About the Lagrangian Representation of the Fokker-Planck Dynamics. I

--- A Non-Renormalized Description ---

F. Sagués* and L. Garrido*

*Departamento de Física Teórica, Universidad de Barcelona
†Departamento de Química Física, Universidad de Barcelona
Diagonal, 647, Barcelona-28

(Received August 20, 1983)

A functional description of the Fokker-Planck dynamics in terms of a Lagrangian is considered. The powerful functional techniques are used in order to generate a non-renormalized perturbation scheme. We study the same question in a Hamiltonian description. It results that both expansions coincide exactly and can be expressed in a natural way according to a set of diagrams uniquely determined. In addition, the characterization of any propagator like the response function is also discussed.

§ 1. Introduction

Whereas an operational description for a stochastic dynamics was initially proposed in the fundamental work of Martin, Siggia and Rose (RSR),\(^1\) and has been used afterwards by many others,\(^2\) we may also think of an alternative description based on the use of functional integrals.\(^3\)

If we restrict ourselves to this last viewpoint, two representations can be considered: a Hamiltonian one and a Lagrangian one. The use of the powerful functional techniques applied to a generating functional built from the first of these two representations is sufficiently well-known,\(^4\) and in this way we can readily establish perturbation expansions in terms of free or elementary propagators. A fundamental feature of this kind of expansions consists in the equivalent role that play the free correlation and response functions; a fact that is diagrammatically reflected in the unavoidable presence of two types of lines.

On the other hand, in a Lagrangian representation, the role of the response function is less evident, and in this sense we want first to generate this kind of non-renormalized expansions, in order to see if it is possible to build them in terms of only one kind of lines. This will be the aim of this paper.

Furthermore, and being more fundamental for us, we want to consider this same question for a perturbation expansion in terms of dressed propagators, which in some sense would be a renormalized expansion.

In this context one can adopt an MSR scheme\(^4\) based on a Dyson equation involving both correlation and response functions. But this coupling makes the calculation very difficult, except for a few fortunate cases in which certain types of fluctuation-dissipation theorems (FDT) can be used.\(^5\)

To avoid this limitation, and in the same context of a renormalized perturbation scheme, we propose again the use of the Lagrangian representation considered here, and

---

* See, for example, the first four chapters of the book by F. Langouche et al. in Ref. 3), and the references given therein related to this point.
we can anticipate that, precisely owing to the representation utilized, we will be able to state a Dyson equation only in terms of one propagator, making no use of any additional relation like the FDT ones. This will be the contents of a companion paper.

The present paper is organized so that in §2 a summary of the fundamental results in the Hamiltonian representation is reviewed. In §3 we present the generating functional from a Lagrangian representation and we find a functional differential equation for it, which is both the connection and the starting point for the next paper. A restriction of this equation to the free dynamics will be considered and solved in §4. In §5 we analyze the unrenormalized perturbation expansion for the correlation function, and in §6 we find an explicit expression for the response function in the Lagrangian representation. We devote the last section to the conclusions.

§ 2. A Hamiltonian functional representation

Let us consider an $m$-dimensional system $q(\tau) = \{q^n(\tau)\}_{n=1,\ldots,m}$, whose conditional probability density $P(q, t/q_0, t_0)$ satisfies a Fokker-Planck equation (FPE),

$$
\frac{\partial}{\partial t} P(q, t/q_0, t_0) = \frac{\partial}{\partial q} \left\{ \left(-f^n(q) + \frac{1}{2} D^{\mu\nu} \frac{\partial}{\partial q^\nu} \right) P(q, t/q_0, t_0) \right\}
$$

(2·1)

with the usual initial condition

$$
P(q, t_0/q_0, t_0) = \delta(q - q_0).
$$

(2·2)

A path integral representation of $P(q, t/q_0, t_0)$ between an initial point, $q(t_0) = q_0$, and a final one, $q(t) = q$, can be established using a pre-point discretization, as

$$
P(q, t/q_0, t_0) = \int \mathcal{D}q(\tau) \mathcal{D}p(\tau) \exp \left\{ i \int_{t_0}^{t} d\tau [p_\mu(\tau) \dot{q}^\mu(\tau) - H(p(\tau), q(\tau))] \right\},
$$

(2·3)

where $H(p, q)$, which is sometimes called Hamiltonian function, is given by

$$
H(p, q) = p_\mu f^n(q) - (i/2) p_\mu p_\nu D^{\mu\nu}.
$$

(2·4)

The non-uniqueness of the discretization is irrelevant in a perturbation expansion*) as long as it is used consistently. In this case we have chosen the pre-point one for the sake of simplicity.

Let us define the propagators of the system, which in terms of a path integral are expressed by

$$
\langle p_{\mu_1}(t_n) \cdots q^{\nu_1}(t_{i'}) \rangle = \int \mathcal{D}q(\tau) \mathcal{D}p(\tau) p_{\mu_1}(t_n) \cdots q^{\nu_1}(t_{i'}) \times \exp \left\{ i \int_{t_0}^{t} d\tau [p_\mu(\tau) \dot{q}^\mu(\tau) - H(p(\tau), q(\tau))] \right\},
$$

(2·5)

where causality requires that the propagator be zero if any instant $t_i$ is greater than any $t_i'$. The measure $\mathcal{D}q(\tau)$ in Eq. (2·5) involves an additional integral over $q(t)$, in order to avoid that the propagators depend on this final point.

With the introduction of a generating functional $Z(J, J^*)$,

*) See the footnote on p. 50.
the propagators are obtained as
\[
\langle \rho_{\mu_n}(t_n) \cdots q_{\nu_1}(t_1') \rangle = \frac{1}{i^{n+m}} \frac{\delta^{n+m} Z(J, J^*)}{\delta^{n+m}(t_n) \cdots \delta^{n+m}(t_1')} |_{J=J^*=0}.
\]

Now, we assume that the drift can be decomposed into a well-defined elementary dynamics as the unperturbed part and its perturbation:
\[
f^\mu(q) = f_0^\mu(q) + F^\mu(q),
\]
where the bracketed index denotes the one without summation convention, which we assume through the paper.

For the elementary dynamics previously introduced, we can write down the corresponding generating functional
\[
Z_0(J, J^*) = \exp \left\{ i \int_{t_0}^t d\tau \left[ f_0^\mu(\tau) q_\mu(\tau) + J^\mu(\tau) \rho_\mu(\tau) \right] \right\},
\]
\[
Z(0, 0) = \int d^m q P(q, t/t_0, t_0) = 1,
\]
(2.6)

\[
\text{If the initial condition is subject to a given probability distribution } P(q_0), \text{ we can introduce this statistical component into the propagators}
\]
(2.13)

\[
\langle \cdots \rangle = \int d^m q_0 \langle \cdots \rangle P(q_0)
\]

that can be obtained from a suitable functional \( Z^\rho(J, J^*) \)
\[ Z^p(J, J^*) = \langle Z(J, J^*) \rangle^p \] (2.14)

satisfying
\[ Z^p(J, J^*) = \exp \left\{ -i \int_{t_0}^t d\tau H_1 \left( \frac{1}{i} \frac{\delta}{\delta J^*(\tau)}, \frac{1}{i} \frac{\delta}{\delta J(\tau)} \right) \right\} Z_0^p(J, J^*). \] (2.15)

From Eq. (2.15) we can readily establish a non-renormalized expansion in terms of the free correlation and free response functions

\[ G_{\mu \nu}(t', t'') = g_{\mu \nu}(t', t'') + \Delta_{\mu \nu}(t', t''), \]

\[ R_{\mu \nu}(t', t'') = R_{\mu \nu}(t' - t'') = \theta(t' - t'') \delta_{\mu \nu} \exp \{ \lambda_{(\mu)}(t' - t'') \}, \] (2.16)

where \( g_{\mu}(t', t'') \) contains all the dependence on the initial conditions

\[ g_{\mu}^{\mu\nu}(t', t'') = \langle q_{\mu} \exp \{ \lambda_{(\mu)}(t' - t_0) \} q_{\mu} \exp \{ \lambda_{(\mu)}(t'' - t_0) \} \rangle^p \]

\[ = \langle q_0^p(t') q_0^p(t'') \rangle^p \] (2.17)

with a slight change in the notation

\[ q_0^\mu(t') = q_{\mu} \exp \{ \lambda_{(\mu)}(t' - t_0) \} \] (2.18)

that will be useful later.

§ 3. Generating functional in a Lagrangian representation

In the previous section we have considered a phase-space functional representation of the Fokker-Planck dynamics. Certainly, we can also deal with a configuration-space functional representation of the same dynamics. Since the Hamiltonian is a quadratic function in \( p(\tau) \), we can perform the transformation from one to another representation by a mere integration over these variables. In this way we obtain

\[ P(q, t/q_0, t_0) = \int \mathcal{D} q(\tau) \exp \left\{ -i \int_{t_0}^t d\tau \left[ \dot{q}^\mu(\tau) - f^\mu(q(\tau)) \right] D_{\mu \nu} \left( \dot{q}^\nu(\tau) - f^\nu(q(\tau)) \right) \right\}, \] (3.1)

where \( D_{\mu \nu} \) are the elements of the inverse of the diffusion matrix.

If we write down the Lagrangian function from the Hamiltonian, as is usually done in classical mechanics, we obtain

\[ \mathcal{L}(q, \dot{q}) = \frac{i}{2} (\dot{q}^\mu - f^\mu(q)) D_{\mu \nu} (\dot{q}^\nu - f^\nu(q)) \] (3.2)

or in terms of a real-value one,

\[ \mathcal{L}_0(q, \dot{q}) = -i \mathcal{L}(q, \dot{q}) = (1/2)(\dot{q}^\mu - f^\mu(q)) D_{\mu \nu} (\dot{q}^\nu - f^\nu(q)). \] (3.3)

This last expression allows us to rewrite Eq. (3.1) in a compact form,

\[ P(q, t/q_0, t_0) = \int \mathcal{D} q(\tau) \exp \left\{ -\int_{t_0}^t d\tau \mathcal{L}_0(q(\tau), \dot{q}(\tau)) \right\}, \] (3.4)

where the Lagrangian is the one that corresponds to the pre-point discretization that we have taken. The propagators defined now are those involving \( q(\tau) \) variables, and once more they can be stated through a path integral.
Nevertheless, response-type propagators can also be easily evaluated, as we will see in §6. Introducing a generating functional depending now on an external source \( J(\tau) \) only

\[
Z(J) = \exp\left\{ \int_{t_0}^{t_f} d\tau \mathcal{L}_\nu(\dot{q}(\tau), q(\tau)) - J(\tau)q^n(\tau) \right\},
\]

\( Z(0) = 1 \),

we obtain in the usual way

\[
\langle q^{\mu_1}(t_n) \cdots q^{\mu_n}(t_1) \rangle = \frac{\delta^n Z(J)}{\partial J_{\mu_1}(t_1) \cdots \partial J_{\mu_n}(t_n)} \bigg|_{J=0}.
\]

3.1. Functional differential equation for \( Z(J) \)

In order to obtain a functional differential equation for \( Z(J) \), we apply now the integration by parts lemma to the integral in (3·6):

\[
\int \mathcal{D}q(\tau) \frac{\delta}{\delta q^n(t')} \exp \left\{ - \int_{t_0}^{t_f} d\tau [\mathcal{L}_\nu(\dot{q}(\tau), q(\tau)) - J(\tau)q^n(\tau)] \right\} = 0, \quad t > t' > t_0.
\]  

According to the decomposition in Eq. (2·8), we may consider in the Lagrangian a free term and a perturbative one. We can further decompose this last one in two contributions:

\[
\mathcal{L}_\nu(q, \dot{q}) = \mathcal{L}_0(q, \dot{q}) + \mathcal{L}_1(q, \dot{q}),
\]

\[
\mathcal{L}_1(q, \dot{q}) = \mathcal{L}_1^I(q) + \mathcal{L}_1^{II}(q, \dot{q}),
\]

where

\[
\mathcal{L}_0(q, \dot{q}) = 1/2 [\dot{q}^\nu - \lambda(\nu)q^\nu] D_{\mu\nu}[\dot{q}^\nu - \lambda(\nu)q^\nu],
\]

\[
\mathcal{L}_1^I(q) = 1/2 F^{\mu}(q) D_{\mu\nu} F^\nu(q),
\]

\[
\mathcal{L}_1^{II}(q, \dot{q}) = - F^{\mu}(q) D_{\mu\nu}[\dot{q}^\nu - \lambda(\nu)q^\nu].
\]

In terms of these contributions we can evaluate the functional derivative in Eq. (3·8) obtaining

\[
\int \mathcal{D}q(\tau) \exp \left\{ - \int_{t_0}^{t_f} d\tau [\mathcal{L}_\nu(\dot{q}(\tau), q(\tau)) - J(\tau)q^n(\tau)] \right\} \times \left[ \tilde{\mathcal{L}}_{\mu\nu}(t')q^\nu(t') - \frac{\partial \mathcal{L}_1^I}{\partial q^n(t')} + \frac{d}{dt'} \frac{\partial \mathcal{L}_1^{II}}{\partial q^n(t')} + J(\tau) \right] = 0,
\]

where the differential operator \( \tilde{\mathcal{L}}(t') \) arises from \( \mathcal{L}_0(q, \dot{q}) \) and it is defined as

\[
\tilde{\mathcal{L}}_{\mu\nu}(t') = D_{\mu\nu}\left( \frac{\partial}{\partial t'} + \lambda(\nu) \right) \left( \frac{\partial}{\partial t'} - \lambda(\nu) \right).
\]

On the other hand, if we define

\[
\mathcal{L}_1^{II}(q, \dot{q}) = \frac{\partial \mathcal{L}_1^I(q, \dot{q})}{\partial q^n} \quad \frac{d}{dt'} \frac{\partial \mathcal{L}_1^I(q, \dot{q})}{\partial q^n},
\]

\[
\mathcal{L}_1^{II}(q, \dot{q}) = \frac{d}{dt'} \frac{\partial \mathcal{L}_1^I(q, \dot{q})}{\partial \dot{q}^n}.
\]
Eq. (3·11) yields the functional equation applicable to \( Z(J) \) which we wanted
\[
\left[ \tilde{\Lambda}_{\mu}(t') \frac{\delta}{\delta J_{\mu}(t')} - \tilde{\mathcal{L}}_{\mu}(t') \frac{\delta}{\delta J_{\mu}(t')} + \frac{d}{dt'} \frac{\delta}{\delta J_{\mu}(t')} \right] Z(J) = 0
\] (3·14)
with the obvious boundary condition
\[
\frac{\delta Z(J)}{\delta J_{\mu}(t_0)} = q_{0\mu} Z(J).
\] (3·15)

§ 4. Reduction to the elementary dynamics

When we particularize Eq. (3·14) to the free dynamics and express this equation in terms of \( Z_0(J) \) we obtain
\[
\tilde{\Lambda}_{\mu}(t') \frac{\delta Z_0(J)}{\delta J_{\mu}(t')} = -J_{\mu}(t') Z_0(J)
\] (4·1)
and the boundary condition becomes
\[
\frac{\delta Z_0(J)}{\delta J_{\mu}(t_0)} = q_{0\mu} Z_0(J).
\] (4·2)
We can easily evaluate \( Z_0(J) \) from (4·1) and (4·2). After some formal manipulations that we will not detail here, one gets \( Z_0(J) \) in the final form
\[
Z_0(J) = \exp \left\{ q_{0\mu} \int_{t_0}^{t} d\tau J_{\mu}(\tau) \exp \left\{ \lambda_{(\mu)}(\tau - t_0) \right\} \right. \\
\left. + \left( \frac{1}{2} \right) \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau' J_{\mu}(\tau) \Delta^{\mu\nu}(\tau, \tau') J_{\nu}(\tau') \right\},
\] (4·3)
where \( \Delta(\tau, \tau') \) is identically given by (2·11).

§ 5. Perturbation expansion in terms of free propagators

In the same way as \( Z(J, J^*) \) could be deduced from \( Z_0(J, J^*) \), \( Z(J) \) can be expressed now in terms of \( Z_0(J) \). More precisely,
\[
Z(J) = \exp \left\{ - \int_{t_0}^{t} d\tau \tilde{\mathcal{L}}_{\mu}(\tau) \left( \frac{\delta}{\delta J_{\mu}(\tau)} \frac{d}{d\tau} \frac{\delta}{\delta J_{\mu}(\tau)} \right) \right\} Z_0(J)
\] (5·1)
or with a prescribed distribution \( P(q_0) \) for the initial conditions
\[
Z^p(J) = \exp \left\{ - \int_{t_0}^{t} d\tau \tilde{\mathcal{L}}_{\mu}(\tau) \left( \frac{\delta}{\delta J_{\mu}(\tau)} \frac{d}{d\tau} \frac{\delta}{\delta J_{\mu}(\tau)} \right) \right\} Z_0^p(J).
\] (5·2)

5.1. Non-renormalized expansion for the correlation function

We pretend here to outline briefly the main features of a non-renormalized perturbation scheme for the correlation function. This propagator involves the average of two variables, and so it can be stated directly from (3·7) as
\[
\mathcal{G}^{\mu\nu}(t'', t') = \langle q^{\mu}(t'') q^{\nu}(t') \rangle^p
\]
Let us begin with a further splitting of the functional $Z_0(J)$ into its deterministic and stochastic parts

$$Z_0^d(J) = \exp \left\{ q_0^\mu \int_{t_0}^t d\tau J_\mu(\tau) \exp \left\{ \beta (\gamma_\mu(\tau - t_0))J_\mu(\tau) \right\} \right\},$$

$$Z_0^s(J) = \exp \left\{ \frac{1}{2} \int_{t_0}^t d\tau \int_{t_0}^t d\tau' J_\mu(\tau) \Delta^{\mu\nu}(\tau, \tau')J_\nu(\tau') \right\}.$$

The first one carries all the dependence on the initial conditions and it incorporates this statistical feature into the propagators. If the free system was assumed to be driven by a deterministic dynamics but subject to a prescribed distribution of initial conditions, $Z_0(J)$ is reduced to $Z_0^d(J)$, and therefore we call it deterministic. On the other hand, $Z_0^s(J)$ is defined in terms of the stochastic propagator $\Delta(\tau, \tau')$, so we can refer to it as the stochastic part of the generating functional. $Z_0^d(J)$ is moreover linear in $J(\tau)$, and allows us to define a functional differential operator $\hat{\phi}(t)$:

$$\hat{\phi}(t) = \left[ Z_0^d(J) \right]^{-1} \frac{\delta}{\delta J_\mu(t)} Z_0^s(J)$$

In terms of these operators we rewrite the correlation function (5.3) in the form:

$$\mathcal{G}^{\mu\nu}(t', t) = \left\langle \hat{\phi}^\mu(t') \hat{\phi}^\nu(t) \exp \left\{ - \int_{t_0}^t d\tau \mathcal{L}_1(\hat{\phi}(\tau), \frac{d}{d\tau} \hat{\phi}(\tau)) \right\} Z_0^s(J) \right\rangle_{J=0}.$$

Taking into account the partition of $\mathcal{L}_1(q, \dot{q})$ in (3.9), we prefer to evaluate the contribution from $\mathcal{L}_1^I(q)$ and $\mathcal{L}_1^II(q, \dot{q})$ in Eq. (5.8) separately.

### 5.2. Contributions from $\mathcal{L}_1^I$

Recalling its definition in (3.10) and assuming a polynomial function $F(q)$, the contributions from $\mathcal{L}_1^I(q)$ involve terms like

$$\langle \hat{\phi}^1(\tau_1) \cdots \hat{\phi}^m(\tau_m) Z_0^s(J) \rangle_{J=0} = \mathcal{P},$$

where we have interchanged, for convenience, the average over $P(q_0)$ and the prescription corresponding to taking the functional derivative evaluated in $J=0$. $Z_0^s(J)$ in (5.5) contains the external source $J(\tau)$ quadratically in the exponent and consequently the $\hat{\phi}(\tau)$ operators in (5.9) groups in pairs each one yielding a stochastic propagator $\Delta(\tau_{j+1}, \tau_{j+2})$. In this way the calculation of contributions like (5.9) is an easy task, and we obtain

$$\langle \hat{\phi}^1(\tau_1) \cdots \hat{\phi}^m(\tau_m) Z_0^s(J) \rangle_{J=0} = \mathcal{P} = \sum_{j=0}^m \sum_{(m-j)} \langle q_0^1(\tau_1) \cdots q_0^m(\tau_j) \rangle \times \sum_{(m-j)} \Delta^{j+1}(\tau_{j+1}, \tau_{j+2}) \cdots \Delta^{m-1}(\tau_{m-1}, \tau_m).$$

$$\times \sum_{(m-j)} \Delta^{j+1}(\tau_{j+1}, \tau_{j+2}) \cdots \Delta^{m-1}(\tau_{m-1}, \tau_m),$$

$$\times \sum_{(m-j)} \Delta^{j+1}(\tau_{j+1}, \tau_{j+2}) \cdots \Delta^{m-1}(\tau_{m-1}, \tau_m),$$
where $\Sigma_{(j)}$ stands for a sum over all the partitions of $m$ indices in two subsets of $(j)$ and $(m-j)$ respectively, and $\Sigma_{(m-j)}$ stands for a sum over all the contractions of $(m-j)$ indices grouped two by two. From the previous considerations, $(m-j)$ must be necessarily even. Furthermore, if $P(q_0)$ is a gaussian distribution, we can write (5.10) in a more compact form

$$\langle \hat{\phi}^1(\tau_1) \cdots \hat{\phi}^m(\tau_m) Z_0^s(J) \rangle_{J=0}^P$$

$$= \sum (g_0^{12}(\tau_1, \tau_2) + \Delta^{12}(\tau_1, \tau_2)) \cdots [g_0^{m-1}(\tau_{m-1}, \tau_m) + \Delta^{m-1}(\tau_{m-1}, \tau_m)]$$

$$= \Phi_0^{12.p}(\tau_1, \tau_2) \cdots \Phi_0^{m-1.p}(\tau_{m-1}, \tau_m)$$

which implies that in this special case $m$ must be also even.

5.3. Contributions from $L_1^{\Pi}$

Let us begin with the definition of $L_1^{\Pi}(q, \dot{q})$ in (3.10) where we introduce the $\hat{\phi}(\tau)$ operators considered previously. If we make an additional use of (5.7) we arrive at

$$L_1^{\Pi}(\phi(\tau), d/d\tau \phi(\tau)) = -F^\mu(\phi(\tau)) D_{\nu \mu} \left( \frac{d}{d\tau} - \lambda(\nu) \right) \frac{\delta}{\delta J_{\nu}(\tau)}.$$  \hspace{1cm} (5.12)

Thus the expansion for $\mathcal{G}^p(t'', t')$ will contain contributions like

$$\left( \frac{\delta}{\delta J_{\nu}(\tau')} D_{\mu \nu} \left( \frac{d}{d\tau} - \lambda(\mu) \right) \frac{\delta}{\delta J_{\nu}(\tau)} Z_0^s(J) \right)_{J=0}$$

that can be easily evaluated from (2.10) and (2.11). After some manipulations, we conclude that a term as (5.13) is responsible for the unavoidable presence of the free response function. Being more precise

$$\left( \frac{\delta}{\delta J_{\nu}(\tau')} D_{\mu \nu} \left( \frac{d}{d\tau} - \lambda(\mu) \right) \frac{\delta}{\delta J_{\nu}(\tau)} Z_0^s(J) \right)_{J=0} = R_{\delta \nu}(\tau' - \tau).$$  \hspace{1cm} (5.14)

We think that this is an important result, because although the response functions have not been introduced until now in the Lagrangian representation and, in fact, it was not necessary, it is also true that the vertices related to $L_1^{\Pi}(q, \dot{q})$ include the free response functions. Owing to this, our situation here is similar to the one considered in a Hamiltonian representation, where both free propagators equally appeared in the non-renormalized perturbation expansion for any propagator, and consequently it is not possible to state a non-renormalized scheme with only one kind of lines, even with a Lagrangian description.

5.4. Expansion for $\mathcal{G}^p(t'', t')$ up to second order

Let us now consider a quadratic perturbation in the drift

$$F^\sigma(q) = \gamma^\sigma_{\nu \nu} q^\nu q^\sigma$$  \hspace{1cm} (5.15)

and we restrict ourselves to a gaussian distribution $P(q_0)$. If we expand the exponential in (5.8) up to second order in $L_1$ we obtain

$$\mathcal{G}^{\mu \nu.p}(t'', t') = \langle \hat{\phi}^\sigma(t'') \hat{\phi}^\sigma(t') Z_0^s(J) \rangle_{J=0}^p$$
\begin{equation}
\left\langle \hat{\phi}^{\mu}(t^\prime)\hat{\phi}^{\nu}(t^\prime) \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau), d/d\tau\hat{\phi}(\tau))Z_0^s(J)|_{J=0} \right\rangle^P
\end{equation}

\begin{equation}
+ (1/2) \left\langle \hat{\phi}^{\mu}(t^\prime)\hat{\phi}^{\nu}(t^\prime) \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau), d/d\tau\hat{\phi}(\tau)) \times \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau^\prime), d/d\tau\hat{\phi}(\tau^\prime))Z_0^s(J)|_{J=0} \right\rangle^P + O(\mathcal{L}_1^3).
\end{equation}

(5.16)

We first consider the two terms in $O(\mathcal{L}_1)$ that we write down explicitly

\begin{equation}
- \left\langle \hat{\phi}^{\mu}(t^\prime)\hat{\phi}^{\nu}(t^\prime) \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau), d/d\tau\hat{\phi}(\tau))Z_0^s(J)|_{J=0} \right\rangle^P,
\end{equation}

(5.17)

\begin{equation}
- \left\langle \hat{\phi}^{\mu}(t^\prime)\hat{\phi}^{\nu}(t^\prime) \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau), d/d\tau\hat{\phi}(\tau))Z_0^s(J)|_{J=0} \right\rangle^P.
\end{equation}

(5.18)

The contribution in (5.18) is clearly the only one of $O(\gamma)$. If we remember now Eq. (5.12) and owing to the quadratic non-linearity assumed in (5.15), it turns out that this contribution equals zero as a consequence of the gaussian character of $P(q_0)$. Let us deal with the next order. The terms that contribute in $O(\gamma^2)$ are (5.17) together with the one that corresponds to $O((\mathcal{L}_1)^2)$ in (5.16). Since (5.17) involves an even number of $\hat{\phi}(\tau)$ operators, it can be written as

\begin{equation}
- (1/2)\gamma_0^2D_{\rho\sigma}\gamma_0^2 \int_{t_0}^{t}\!dt\, \sum^{\infty}_{n, m=0} \mathcal{G}_0^{\mu, \nu}(t^\prime, t^\prime)\mathcal{G}_0^{a\beta, p}(\tau, \tau)\mathcal{G}_0^{\gamma, \gamma}(\tau, \tau).
\end{equation}

(5.19)

On the other hand, the term in $O((\mathcal{L}_1)^2)$ can be conveniently reordered and split further in two contributions:

\begin{equation}
(1/2)\gamma_0^2D_{\rho\sigma}\gamma_0^2 \left\langle \hat{\phi}^{\mu}(t^\prime)\hat{\phi}^{\nu}(t^\prime) \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau), d/d\tau\hat{\phi}(\tau)) \frac{d}{d\tau} - \lambda(\tau) \right\rangle \int_{t_0}^{t}\!dt\, \hat{\phi}^{\gamma}(\tau^\prime)\hat{\phi}^{\gamma}(\tau^\prime)
\end{equation}

\begin{equation}
\times \int_{t_0}^{t}\!dt\, J_\rho(\tau^\prime)R_\sigma(\tau^\prime - \tau') \int_{t_0}^{t}\!dt\, J_\omega(\tau^\prime)\Delta^{\omega\rho}(\tau^\prime, \tau)Z_0^s(J)|_{J=0} \right\rangle^P
\end{equation}

(5.20)

and

\begin{equation}
(1/2)\gamma_0^2D_{\rho\sigma}\gamma_0^2 \left\langle \hat{\phi}^{\mu}(t^\prime)\hat{\phi}^{\nu}(t^\prime) \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau), d/d\tau\hat{\phi}(\tau)) \frac{d}{d\tau} - \lambda(\tau) \right\rangle
\end{equation}

\begin{equation}
\times \int_{t_0}^{t}\!dt\, \hat{\phi}^{\gamma}(\tau^\prime)\hat{\phi}^{\gamma}(\tau^\prime)\hat{\phi}^{\gamma}(\tau^\prime)R_\sigma(\tau - \tau')Z_0^s(J)|_{J=0} \right\rangle^P.
\end{equation}

(5.21)

If we remember now the expression for the free response function in (2.16), we can readily establish

\begin{equation}
\left\langle \frac{d}{d\tau} - \lambda(\tau) \right\rangle R_\sigma(\tau - \tau') = \delta_\sigma \delta(\tau - \tau') \exp\{\lambda(\tau)\}(\tau - \tau') \end{equation}

(5.22)

and consequently we rewrite (5.21) as

\begin{equation}
(1/2)\gamma_0^2D_{\rho\sigma}\gamma_0^2 \left\langle \hat{\phi}^{\mu}(t^\prime)\hat{\phi}^{\nu}(t^\prime) \int_{t_0}^{t}\!dt\, \mathcal{L}_1(\hat{\phi}(\tau), d/d\tau\hat{\phi}(\tau)) \hat{\phi}^{\gamma}(\tau)\hat{\phi}^{\gamma}(\tau)\hat{\phi}^{\gamma}(\tau)Z_0^s(J)|_{J=0} \right\rangle^P
\end{equation}

(5.23)

that coincides with (5.19) but it appears with an opposite sign, so they cancel each other.
Thus the only contribution in $O(\gamma^2)$ corresponds to (5·20).

This is also an important result, because it shows that terms like (5·21) which do not exist in the Hamiltonian context will cancel each other in the expansion considered here, and because of that we can conclude that the non-renormalized expansion actually considered adopts a final form which is identical to the one that could have been obtained starting from the Hamiltonian description. Thus, and in an obvious way, we can say that the diagrammatic transcription of such an expansion is univocally determined. In fact, the previous considerations come from the expansion up to second order, but it is easy to show that they can be generalized to any order in the perturbation.

Anyway, we believe that the use of functional techniques in order to generate non-renormalized perturbation schemes is a simpler and clearer method than the one available from Wick's theorem. 7)

§ 6. The response function in a Lagrangian representation

In this section we want to derive an alternative expression for the response function, defined as usual by

$$R_\gamma''(t'', t') = -i \langle q''(t'') \hat{p}_\gamma(t') \rangle$$

(6·1)

and adapted to the Lagrangian representation.

In a Hamiltonian level of description we can make explicit the average in (6·1);

$$\langle q''(t'') \hat{p}_\gamma(t') \rangle = \int \mathcal{D} q(\tau) \mathcal{D} p(\tau) q''(t'') \hat{p}_\gamma(t')$$

$$\times \exp \left\{ i \int_{t_0}^t d\tau \left[ p_\gamma(\tau) \dot{q}''(\tau) - H(p(\tau), q(\tau)) \right] \right\}. \quad (6·2)$$

Initially, we restrict ourselves to a one-dimensional system and consider, in this case, the discretized version of Eq. (6·2)

$$\langle q''(t'') \hat{p}_\gamma(t') \rangle = \lim_{n \to \infty} \int \left( \prod_{j=1}^{n+1} dq_j \right) q_1 \int \left( \prod_{j=1}^{n+1} dp_j / 2\pi \right)$$

$$\times \exp \left\{ i \epsilon \sum_{j=h}^{n+1} \left[ p_j \frac{q_j - q_{j-1}}{\epsilon} - H(p_j, q_{j-1}) \right] \right\} \int (dp_h / 2\pi)p_h$$

$$\times \exp \left\{ i \epsilon \left[ p_h \frac{q_h - q_{h-1}}{\epsilon} - H(p_h, q_{h-1}) \right] \right\} \quad (6·3)$$

with the indices $l$ and $k$ within the partition $(1, \cdots, n+1)$ and with $l \leq k$, to satisfy the causality requirement. Performing explicitly the last integration in (6·3), we will arrive at

$$\langle q''(t'') \hat{p}_\gamma(t') \rangle = \lim_{n \to \infty} \int \left( \prod_{j=1}^{n+1} dq_j \right) q_1 \int \frac{i}{\epsilon} \left[ \frac{q_h - q_{h-1}}{\epsilon} - f(q_{h-1}) \right]$$

$$\times \int \left( \prod_{j=h}^{n+1} dp_j / 2\pi \right) \exp \left\{ i \epsilon \sum_{j=1}^{n+1} \left[ p_j \frac{q_j - q_{j-1}}{\epsilon} - H(p_j, q_{j-1}) \right] \right\}$$
that can be further elaborated by integration over the remaining $p_j$ variables, yielding

$$
\langle q(t'')p(t') \rangle = \lim_{n \to \infty} \int \left( \prod_{j=1}^{n+1} dq_j \right) \frac{i}{D} \left[ \frac{q_k - q_{k-1}}{\epsilon} - f(q_{k-1}) \right] \times \prod_{j=1}^{n+1} \left( \frac{1}{2\pi\epsilon D} \right)^{1/2} \exp \left\{ -\frac{\epsilon}{2D} \left[ \frac{q_j - q_{j-1}}{\epsilon} - f(q_{j-1}) \right]^2 \right\}.
$$

(6.5)

This average can now be expressed in terms of a path integral as

$$
\langle q(t'')p(t') \rangle = (i/D) \int \mathcal{D} q(\tau) q(t'') \left[ \dot{q}(t') - f(q(t')) \right] \times \exp \left\{ -\int_{t_0}^{t} d\tau \mathcal{L}_\mathcal{E}(q(\tau), \dot{q}(\tau)) \right\}
$$

(6.6)

which admits an immediate generalization to $m$-dimensional systems in the form

$$
\langle q''(t'')p''(t') \rangle = iD_{\nu\sigma} \int \mathcal{D} q(\tau) q''(t'') \left[ \dot{q}''(t') - f''(q(t')) \right] \exp \left\{ -\int_{t_0}^{t} d\tau \mathcal{L}_\mathcal{E}(q(\tau), \dot{q}(\tau)) \right\}.
$$

(6.7)

Thus we can write the response function using propagators evaluated in the Lagrangian representation. More precisely, and according to (6.7),

$$
R_{\nu''}(t'', t') = D_{\nu\sigma} \left[ \frac{\partial}{\partial t'} \langle q''(t'')q''(t') \rangle - \langle q''(t'')f''(q(t')) \rangle \right]
$$

(6.8)

which is the final form we wanted. This procedure could obviously be repeated for any propagator originally defined with $p$ variables.

§ 7. Conclusions

The presence of the free response functions in our non-renormalized perturbation scheme has been verified. In the case that the treatment is carried out on a Lagrangian representation this implies that, as well as when the representation is Hamiltonian, the diagrams consist of two kinds of lines. Furthermore, both expansions coincide exactly, and can be expressed in a natural way according to a set of diagrams uniquely determined.

An explicit expression for the response function in the Lagrangian description used, has been found, thus permitting us to write a generalized expression for any average originally defined with $p$ variables. As a consequence, any propagator, and therefore the Fokker-Planck dynamics itself, may be equally characterized by both representations.

With this latter conclusion in mind we want to continue our work further in order to test the usefulness of a Lagrangian description in a renormalized perturbation scheme, as we believe that in this case that description is a more definite and advantageous alternative to the MSR schemes implicitly based themselves on a Hamiltonian representation. This feature will be discussed in the next paper.
About the Lagrangian Representation of the Fokker-Planck Dynamics. I

References

5) L. Garrido and M. San Miguel, see Ref. 2).