The equilibrium condition of two Reissner-Nordström black holes is studied in terms of the Cosgrove solution of the Einstein-Maxwell equations. It is found that the masses and the charges must satisfy two constraints, which would allow only the extreme black holes.

A newly discovered Bäcklund transformation (Neugebauer's transformation) for the vacuum Einstein equations presented an interesting family of stationary and axisymmetric solutions, i.e., what is called "multiple Kerr-NUT solutions". The event horizons of Kerr-NUT particles are aligned along the symmetry axis at any separation. If the spacetime is regular on the axis, the metric can be interpreted as a stationary gravitational field of many black holes balanced by their gravitational spin-spin interaction. It was shown in Refs. 2) and 3) that the regularity is assured under some constraints to the parameters. The N-Kerr-NUT metric has $4N$ parameters (both the Kerr-type and the NUT-type rotation parameters are included). The condition for regularity, however, reduces the number of free parameters to $2N$, except the position of center of mass of the system. Then, we can assign the free parameters to the mass and the angular momentum of each black hole, which may be defined in terms of the Komar integrals over each horizon. If these parameters are given, the metric describing stationary system of many black holes is completely determined (e.g., the spatial separation between horizons is fixed).

The equilibrium condition for charged black holes will be realized in a slightly different manner. For example, according to the classical theory, the condition for static balance of two masses $M_a$ ($a=1,2$) with charges $Q_a$ is

$$M_1M_2=Q_1Q_2,$$  \hspace{1cm} (1)

and the separation is arbitrary. We use units such that $c=1$, $G=1$. The static equilibrium in general relativity was studied up to the post-post-Newtonian approximation. Bonnor conjectured that the classical condition (1) is sufficient for the equilibrium, while Ohta and Kimura claimed more restrictive condition

$$M_a=Q_a \quad (a=1,2)$$  \hspace{1cm} (2)

In this letter we want to investigate this problem from a different point of view, i.e., by using an exact solution of the static and axisymmetric Einstein-Maxwell equations (two Reissner-Nordström black holes).

In terms of the Ernst potential $\varepsilon$ and the complex electromagnetic potential $\Phi$, Cosgrove obtained the electrovac generalization of Neugebauer's transformation. When the seed metric is the Minkowski spacetime ($\varepsilon=1$, $\Phi=0$), two successive Cosgrove's transformation generates the Kerr-Newman-NUT metric with magnetic charge:

$$\varepsilon=1-2iN/D, \quad \Phi=-2N'/D,$$  \hspace{1cm} (3)

where

$$D=\Delta_{123}, \quad N=(K_i-K_i)R_i\Delta_{23},$$  \hspace{1cm} (4)

In these expressions we have used the 3 by 3 determinants

$$\Delta_{ijk}=\det(\phi_i, \phi_j, \phi_k)$$  \hspace{1cm} (5)

of the vectors $\phi_i$, whose transposed forms are $(Q_i, R_i, S_i)$, and

$$\Delta_{ij}=R_iS_j-R_jS_i, \quad \Delta'_{ij}=Q_iR_i-Q_jR_j,$$  \hspace{1cm} (6)

where

$$Q_i=-i[R_i^*r(K_i^*)-R_i(K_i^*-z)],$$  \hspace{1cm} (7)

$$S_i=2ir(K_i^*),$$  \hspace{1cm} (8)

and

$$\Delta_{123}=R_1R_2R_3.$$  \hspace{1cm} (9)
\[ Q_s = -i[R_s^*r(K_i) - R_s(K_i - z)], \]
\[ S_s = 0, \]
\[ Q_s = R_s^*r(K_i) - R_s(K_i - z), \]
\[ S_s = 2\pi r(K_i)[R_sR_s^* - R_sR_s^*], \]
\[ r(\tau) = [(x - r)^2 + \rho^2]^{1/2} \]

and \( R_s, R_s, R_s = (iR_s) \) and \( K_i \) are arbitrary complex parameters. The potentials are functions of cylindrical coordinates \( \rho \) and \( z \). The allowed range of the parameters, however, is restricted to the case beyond the extreme (no horizon). We introduce a familiar parametrization;

\[ K_i, K_i^* = z_i \pm i(a_i^2 + e_i^2 - m_i^2)^{1/2}, \]
\[ R_s, R_s^* = e_i^{-1}[a_i^2 + e_i^2 - m_i^2]^{1/2} - a_i \]
\[ \times \exp[\pm i(a_i + \beta_i)], \]
\[ R_s, R_s^* = \pm ie_i^{-1}m_i \exp(\mp i\beta_i), \]

and \( a_i^2 + e_i^2 > m_i^2 \). For electrostatic case rotation parameters \( a_i, a_i \) and magnetic charge parameter \( \beta_i \) should vanish. Reissner-Nordström black hole is obtained by the following analytic continuations of the parameters \( e_i^2 < m_i^2 \):

\[ f = 1 + [i(\text{N}^*D - ND^*) + 4\text{N}'\text{N}'^*/DD^*] \]

The three parameters \( m_i, e_i, z_i \) mean the mass, the electric charge and the position on the \( z \)-axis, respectively.

Neugebauer and Kramer\(^\text{40}\) found explicit forms of \( \varepsilon \) and \( \Phi \) generated by \( 2N \) successive application of Cosgrove's transformation. When \( N=2 \) (two Kerr-Newman-NUT particles), the expression given by Guo and Ernst\(^\text{10}\) is

\[ D = 2\delta_{32}\delta_{14}(K_i^* - K_i)(K_i^* - K_i), \]
\[ + 2\delta_{12}\delta_{34}(K_i - K_i^*)(K_i^* - K_i), \]
\[ N = 2\delta_{32}\delta_{14}(K_i^* - K_i)(K_i - K_i^*)(K_i^* - K_i), \]
\[ + 2\delta_{12}\delta_{34}(K_i - K_i^*)(K_i - K_i^*)(K_i^* - K_i) \]
\[ + 2\delta_{12}\delta_{34}(K_i - K_i^*)(K_i^* - K_i)(K_i^* - K_i^*), \]
\[ + (\delta_{13}\delta_{24} - \delta_{14}\delta_{23}), \]
\[ \times R_s(K_i - K_i^*)(K_i^* - K_i)(K_i^* - K_i^*), \]

and \( N' \) has the same form as \( N \) except that primes appear on the two-index deltas. We rewrite the newly introduced parameters \( K_s, R_j (j = 1 \rightarrow 3) \) in the same way as Eq. (9), but with \( m_s, e_s, z_s \) replaced by \( m_s, e_s, z_s \), which characterize added Reissner-Nordström black hole. Using this reparametrization and Eq. (10), we will obtain the condition for equilibrium of two static black holes due to the Coulomb-type repulsion in the following.

The line element is given by the form

\[ ds^2 = -f dt^2 + f^{-1}[\rho^2 d\phi^2 + \rho^2(dx^2 + dzd^2)]. \]

The formulas for calculating the metric functions are

\[ f = 1 + (i(\text{N}^*D - ND^*) + 4\text{N}'\text{N}'^*/DD^*) \]

and

\[ \frac{2\partial \gamma}{\partial z} = \rho f^{-1}(\partial A_i/\partial \phi)(\partial A_i/\partial z) \]

where the electric potential \( A_i \) is

\[ A_i = -\text{N}'/D - \text{N}'^*/D^*. \]

In this coordinate system two horizons \( H_a \) shrink to the segments on the \( z \)-axis \( (H_a; K_i^* \geq z \geq K_i, H_b; K_i^* \geq z \geq K_i, \) and we assume \( K_i > K_i^* \). On the horizons \( f \) should vanish, thus the behavior of \( f \) will be

\[ f \approx \tilde{f}_a[\gamma(K_i^*) + z - K_a^*], \quad (a = 1, 2) \]

near the point \( \rho = 0, z = K_a^* \), and

\[ f \approx \tilde{f}_a[\gamma(K_i) - z + K_a] \]

near the point \( \rho = 0, z = K_a \). Here \( \tilde{f}_a \) and \( f_a \) are constants depending on the parameters \( m_a, e_a \) and \( z_a(z_i - z_a = 2z_a) \). From Eq. (13) we note that the behavior of \( \gamma \) is \( \gamma = \gamma_0 + O(\rho^2) \) (\( \gamma_0 = \text{constant} \) near the \( z \)-axis, except the horizons. The constant \( \gamma_0 \) may be different at each region of the axis, and we denote the value at upper region \( (z \geq K_a^*) \), middle region \( (K_i^* \geq z \geq K_i^*) \) and lower region \( (z \leq K_i) \) by \( \gamma_u, \gamma_m \) and \( \gamma_l \), respectively. The condition for equilibrium (i.e., absence of the conical singularity) is

\[ \gamma_u = \gamma_m = \gamma_l. \]

Because of the complicated form of \( f \) and \( A_i \), exact integration of Eq. (13) for obtaining the
value $\gamma_0$ is very difficult. This obstacle will be surmounted by using a property of black hole that "surface gravity" is constant over the horizon.\textsuperscript{11} We denote it on the horizon $H_a$ by $x_a$ ($a=1, 2$). Because black hole is static, we get
\begin{equation}
 x_a^2 = (\partial f/\partial z)^2 / 4 \epsilon^{2z_a}.
\end{equation}

on $H_a$. Then, from Eq. (15), the behavior of $\gamma$ on $H_a$ close to the point $z = K_a$ is given by
\begin{equation}
 e^{2\gamma} \simeq (f_a/x_a)^2 [r(K_a) + z - K_a]/2(r(K_a) - z). \tag{19}
\end{equation}
The same behavior of $\gamma$ as Eq. (19) appears, when Eq. (13) is integrated near the point $\rho = 0$, $z = K_a$ (since $f \approx 0$, the second term of the right-hand side of Eq. (13) can be neglected). The result is
\begin{equation}
 e^{2\gamma} \simeq C_a [r(K_a) + z - K_a]/2r(K_a), \tag{20}
\end{equation}
where $C_1 = \exp(2\gamma_a)$ and $C_2 = \exp(2\gamma_a)$. Comparing Eq. (20) with Eq. (19), we have
\begin{align}
 \bar{u}_1 &= (K_a - K_1)(R_s + R_2)(1 + R_1 + R_s - R_3 R_s^*) + (K_2 - K_s)(R_s - R_3)(1 + R_1 + R_s - R_3 R_s^*) \\
 \bar{v}_1 &= (K_a - K_2)(K_s - K_1)(1 - R_1 - R_2 - R_3 R_s^*) + (K_2 - K_s)(R_s - R_3)(1 - R_1 - R_s - R_3 R_s^*) \\
 \bar{W}_1 &= [(K_a - K_2)(K_s - K_1)(R_s + R_2)(1 + R_1 + R_s - R_3 R_s^*) - (K_2 - K_s)(K_s - K_1)(1 - R_1 - R_s - R_3 R_s^*)] \\
 &\times [(K_a - K_1)(R_s + R_2)(1 + R_1 + R_s - R_3 R_s^*)] \\
 \bar{W}_2 &= [(K_a - K_1)(K_s - K_2)(1 - R_1 - R_s - R_3 R_s^*)] \\
 &\times [(K_a - K_1)(R_s + R_2)(1 + R_1 + R_s - R_3 R_s^*)] \\
 \bar{W}_3 &= [(K_a - K_1)(K_s - K_2)(1 - R_1 - R_s - R_3 R_s^*)] \\
 &\times [(K_a - K_1)(R_s + R_2)(1 + R_1 + R_s - R_3 R_s^*)].
\end{align}

and $u$, $v$, and $W$ have the same form as $\bar{u}_1$, $\bar{v}_1$, and $\bar{W}_1$, respectively, except the exchange of the parameters such that $K_1$, $K_2$, $R_1$, $R_2$, $R_3$, $R_4$, $R_5$, $R_6$, $K_1$, $K_2$, $R_1$, $R_2$, $R_3$, $R_4$, $R_5$, $R_6$. If we exchange the parameters in such a way that $u$, $v$, $W_1$, $W_2$, $W_3$, $\bar{u}_1$, $\bar{v}_1$, $\bar{W}_1$, $\bar{W}_2$, $\bar{W}_3$, we can obtain $u_2$, $v_2$, $u_3$, $v_3$, $\bar{u}_2$, $\bar{v}_2$, $\bar{u}_3$, $\bar{v}_3$, $\bar{u}_4$, $\bar{v}_4$, $\bar{u}_5$, $\bar{v}_5$, respectively.

Contrary to the metric which consists of black holes with no charge, the relation $\gamma_0 = \gamma_1$ is not automatically satisfied. Therefore, absence of the conical stress on the axis requires two constraints to the parameters. This situation is quite different from the classical one, in which only one constraint (i.e., Eq. (1)) is needed. One possible solution of the equation $\gamma_0 = \gamma_1$ is
\begin{equation}
 m_1 e_2 = m_2 e_1, \tag{26}
\end{equation}
since $\bar{u}_1 \bar{v}_1 \bar{u}_2 \bar{v}_2 - u_1 v_1 u_2 v_2 \propto (m_1 e_2 - m_2 e_1)^2$. In this case the metric is reduced to the Ernst generalization\textsuperscript{7} of two Schwarzschild black holes (i.e., $\epsilon = 1 - 2q^2 \Phi$, $q = \epsilon_1 / m_1 = \epsilon_2 / m_2$), and calculation of $\gamma_m$ becomes simple, namely
\begin{align}
 2(\gamma_m - \gamma) &= \ln[4z_0^2 - (m_1^2 - \epsilon_1^2 + m_2^2 - \epsilon_2^2)^2] \\
 &- \ln[4z_0^2 - (m_1^2 - \epsilon_1^2 - m_2^2 - \epsilon_2^2)^2]. 
\end{align}

Thus, we arrive at the final result ($m_2 = e_2$), which coincides with the condition given by Ohta and Kimura.

The possibility of another choice of the parametric relation satisfying the condition $\gamma_0 = \gamma_1$ is not clear as yet. If it is possible, we may obtain a regular metric which consists of two nonextreme Reissner-Nordström black holes.

We give a final comment concerning the
definition of mass and charge. The mass of black hole is usually defined in terms of a surface integral over the horizon. For the line element (11) the mass \( M_a \) of black hole with the horizon \( H_a \) can be written in the form

\[
M_a = \frac{1}{4} \int_{H_a} \lim_{\rho \to 0} (\rho f^{-1} \partial e / \partial \rho) dz .
\]  

(28)

On the other hand the charge \( Q_a \) of black hole can be evaluated from the flux integral over \( H_a \), i.e.,

\[
Q_a = -\frac{1}{2} \int_{H_a} \lim_{\rho \to 0} (\rho f^{-1} \partial A_4 / \partial \rho) dz .
\]

(29)

If \( e_4/m_4 = e_5/m_5 (= q) \), \( f \) and \( A_4 \) are connected with each other through the function \( \xi \),

\[
f = \frac{1}{1 + (1 - q^2) \xi^2}/(1 + \xi)^2 ,
\]

\[
A_4 = q \xi/(1 + \xi) .
\]

(30)

The behavior of \( \xi \) near \( H_a \) \((1 - q^2)^{1/2} \xi = 1 + O(\rho^2)) \) leads to \( M_a = m_a \) and \( Q_a = e_a \). Thus, in this case, the physical interpretation of the parameters \( m_a, e_a \) becomes obvious, even for the system of two interacting black holes.