Successive Screw Approximation in Ising Lattice Gauge Theory

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Kramers and Wannier's successive screw technique is applied to the three-dimensional Ising lattice gauge theory. The largest eigenvalue of the transfer matrix is directly calculated on a computer, independently of Monte Carlo simulation. Anomalous behavior of the specific heat implies the existence of a phase transition.

According to Wilson's ideas\(^1\) on a mechanism for total confinement of quarks, Balian and others\(^2\) discussed the existence of a phase transition for the gauge-invariant Ising systems (a discrete gauge group \(Z_2\) with two elements \(\{1, -1\}\)) in dimensions \(d=3\) and \(4\). After that Creutz and others\(^3\) evaluated the path integral for the lattice gauge theory by Monte Carlo simulations. In fact, the computer experiments have provided a powerful and useful tool for solving the problem, as well as for studying statistical spin systems.

Remembering that there is no rigorous theory except for the two-dimensional Ising spin system,\(^4,5\) and considering that the relevant results on the existence of a phase transition for some other systems have been obtained so far only by computer experiments, we should ask if there is another means of solving the problem.

Because of recent extra-high performance computers, we are strongly tempted to treat the eigenvalue problem of statistical models directly by the use of a huge computer, quite unlike the computer experiment. In the previous papers,\(^6\) the eigenvalue problem of the three-dimensional Ising spin systems was treated in Kramers and Wannier's successive screw approximation\(^6\) on a computer.

In the present work, we will apply the successive screw technique to the Ising lattice gauge theory and evaluate the largest eigenvalue of the transfer matrix by the direct use of a huge computer.

Let us consider a hypercubical lattice in \(d\)-dimensions with unit spacing. The lattice consists of \(N\) sites, and we eventually let \(N \to \infty\). A variable \(\mu_{ij} (=\pm 1)\) is assigned to each link \((ij)\) of neighboring sites, and a set of four neighboring links is a plaquette \((ijkl)\). We compute a partition function in the pure-gauge field case

\[
Z = 2^{N_d} \sum_{\{\mu_{ij} = \pm 1\} \text{ plaquettes}} \exp(\beta \sum_{(ijkl)} \mu_{ij} \mu_{jk} \mu_{kl} \mu_{li}),
\]

where \(\beta\) is the coupling parameter which is inversely proportional to the temperature.

We first apply Kramers and Wannier's successive screw method to the two-dimensional Ising gauge theory. Consider a torus surface, on which \(mn\) sites are distributed along a continuous line twisting its way in screw-like fashion. Each pitch of a screw consists of \(m\) sites, and the \((n+1)\)th pitch is regarded as equivalent to the first pitch as a periodic boundary condition. The \(mn = N\) sites in all are distributed throughout a square lattice. We eventually let \(n \to \infty\). Figure 1 gives us the equation, which corresponds to Eq. (14) of Ref. 6),

\[
\sum_{\mu_{-m+1} \sim \mu_{-m}} \exp(\beta \mu_0 \mu_{-1} \mu_{-2} \mu_{-3} \mu_{-4} \mu_{-5}) A(\mu_{-1} \mu_{-2} \mu_{-3} \mu_{-4} \mu_{-5} \mu_{-6} \mu_{-7} \mu_{-8} \mu_{-9} \mu_{-10} \mu_{-11} \mu_{-12} \mu_{-13} \mu_{-14} \mu_{-15} \mu_{-16} \mu_{-17} \mu_{-18} \mu_{-19} \mu_{-20} \mu_{-21}) = \lambda A(\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8 \mu_9 \mu_{10} \mu_{11} \mu_{12} \mu_{13} \mu_{14} \mu_{15} \mu_{16} \mu_{17} \mu_{18} \mu_{19} \mu_{20} \mu_{21})
\]

where \(A(\cdots)\) stands for the probability referring to the configuration\(\cdots\). Our problem is to find
the largest eigenvalue $\lambda_{\text{max}}$ of the $2^m \times 2^m$ transfer matrix $\mathcal{M}(\beta)$:

$$\mathcal{M}(\beta)A(\beta) = \lambda(\beta)A(\beta).$$  \hspace{1cm} (3)

The partition function $Z$ is given by $\lambda_{\text{max}}$ as

$$\lim_{N \to \infty} N^{-1} \ln Z = \ln \lambda_{\text{max}}.$$  \hspace{1cm} (4)

In the pure gauge field case, the matrix $\mathcal{M}(\beta)$ can be reduced to the two $2^m \times 2^m$ irreducible matrices $V_r(\beta)$.\(^7\) Insofar as we discuss the thermal properties, we have only to find the largest eigenvalue of $V_r(\beta)$:

$$V_r(\beta) = \begin{pmatrix}
a a^{-1} a a^{-1} & \cdots & a a^{-1} a a^{-1} \\
\cdots & \cdots & \cdots \\
a a^{-1} a a^{-1} & \cdots & a a^{-1} a a^{-1}
\end{pmatrix},$$  \hspace{1cm} (5)

where $a = e^\beta$. The largest eigenvalue $\lambda_{\text{max}}(n \to \infty)$ can be calculated as

$$\lambda_{\text{max}} = 2(a + a^{-1}) = 4 \cosh \beta,$$  \hspace{1cm} (6)

irrespective of $m$. Thus the partition function shows no phase transition for nonzero temperature. We knew already that the two-dimensional Ising gauge theory is trivial,\(^2\) and Kogut\(^3\) proved the equivalence of the two-dimensional Ising gauge model to the one-dimensional Ising spin system.

Let us now consider the three-dimensional Ising gauge theory. Suppose a multilayer torus consists of $n$ layers. Over the surface of each torus layer, $m$ sites are distributed along a continuous line twisting its way in screw-like fashion. Each torus layer is continuously connected with the next torus layer (see Fig. 1 of Ref. 5), so that $m$ sites in all are distributed along a continuous line throughout a simple cubic lattice. As a periodic boundary condition, the $(n+1)$th layer is regarded as equivalent to the first layer, and eventually we let $n \to \infty$. From Fig. 2, we have the equation similarly

$$\sum_{\mu_{-1,0,0,1}} \exp \beta(\mu_{0,0,m,0,0} \mu_{0,0,m,0,0})$$

$$+ \mu_{1,0,m,0} \mu_{1,0,m,0} + \mu_{0,0,m,0} \mu_{0,0,m,0}$$

$$\times A(\mu_{-1,0,0,0,1}, \mu_{-1,0,0,0,1}, \mu_{-1,0,0,0,1}, \mu_{-1,0,0,0,1})$$

$$= \lambda A(\mu_{0,0,m,0,0} \mu_{0,0,m,0,0} \mu_{0,0,m,0,0} \mu_{0,0,m,0,0}).$$  \hspace{1cm} (7)

The $2^2m \times 2^2m$ transfer matrix $\mathcal{M}(\beta)$ in the three-dimensional case is somewhat complicated. In the same way as before, the $2^3m-1 \times 2^3m-1$ reduced matrix $V_r(\beta)$ is written as

$$V_r(\beta) = 
\begin{pmatrix}
f_1 & g_1 & f_2 & g_2 & f_3 & \cdots \\
g_1 & f_1 & g_2 & f_2 & g_3 & \cdots \\
f_2 & g_2 & f_3 & g_3 & f_4 & \cdots \\
g_2 & f_3 & g_3 & f_4 & g_5 & \cdots \\
f_3 & g_3 & f_4 & g_5 & f_6 & \cdots \\
g_3 & f_4 & g_5 & f_6 & g_7 & \cdots \\
f_4 & g_5 & f_6 & g_7 & f_8 & \cdots \\
g_5 & f_6 & g_7 & f_8 & g_9 & \cdots \\
f_6 & g_7 & f_8 & g_9 & f_{10} & \cdots \\
g_7 & f_8 & g_9 & f_{10} & g_{11} & \cdots \\
de & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.'  \hspace{1cm} (8)

Fig. 2. Building up the infinite screw for the three-dimensional simple cubic lattice.
where \( f, g, f', g' \) \((i = 1 \text{ to } 4)\) are given by

\[
\begin{align*}
  f_i &= \left[ a^2 a a a^{-1} a^3 a a a^{-1} \right], \\
  g_i &= \left[ a^3 a^{-1} a a a^{-1} a^2 a \right], \\
  f_i' &= \left[ a^3 a^{-1} a a a^{-1} a^2 a \right], \\
  g_i' &= \left[ a^2 a a a^{-1} a^3 a a a^{-1} \right].
\end{align*}
\]

The largest eigenvalue \( \lambda_{\text{max}} \) of the transfer matrix (8) can be calculated by means of iteration on a computer. The size of core storage of the available large-scale machines restricts our computation to the cases \( l m n = 2 \times 2 \times \infty \) and \( 2 \times 3 \times \infty \) at the present time.

Then the behavior of the specific heat \( C_v \) is obtained by

\[
C_v R = \beta^2 d \ln \lambda_{\text{max}} / d \beta.
\]

Figure 3 shows the \( C_v / R \) versus \( \beta \) curves for these three-dimensional lattices and also the curve in the two-dimensional case for comparison. The inverse temperature, at which the specific heat becomes maximum is given by \( \beta_{c,x} = 0.658 \) and \( 0.668 \) for the \( 2 \times 2 \times \infty \) and \( 2 \times 3 \times \infty \) lattices respectively. The fact that a sharp anomaly of the specific heat has appeared even in the case of rather small values \( l \) and \( m \), strongly suggests the existence of a phase transition in the three-dimensional Ising lattice gauge theory.

If we compare the anomalous behavior of the specific heat in the three-dimensional lattice gauge theory with that in the three-dimensional Ising spin systems \(^b\) for the same small values \( l \) and \( m \), the specific heat in the lattice gauge theory shows a more rapid increase, because of the features of plaquette interactions other than spin-spin interactions.

Wegner's dual relation \(^b\) combined with the \( P\text{a}d\mathring{e} \) approximant value of the critical temperature in the three-dimensional Ising spin system \(^b\) leads to the critical coupling in the three-dimensional Ising gauge theory, that is

\[
\beta_c = 0.7613.
\]

Although from Fig. 3 the calculated \( \beta_{c,x} \) seems to consistently approach \( \beta_c \) with increasing \( l \) and \( m \), it is desirable yet to be confirmed by the calculation for larger \( l \) and \( m \). As a matter of fact, however, in order to deal with the \( 3 \times 3 \times \infty \) gauge lattice, about 1 G bytes of storage are
required. We may hope that such a giant computer will be in practical use soon.

As for the four-dimensional Ising lattice gauge theory, the very least lattice $(2 \times 2 \times 2 \times \infty)$ needs approximately $32 \ G$ bytes of storage, which would become available in several years.

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4) L. Onsager, Phys. Rev. 65 (1944), 117.
8) J. B. Kogut, Rev. Mod. Phys. 51 (1979), 659.