t)$ should have.

As previously noted our $y^{(\mu)}(s, t)$ is subject to the Dirichlet condition, while the usual position vector $x^{(\mu)}(s, t)$ is within the Neumann problem. In this sense, $x^{(\mu)}(s, t)$ and $y^{(\mu)}(s, t)$ are dual each other. Such an $x^{(\mu)}(s, t)$ is a posteriori found from the $y^{(\mu)}(s, t)$ by the condition that

$$
z^{(\mu)}(s, t) = x^{(\mu)}(s, t) + iy^{(\mu)}(s, t)
$$

(3·1)

is holomorphic with respect to $u = s + it$. Incidentally the Virasoro gauge (2·4) can be rewritten in terms of $z^{(\mu)}(s, t)$ in the form

$$
(dz(s, t) / du)^2 = 0,
$$

(3·2)

which is well-known Weierstrass' characteristic equation for minimal surfaces.\textsuperscript{8)}

The dual position $y^{(\mu)}(s, t)$ should be said to be mostly suited for two-dimensional
problems. But we can still use it for the usual point-particle propagation.11) For instance, let us pick out the simplest momentum-space Feynman propagator

\[ \Delta[p, m^2] = 1/(p^2 + m^2), \quad (3\cdot3) \]

where \( p^2 = p^2 - (p^{(0)})^2 \). We first alternatively rewrite (3\cdot3) as

\[ \Delta[p, m^2] = \int_0^\infty d\beta K[\beta; p] \quad (3\cdot4) \]

with

\[ K[\beta; p] = \frac{1}{2} e^{-\frac{1}{2} \beta m^2} e^{-\frac{1}{2} \beta p^2}. \quad (3\cdot5) \]

At this stage we can introduce dual positions \( y_2 \) and \( y_1 \) by

\[ \beta p = y_2 - y_1. \quad (3\cdot6)^* \]

Then, Eq. (3\cdot5) can be reformed

\[ K[\beta; p] = \frac{1}{2} e^{-\frac{1}{2} \beta m^2} e^{-\frac{1}{2} \beta p^2} k[y_2 - y_1; \beta], \quad (3\cdot7) \]

where \( k[y_2 - y_1; \beta] \) can be given as a path-integral

\[ k[y_2 - y_1; \beta] = \frac{1}{N} \int \mathcal{D}y(t) \exp \left[ -\frac{1}{2} \int_0^\beta dt (dy)^2 \right] \quad (3\cdot8) \]

with the proviso

\[ y(\beta) = y_2 \quad \text{and} \quad y(0) = y_1. \quad (3\cdot9) \]

For, if we change the integration variable from \( y(t) \) to \( y^0(t) \) defined by

\[ y(t) = y^l(t) + y^0(t), \quad (3\cdot10) \]

\( y^l(t) \) being the classical dual position linear in \( t \) and subject to (3\cdot9), and \( y^0(t) \) being a deviation whose boundary condition is accordingly

\[ y^0 = 0 \quad \text{at} \quad t = 0 \quad \text{and} \quad t = \beta \quad (3\cdot11) \]

we can rewrite (3\cdot8) as

\[ k[y_2 - y_1; \beta] = \frac{g(\beta)}{N} e^{-\frac{1}{2} \beta (y_2 - y_1)^2/\beta}, \quad (3\cdot12a) \]

and it is easily seen

\[ g(\beta) = \int \mathcal{D}y^0 \exp \left[ -\frac{1}{2} \int_0^\beta dt (dy^0)^2 \right] \sim 1/\beta^2 \quad (3\cdot12b) \]

in space of four dimensions. It should be remarked at this stage that we have introduced the path-integral to represent directly the momentum-space propagator without use of the Fourier transformation.

* If \( y_1 \) are regarded as potentials, then \( \beta \) is something like a resistance, \( p \) a current.
We now go on to regard $y(s, t)$ defined in §2 as a generalisation of $y(t)$ here. Corresponding to the boundary values (3·9), we would have the preassigned values $y(s, \beta)$ and $y(s, 0)$, whereby $\beta$ should play quite the same role. The parameter $l$ is however a new one. It is to be hoped by the principle of continuity that any quantities in the generalised model should be reducible to those in the point-particle model when $l \to 0$. Before proceeding, we here give two remarks: First, the mass-squared appearing in $\mathcal{A}[p, m^2]$ of (3·3) is derived via

$$m^2 = -(2/\beta) \lim_{\beta \to 0} K[\beta; p].$$

(3·13)

Second, if we change the metric as $dt^2 \to g(t)dt^2$, then the kernel (3·8) may take the form

$$\frac{1}{N} \int \mathcal{D}y \mathcal{D}g \exp \left\{ -\frac{1}{2} \int_0^a dt \left[ \frac{1}{\sqrt{g}} (dy)^2 + \sqrt{g} \right] \right\}$$

(3·14)

or something like that since originally

$$\int_0^at (dy)^2$$

measures the length of the possible path.

Now we shall begin to give a brief description how the above is generalised: Supposing that the Lagrangean $\mathcal{L}$ is written by $y(s, t)$ and $\varphi(s, t)$, we make the definition

$$K[y(s, \beta), y(s, 0); \beta/l] = (1/N) \int \mathcal{D}y \mathcal{D}\varphi \exp \left[ -\int \mathcal{L} ds dt \right].$$

(3·15)

Then corresponding to (3·4), we can define the propagator

$$\mathcal{A}[p(s, \beta), p(s, 0)] = \int_0^\infty d(\beta/l) K[\beta/l; p(s, \beta), p(s, 0)].$$

(3·16)

The relation between $p(s, t)$ and $y(s, t)$ at the boundaries is given by

$$p(s, \beta) = \partial s y(s, t)|_{t=\beta},$$

$$p(s, 0) = -\partial s y(s, t)|_{t=0},$$

(3·17)

where $\partial s$ implies a derivative along the contour (the harmonic $y(s, t)$ and $x(s, t)$ satisfy the Cauchy-Riemann relations so that $p(s, t)$ is alternatively given by a normal derivative if we make use of $x(s, t)$ instead of $y(s, t)$). Then the big kernel in the rhs of (3·16) may be written as

$$K[\beta/l; p(s, \beta), p(s, 0)] = m(\beta/l) k[y(s, \beta), y(s, 0); \beta/l]|_{3·17},$$

(3·18)

where $m(\beta/l)$ is a factor depending only on $\beta/l$.

It will be observed that $p(s, \beta)$ and $p(s, 0)$ satisfy each of the conservation laws

$$\int_0^{\alpha} dp(s, \beta) = 0, \quad \int_0^{\alpha} dp(s, 0) = 0$$

(3·19)

because $y(s, t)$ has a periodicity with respect to the polar angle $s$. One point which decisively differs from that in the case of the point-particle propagators is that in the present case no momentum is propagated from $t=0$ to $t=\beta$ (or it is a vacuum that is propagated), and the external momentum is conserved on each boundary of the doubly-
In the case considered in §2, we may give the momentum distribution, for example at \( t = \beta \), defining \( \lambda = a_0(\pi/l)s \) as follows:

\[
p(s, \beta) = \partial_s y = \left(-\sin \frac{s}{l}, \cos \frac{s}{l}, 0\right),
\]

which further says that the momenta incident at every pair of antipodal points on the circle face to the opposite direction symmetrically to the line \( y(1) = y(2) \). Another remarkable fact about (3·20) is that the magnitude of the momentum does not depend on \( \beta \) while the radius \( a_0 \), as seen from (2·14), would become large as \( \beta \) does so (if not, the catenoid might be squashed as \( \beta \to \infty \)).

§ 4. A contribution from the Liouville action

To specify the Lagrangean appearing in (3·15), we follow Marnelius' idea. We then consider some contribution from the Liouville part of the Lagrangean. We do not fully work out the necessary functional-integration, but by a rough approximation we shall show how the Pomeron intercept can be free from the dimension by the presence of \( \varphi \), whereby we guess the value of \( C \) in (1·3).

We first collect here three important equations found in §2.

\[
(\partial_{s}^{2} + \partial_{t}^{2})y(s, t) = 0, \tag{4·1}
\]
\[
(\partial_{s}^{2} + \partial_{t}^{2})\varphi + Ke^{\varphi} = 0 \tag{4·2}
\]

and

\[
K + \mu^2 e^{-2\varphi} = 0. \tag{4·3}
\]

Hence one can propose the following Lagrangean:

\[
\mathcal{L} = \mathcal{L}_0 + C\mathcal{L}_1, \tag{4·4}\]

where

\[
\mathcal{L}_0 = \frac{1}{4\pi}(\partial_{s}\varphi)^2 + (\partial_{t}\varphi)^2 \tag{4·5}
\]

and

\[
\mathcal{L}_1 = \frac{1}{2}(\partial_{s}\varphi)^2 + (\partial_{t}\varphi)^2 - Ke^{\varphi} - 2\mu\sqrt{-K}. \tag{4·6}
\]

For, the variations of (4·6) with respect to \( \varphi \) and \( K \) yield (4·2) and (4·3) respectively. Note however that if we consider the problem in the section where (4·3) or the Codazzi equation holds, \( \mathcal{L}_1 \) of (4·6) is simply rewritten

\[
\mathcal{L}_1 = \frac{1}{2}(\partial_{s}\varphi)^2 + (\partial_{t}\varphi)^2 - \mu^2 e^{-\varphi}. \tag{4·7}
\]

Accordingly the kernel (3·15) may become

\[
k[y(s, \beta), y(s, 0); \beta/l] = (1/N) \int D^2 y(s, t) D\varphi(s, t)\]
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\[ \times \exp \left[ -\frac{1}{4\pi} \int C \, dsdt \left( (\partial_s y)^2 + (\partial_t y)^2 \right) \right. \]

\[ \left. - C \int C \, dsdt \left\{ \frac{1}{2} \left[ (\partial_s \varphi)^2 + (\partial_t \varphi)^2 \right] - \mu^2 e^{-\varphi} \right\} \right]. \]  

(4.8)

As to the \( y(s, t) \), similarly to (3.10) we can decompose it such that

\[ y(s, t) = y^h(s, t) + y^0(s, t), \]

(4.9)

where \( y^h(s, t) \) is the one subject to (4.1) and \( y^0(s, t) \) is a fluctuation. The boundary conditions they observe are as follows:

\[ \begin{align*}
  y^h(s, t) & \rightarrow \left\{ \begin{array}{l}
  y(s, 0) \quad \text{at} \quad t = 0, \\
  y(s, \beta) \quad \text{at} \quad t = \beta,
  \end{array} \right. \\
  y^0(s, t) & \rightarrow 0 \quad \text{at} \quad t = 0 \text{ and } \beta.
\end{align*} \]  

(4.10)

By this change of variable, kernel (4.8) splits as

\[ k[y(s, \beta), y(s, 0); \beta/l] = (1/N) \exp \left\{ -\frac{1}{2\pi} D[y^h; C^*] \right\} k^0[\varphi; \beta/l], \]

(4.11)

where

\[ k^0[\varphi; \beta/l] = \int D y^0 \varphi \exp \left\{ -\frac{1}{2\pi} D[y^0; C^*] - C[D[\varphi; C^*] - \mu^2 \int dsdt e^{-\varphi}] \right\}, \]

(4.12)

\( D[f, C^*] \) denoting the Dirichlet integral

\[ D[f, C^*] = \frac{1}{2} \int C \, dsdt \left( (\partial_s f)^2 + (\partial_t f)^2 \right). \]

Out of the factors in (4.12), the kernel defined by

\[ k^0[\beta/l] = \int D y^0 \exp \left\{ -\frac{1}{2\pi} D[y^0; C^*] \right\} \]

(4.13)

is a familiar functional-integral appearing in the case of the Pomeron propagator in the old string model, and it was once calculated by the present author by use of the zeta-function regularisation. We shall give a brief account of the manipulation in the Appendix. The result itself is simple and expressed as

\[ k^0[\beta/l] = \eta(\beta/l)^{-\delta}, \]

(4.14)

where \( \eta \) is the Dedekind eta function:

\[ \eta(\tau) = e^{\pi i \tau/12} \prod_{\nu=1}^{m} (1 - e^{2\pi i \nu \tau}). \]

(4.15)

In (4.14), \( \delta \) denotes the number of the transverse degrees of dimension, and if we take the Faddeev-Popov ghosts into consideration, it is given as

\[ \delta = D - 2, \]

(4.16)

where \( D \) denotes the space-time dimension.

In the old string model, the ground-state mass squared of the Pomeron trajectory \( \alpha(s) \)
\[ m^2 = -\alpha(0)(= -2) = \lim_{l \to 0} \left( \frac{\beta}{l} \pi \right)^{-1} \log k_0^o[\beta/l] \]  

(4.17)

being a generalised version of (3.13). The limit of \( l \to 0 \) has been taken because the ground state mode belongs to the point-particle modes. The behaviour of \( k_0^o[\beta/l] \) for very small \( l \) is known with the aid of the following estimation:

\[
\lim_{l \to 0} \prod_{\nu=1}^{\infty} (1 - e^{-2\pi \nu l/\beta})^{-1} \quad \longrightarrow \quad \exp \left( \frac{\pi^2}{6} \frac{1}{1 - \exp(-2\pi l/\beta)} \right) \sim e^{(\pi/12)(\beta/\lambda)}.
\]

(4.18)

In effect we see immediately from (4.17) that

\[-2 = -\delta/12,
\]

(4.19)

which implies

\[ D = 26, \]

(4.20)

just being the critical dimension of the old bosonic model. One of possible shortcomings of (4.19) is that the space-time dimension is forced to depend on the intercept. What is then expected of the Polyakov model is a removing of this defect. Therefore we turn to take account of the remaining factor in (4.12):

\[ k_1^o[\varphi; \beta/l] = \int \mathcal{D} \varphi \exp \left\{ -\sum_{\nu=1}^{\infty} (\varphi[C^*] + C_{\mu}^2 \int d\sigma t d t e^{-\varphi}) \right\}. \]

(4.21)

As far as we know, there has not yet been developed any general method of evaluation of this integral. However, as far as we are concerned here, we need only a small \( l \) limit of the integral, and in our case we shall be able to take advantage of some explicit expressions obtained in §2.

Before taking the limit for small \( l \), we first see how \( k_1^o \) behaves as \( \beta \to 0 \) because if \( \beta \) is sufficiently small the "paths" are mostly within the most important "path" without fluctuation. Then we should have an estimation

\[ k_1^o \sim \exp \left\{ -\sum_{\nu=1}^{\infty} (\varphi[C^*] + C_{\mu}^2 \int d\sigma t d t e^{-\varphi}) \right\} \]

(4.22)

for small \( \beta \). Let us now invoke the concrete expression (2.16) in (4.22). Then

\[ k_1^o \longrightarrow \exp \left\{ -4C \pi (\pi \beta/l - 4 \tanh(\beta \pi/2l)) \right\} \sim \exp(4C \pi^2 \beta/l). \]

(4.23)

If we use this result in (4.17) by replacing \( k_0^o \) in (4.17) by \( k_0^o k_1^o \), then we have, in place of (4.19), the following relation:

\[-\alpha(0) = -\left( \frac{\delta}{12} + 4C \pi \right). \]

(4.24)

Now, by the requirement that \( \alpha(0)(=2) \) should not be dependent on the space-time dimension, we are to obtain
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\[ C = \frac{(\delta' - \delta)}{48\pi}, \]  

(4.25)

where \( \delta' \) is a constant. And if \( a(0) = 2 \) we necessarily have \( \delta' = 24 \). Hence, Eq. (4.25), in the light of (4.20), can be written

\[ C = \frac{24 - (D-2)}{48\pi}, \]  

(4.26)

which thus verifies (1.4).

Before closing this section, we should like to record two remarks: One is concerning \( \varphi \) (no explicit separation of the background from the fluctuation part is done) —— The part of \( \varphi \) which contributes to \( k_{10} \), at least in the limit, should satisfy the boundary condition (2.33) or something like that, and thus has a much different aspect than the case of \( y^0(s,t) \). The other is about \( k_{10} \) (no explicit performance of the integration of it is done) —— \( k_{10} \) might be a modular form invariant under the group generated by \( \tau' \rightarrow \tau' + 1 \) and \( \tau' \rightarrow -1/\tau' \) just like \( k_0 \) given by (4.14) where \( \tau' = il/\beta \). In fact, the critical dimensionality is related to this property: \( \eta(\tau')^{-\delta} \) given by (4.14) has a pole at \( \tau' \rightarrow i\infty \), but the pole is a simple pole only when \( \delta = 24 \) because the local variable at \( \tau' \rightarrow i\infty \) is \( \exp(2\pi i\tau') \).

§ 5. Concluding remarks and discussion

The model we described in §2, though it is stated only in Euclidean space of three dimensions, shows clearly how the Liouville equation is associated with the propagation of a closed string, and, at the same time, what role the Virasoro gauge could play in considering the Liouville equation for \( \varphi(s,t) \) along with the Laplace equation for \( y(s,t) \). This naturally leads to the view that Polyakov's way is very convincing of taking explicit account of \( \varphi \) and applying the variational principle to \( \varphi \) in addition to \( y \).

It should be noted however that the Liouville equation we encountered in the present work looks different by sign somewhere from that in several literatures if \( \mu \) is real. In general, the total curvature \( K \), (2.25), can be of both signs. For example, if the surface is closed to be a sphere, \( K \) is positive and constant (for example see Ref. 15)). In our model, however, the \( K \) in the Lagrangean (4.6) is not meant to be a constant, and in the specific model of §2, it is explicitly given by (2.28), and always negative. Incidentally we may interpolate a remark here that the action integral

\[ \int dsdtK e^\eta \]  

(5.1)

coming from the second term of the rhs of (4.6) measures the total curvature of a geodesic triangle,\(^7\) and can be converted to a boundary-integral over the geodesic curvature by the Gauss-Bonnet theorem. The third term of the rhs of (4.6) is related to the torsion of the minimal surface.\(^7\)

In the model of §2, we may easily release the restriction that the number of space dimension is three, even though we may keep the assumption that one of the coordinates is proportional to the propertime. If we further need to augment the variety of the preassigned momentum incident to the boundaries, we should remove the assumption that

\footnote{\[ A \] relevance of the Liouville equation in the dual model was once pointed out by Omnès in a different context.\(^1\)}
the surface is that of revolution. The dual position $y^{(n)}(s,t)$ for fixed $t$ need not constitute a circle, but may be expanded as

$$y^{(n)}(s,t) = \sum_{n=0}^{\infty} \left[ a_{n}(t) \cos \frac{\pi}{l} s + b_{n}(t) \sin \frac{\pi}{l} s \right]. \quad (5.2)$$

The coefficient vectors $a_{n}(t)$ and $b_{n}(t)$ are determined by the boundary values $a_{n}(0), a_{n}(\beta), b_{n}(0)$ and $b_{n}(\beta)$ as follows:

$$a_{n}(t) = \frac{1}{2} \sum_{k} \text{sech} \frac{\lambda_{n} \beta}{2l} \left[ \{a_{k}(\beta)\} e^{(\lambda_{k}/2 \pi/1)(t-\beta/2)} + \{a_{k}(0)\} e^{-(\lambda_{k}/2 \pi/1)(t-\beta/2)} \right], \quad (5.3)$$

Returning to our model in §2, we should also note that we can use, in place of the initial condition (2·33), the boundary values $\varphi(0)$ and $\varphi(\beta)$ to determine $\varphi(t)$. This fact may imply, if we refer again to (2·4b), that we are to use the values of $\partial_{s}y(s,t)$ or $\partial_{t}y(s,t)$ at the boundaries in addition to the Dirichlet conditions. In terms of $x(s,t)$ introduced by (3·1), this is a matter concerning

$$(\partial_{s}x)^{2} = (\partial_{t}x)^{2} = e^{\varphi}, \quad (5.4)$$

which holds in the extremum if (2·4b) does because of the Cauchy-Riemann relations. In this respect, we shall close this paper by writing down an expression for $x(s,t)$ relevant to the model in §2 for interest. The holomorphic position $z(u = s + it)$ of (3·1) is

$$z = \left( \frac{i \gamma l}{\pi} \cos \frac{\pi}{l}(u - i \beta), \frac{i \gamma l}{\pi} \sin \frac{\pi}{l}(u - i \beta) \right), \quad \gamma u \quad (5.5)$$

up to trivial terms, so that $x(s,t)$ is given by

$$x = \left( -\rho'(t) \sin \frac{\pi}{l} s, \rho'(t) \cos \frac{\pi}{l} s, \gamma s \right), \quad (5.6)$$

where

$$\rho'(t) = \frac{\gamma l}{\pi} \sinh \frac{\pi}{l} \left( t - \frac{\beta}{2} \right) \quad (5.7)$$

and constitutes a skew-helicoid. It is apparent that $\rho'(t)$ satisfies (2·11), but is not subject to the boundary condition (2·10). There may be some other differences in character between the catenoid and the skew-helicoid, but the latter is still a surface of least area, and thus we finally remark that there will be given rise to another story where $y$ constitutes a skew-helicoid and $x$ does a catenoid.
Appendix

We here give a brief summary of a method of deriving (4·14). See Ref. 13) for further details.*)

Let \( \Delta = \partial_x^2 + \partial_y^2 \). Then the rhs of (4·13) is first evaluated as

\[
\kappa_0^0[\beta/l] = (2\pi)^{2+n} (\det \Delta)^{-\delta/2},
\]

(A·1)

in which \( j \) is a scaling constant, and \( Z_0(0) \) and \( \det \Delta \) are defined as

\[
Z_0(0) = \lim_{k \to 0} \frac{1}{\Gamma(k)} \int_0^\infty d\tau \tau^{k-1} \text{Tr} e^{-\tau A}
\]

(A·2)

and

\[
\det \Delta = \exp \left\{ - \int_0^\infty d\tau \tau^{-1} \text{Tr} e^{-\tau A} \right\}.
\]

(A·3)

Let \( \lambda_\mu \) and \( \lambda_{\mu\nu} (\mu, \nu = 1, 2, 3, \ldots \in \text{infini}) \) be the eigenvalues of \( \Delta \) for \( C^* \). Define the zeta-function by

\[
Z_0(k) = \frac{1}{2} \sum_{\mu = 1}^2 \lambda_\mu^k + \sum_{\mu, \nu = 1}^3 \frac{1}{\lambda_{\mu\nu}^k}.
\]

(A·4)

Then the rhs of (A·3) can be written as

\[
\det \Delta = \exp(-Z_0'(0))
\]

(A·5)

and we can also see

\[
Z_0(k = 0) = Z_0(0).
\]

(A·6)

By use of such formulae as Kronecker's first limit formula, we are eventually led to

\[
Z_0(0) = 0 \quad \text{and} \quad Z_0'(0) = - \log |\eta(i\lambda/\beta)|.
\]

(A·7)

References


*) On this occasion we should like to correct an obvious error in (3·5) of Ref. 13): \( s \) and \( t \) there should be interchanged with each other.