Quark Spin-Flavor Layered Structure of High Density Matter with Condensed $\pi^0$ Field in Chiral Bag Model

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High density matter of quarks accompanying a condensed $\pi^0$ field is studied in chiral bag model, as a new phase arising from the neutron matter under $\pi^0$ condensation. It is shown that this phase possibly occurs as an intermediate stage prior to the phase transition to uniform quark matter. Structure of the phase is characterized by the following features; the layers of the two-dimensional quark matter with a specific spin-flavor order and the condensed $\pi^0$ field existent as the Nambu-Goldstone mode between the adjacent layers. Such quark configuration is caused due to the pion quark coupling at the layer (bag) surface which drastically lowers quark energy. Energy properties of the system are examined and stability of this phase is discussed.

§ 1. Introduction

Since quarks have been recognized as real constituents of hadrons, it becomes important to study typical problems in hadronic level from the viewpoint of quark model motivated by the QCD (quantum chromodynamics). Among them baryon structure and nuclear force are of primary importance because of direct relevance to the very existence of the nuclear system, and a number of works have been already reported on these problems. Pionic degrees of freedom are essential for description of various facets in hadronic level, as inferred from the well-known fact that the pion exchange plays the role indispensable for nuclear binding. Therefore both quark and pion degrees of freedom should be taken into account for realistic understanding of the nuclear system.

In hadronic matter pion condensation is one of the interesting aspects in the above-mentioned context. In this phase where the pion dynamics manifests itself, the quark degrees of freedom come into play more or less explicitly for a well-developed condensation at high density. The chiral bag model, which was initiated by Chodos and Thorn and by Inoue and Maskawa, is a workable framework to describe the two degrees of freedom simultaneously. The chiral bag model has been further developed by Jaffe, by the SUNY-Saclay group and by Théberge, Thomas and Miller. In this paper, we study the baryonic ALS (alternating-layer-spin) structure equivalent to a typical $\pi^0$ condensation from a viewpoint of the chiral bag model. Here we deal with neutron matter and its structure change at zero temperature.

In the ALS structure nucleons ($n$ and $p$) form the layers perpendicular to the direction of the condensed $\pi^0$ momentum (spin quantization axis) and their spins aligning in the combination ($n \uparrow, p \downarrow$) or ($n \downarrow, p \uparrow$) change alternately layer by layer. The ALS structure of nucleons with such a specific order of the localized spin-isospin provides the source of a standing-wave type of condensed $\pi^0$ field to bring about the attractive effect of the $\pi$-nucleon $P$-wave interaction; we can also express this as a phase where the OPE...
tensor force is most efficiently utilized among the nonuniform nucleon configurations studied earlier.\(^8\) For realization of pion condensation, the coherent mixing of isobar \(\Lambda(1232)\) into nucleon \((N)\) states is essential to enhance the attractive OPE effect, because short-range effects such as \(\rho\)-meson exchange and repulsive core act suppressively. Recently possible realization of the ALS structure in neutron matter at high density has been shown by calculations taking into account both the effects of \(\Lambda(1232)\) and short-range effects,\(^9\)-\(^{12}\) although the results are sensitive to the strength of the \(\rho\cdot N\) tensor coupling.

When baryon number density \(\rho\) becomes much higher than the nuclear density \(\rho_0 \cong 0.17\) fm\(^{-3}\), hadronic matter turns into quark matter although its transition density predicted so far is still in large uncertainty at present, e.g. around \(5\rho_0 \sim 10\rho_0\)\(^{13}\),\(^{14}\) in symmetric nuclear matter and possibly higher in neutron matter.\(^{15}\) Baym\(^{16}\) pointed out, by taking analogy with percolation, the possibility of a phase appearing prior to this transition; matter is composed of an infinite network of interconnected bags where color currents run through but quarks remain still in localized trios. We may expect other versions of intermediate stages to quark matter which reflect characteristic aspects of matter in hadronic level at high density. If a specific order such as pion condensation is realized, partial fusion of the baryon bags probably occurs in a specific manner with persistence of this order.

A scenario we study here is described as follows. When the hadronic ALS structure becomes well-developed as \(\rho\) goes high, as illustrated in the change (a)\(\rightarrow\) (b) in Fig. 1, the shrinkage of the layer distance \(d\) is very gentle while that of the layer width is rather remarkable.\(^9\)-\(^{11}\) Then density gaps appear between the layers and the resulting \(\pi^0\) field becomes of plateau shape in these gaps. The two-dimensional density of baryons in the layers, \(\rho_i = \rho d\), becomes high enough for each baryonic layer to melt into a layer of two-dimensional quark matter, a kind of interconnected bags. The condensed \(\pi^0\) field with momentum \(\pi/d\) in the \(z\)-direction persists as the Nambu-Goldstone mode\(^{17}\),\(^{18}\) in the region between the adjacent layers. Such aspect is illustrated in Fig. 1(c). Actually the boundary of the layers is neither completely sharp nor static, but it should be regarded as dynamical. As the first step to the scenario, however, we treat a model in which the layer surface is static and completely sharp for simplicity.

In this model the possibility is shown that quarks \((d\) and \(u)\) in the layers take their spin-flavor combination in the ALS-

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Fig. 1. Profile of change from the hadronic ALS structure to the quark ALSF structure; (a) a moderately developed ALS one in neutron matter, (b) its well-developed version and (c) the quark ALSF structure with layer thickness \(2b\). The condensed \(\pi^0\) field \(\pi^0(z)\) with the momentum along the \(z\)-axis is shown by the solid lines. The arrows along the \(z\)-axis indicate the resultant spin of each layer.
type order, \((d \uparrow, u \downarrow)\) or \((d \downarrow, u \uparrow)\), due to the coupling with the condensed \(\pi^0\) field at the layer (bag) surface. This aspect that we call the quark ALSF (alternating layer spin-flavor) structure provides us with an example to which the chiral bag model can be applied nonperturbatively. In one-baryon problem, nonperturbative approach in the chiral bag model has been made by taking the hedgehog solution.\(^{13,46}\) In such an approach a serious problem has been pointed out in connection with instability of nucleon structure due to the strong effect of pion field.\(^{46}\) Recently several attempts to restore the stability have been proposed.\(^{19-21}\) It is another aim of the present study to see how this stability problem arises in the aspect of matter where pions play an essential role.

In the next section a theoretical framework to describe the model is presented for the single-particle eigenstates of quarks, the \(\pi^0\) field equation and the boundary conditions at the layer surface. In §3, we show the results of calculations on the quark eigenstates and the condensed \(\pi^0\) field and discuss the reason why drastic decrease of quark eigenvalue arises. Then energy properties of the system are shown and discussed. We point out the necessity of introducing some additional repulsive contribution to restore stability of the quark ALSF structure. The last section is devoted to concluding remarks.

§ 2. Formulation of quark ALSF structure with condensed \(\pi^0\) field

In this section we present a framework to describe the quark ALSF structure accompanying the condensed \(\pi^0\) field. At first we give the quark single-particle states inside the layers without pion field and then consider how they are affected by pion field. The condensed \(\pi^0\) field described as a classical field is introduced between the adjacent layers; this field with the character of the Nambu-Goldstone boson is coupled to quarks inside the layers at the boundary. Through the eigenvalue equation modified by the presence of the \(\pi^0\) field the quark single-particle energy turns out to be lowered than that without \(\pi^0\) field, if quarks in the layers take their spin-flavor configuration with the ALSF order. The derivative of pion field at the boundary is given by the continuity of the axial vector current at the layer surface for the given quark ALSF structure. Thus the problem to be solved becomes a self-consistent one.

2.1. Quark single-particle states inside the layers without pion field

We start with the description of quark single-particle states inside the layers in the absence of pion field, just as in the original MIT bag model\(^{22}\) in the problem of hadron structure. Because of the periodicity of the same layered structure, it is sufficient to consider only the region \(-b \leq z \leq b\). When we treat the \(l\)-th layer, \(\bar{z} \equiv z - ld\) is taken in place of \(z\). Inside the layer the quark field \(\psi\) with mass \(m\) obeys the free Dirac equation

\[(\gamma_\mu \partial_\mu + m)\psi = 0\]  \((2.1)\)

with the boundary condition of the MIT bag model;

\[n_\mu \gamma_\mu \psi = \psi,\]  \((2.2)\)

where \(n_\mu = (0, 0, \pm 1, 0) \equiv \pm \bar{z}\) is the outward normal at the layer (bag) surface \((z = \pm b)\). Because of the geometrical structure of the layers the solutions have the following form:

\(*\) The following notations are used; \(x_\mu = (r, \theta)\) and \(\gamma_\mu = (-i \beta a, \beta)\) for \(\mu = 1-4\).
\[ \phi_a = e^{-i \omega t} (e^{i q \cdot r_1 / \sqrt{Q_0}}) u(z) f_a c_a, \quad (2.3) \]

where \( \omega \) is the single-particle energy, \( r_1 = (x, y), Q_0 \) the two dimensional normalization volume, \( q = (q_x, q_y) \) the momentum of quarks in the two-dimensional Fermi gas, and \( f_a c_a \) denotes the flavor (color) state. We specify \( s_a \) representing the spin state when the solutions of the spinor \( u(z) \) are determined. Substitution of (2.3) into Eqs. (2.1) and (2.2) leads to the equations for \( u_a(z) \) with \( x = 1 \sim 4 \) (the \( x \)-th component of spinor \( u(z) \)). From (2.1) we obtain \( d^2 u_a(z)/dz^2 = -q_a^2 u_a(z) \) with \( q_a = \sqrt{\omega^2 - q_1^2 - m^2} \), where \( q_a \) is the quark momentum in the \( z \)-direction which is determined by an eigenvalue equation. Then we have

\[ u_a(z) = A_a \cos q_a z + B_a \sin q_a z. \quad (2.4) \]

The boundary condition (2.2) is rewritten as

\[ u_3(\pm b) = \pm i u_1(\pm b), \quad u_4(\pm b) = \mp i u_2(\pm b). \quad (2.5) \]

As the first type (c-type) solution we choose

\[ u_1(z) = A_1 \cos q_a z \quad \text{or} \quad u_2(z) = A_2 \cos q_a z \quad (2.6a) \]

and as the second type (s-type) solution we choose

\[ u_1(z) = B_1 \sin q_a z \quad \text{or} \quad u_2(z) = B_2 \sin q_a z. \quad (2.6b) \]

By substituting (2.6a) into Eqs. (2.1) and (2.5), we obtain the spinors with the subscript \( c \) and the superscript \( s_a = 1 \) \((s_a = 2)\) corresponding to the former (latter);

\[ u_c^{(s_a=1)}(z) = A_1 \begin{bmatrix} \cos q_a z \\ i q_1 \sin q_a z / W_c \\ i \sin q_a z / X \\ q_1 \cos q_a z / W_c \end{bmatrix}, \quad u_c^{(s_a=2)}(z) = A_2 \begin{bmatrix} -i q_1 \sin q_a z / W_c \\ \cos q_a z \\ q_1 \cos q_a z / W_c \\ -i \sin q_a z / X \end{bmatrix}, \quad (2.7) \]

where \( X = \tan q_a b, \quad b = q_x \pm i q_y \) and \( W_c = \omega + q_a X - m \). Similarly for (2.6b) we obtain the spinors with the subscript \( s \) and the superscript \( s_a=1 \) \((s_a=2)\),

\[ u_s^{(s_a=1)}(z) = B_1 \begin{bmatrix} \sin q_a z \\ i q_1 X \cos q_a z / W_s \\ i X \cos q_a z \\ q_1 \sin q_a z / W_s \end{bmatrix}, \quad u_s^{(s_a=2)}(z) = B_2 \begin{bmatrix} -i q_1 X \cos q_a z / W_s \\ \sin q_a z \\ q_1 \cos q_a z / W_s \\ -i X \cos q_a z \end{bmatrix}, \quad (2.8) \]

where \( X = \tan q_a b \) and \( W_s = \omega - q_a X + m \). The eigenvalue equations of \( q_a \) become the same for both c-type and s-type spinors,

\[ q_a (X - 1/X) = 2m, \quad (2.9) \]

from which we have

\[ X = \tan q_a b = (m \pm \sqrt{m^2 + q_a^2}) / q_a. \quad (2.10) \]

For massless quark \((m=0)\) \( X = \pm 1 \) is obtained. Therefore the eigenvalues of \( q_a \) have two series,
where \( n=0, 1, 2, \ldots \). The same eigenvalues have been given in a work done in a different context.\(^{23}\) When \( |q_z| \rightarrow 0 \), the meaning of \( s_a \) becomes clear; the spinor with \( s_a=1 \) (\( s_a=2 \)) corresponds to the one with spin up (down). In this case we have to employ the \( c \)-type spinors (2.7) for \( X^+ \) and the \( s \)-type ones (2.8) for \( X^- \). Even for nonzero \( |q_z| \) it is suitable to adopt this prescription, although we can also use either the \( c \)-type or the \( s \)-type spinor for both \( X^+ \) and \( X^- \). In Fig. 2 the upper and lower nonvanishing components of \( u_c^{(s_a=1)}(z) \) with \( |q_z|=0 \) are shown. The first excited state is higher than the lowest one by \( \pi/2b \lesssim 600 \text{ MeV} \) for \( b \lesssim 0.5 \text{ fm} \), as seen from (2.11). We may restrict ourselves to the lowest state. The essential aspects for nonzero \( m \) of \( u \) and \( d \) quarks are the same because of its smallness.

2.2. Quark configuration in the layer

We suppose the transition that neutrons in the layers turn into the two-dimensional quark (\( u, d \)) matter, as illustrated in Fig. 1. We denote \( u \) quarks with the spin state \( s_a=1 \) (\( s_a=2 \)) symbolically by \( u \uparrow \) (\( u \downarrow \)) and similarly for \( d \) quarks, although they have not so literal meaning for large \( |q_z| \). We consider the neutron layer composed of spin-up neutrons which are described as the symmetrized spin-flavor combination of \( |d \uparrow, d \uparrow, u \uparrow \rangle \) and \( |d \uparrow, d \uparrow, d \downarrow \rangle \), multiplied by the anti-symmetrized color state. Let \( N_L \) be the number of neutrons in a layer. As possible configurations realized from the neutron ALS structure, at least we can consider the following two cases. One is the configuration composed of \( |d \uparrow, d \uparrow, u \uparrow \rangle \) in which the number of \( d \uparrow \) quark \( N(d \uparrow)=2N_L/3 \) and that of \( u \uparrow \) quark \( N(u \uparrow)=N_L/3 \) in each color state (Fig. 3(a)). The other is the one composed of \( |u \uparrow, d \uparrow, d \downarrow \rangle \) in which all the numbers of \( u \uparrow \) quarks, \( d \uparrow \) quarks, and \( d \downarrow \) quarks are the same.
quarks are \( N_j / 3 \) in each color state (Fig. 3(b)). We denote the former (latter) as Configuration I(II). The Fermi momentum \( q_{\perp}(a) \) (here \( a \) means a set of flavor, spin, and color state) is related to the quark number density in the layer,

\[
\rho_{\perp}(a) = N_{\perp}(a)/Q_{\perp} = \frac{q_{\perp}(a)}{(2\pi)^2} = \frac{q_{\perp}(a)^2}{4\pi}.
\] (2.12a)

The baryon number density in the layer is given by

\[
\rho_{\perp} = \rho d = \frac{(\text{OCC})}{3},
\] (2.12b)

where (OCC) means the occupied states. As for the spin-down neutron layers, the same argument is applied by reversing all the spin directions. In this paper where we restrict ourselves to the problems of the sector of \( u \) and \( d \) quarks, the chemical equilibrium is not taken into consideration because \( s \) quarks as well as leptons are relevant to this problem at high density. As far as quark kinetic energy is concerned, it is favorable to choose Configuration II in the absence of pion field. As shown later, however, when the condensed \( \pi^0 \) field strongly affects quark states it is favorable to choose Configuration I, because all the quarks utilize the attractive effect of the \( \pi^0 \) field in this configuration.

2.3. Quark single-particle states in the layer in the presence of condensed \( \pi^0 \) field

We start with the Lagrangian density of the chiral bag model,

\[
\mathcal{L} = \left[ -\bar{\psi} i \gamma_\mu \partial^\mu \psi - m \bar{\psi} \psi - B + \partial_\mu (\lambda_\mu \bar{\psi} (\sigma + i\tau \cdot \pi \gamma_5) \psi) \right] \theta_{\text{in}}
\]

\[
+ \left[ -\frac{1}{2} (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 + m_{\pi}^2 f_{\pi} (\sigma - f_{\pi}) \right] \theta_{\text{out}},
\] (2.13)

where \( B \) is the bag constant implying the difference between the vacuum energies inside and outside the bag, and \( \sigma \) and \( \pi (\pi^i, i = 1 \sim 3) \) denote scalar and pion fields, respectively. We assume that the pion field is regarded as the Nambu-Goldstone mode existent only outside the bag. \( \lambda_\mu \) is the Lagrange multiplier which is determined so as to provide the boundary conditions of the chiral bag model. The term with the pion mass \( (m_{\pi}) \) implies the PCAC (partially conserved axial-vector current), where \( f_{\pi} \) is the pion decay constant, \( \theta_{\text{in}} \) the step function being unity inside the bag and vanishes outside, and \( \theta_{\text{out}} = 1 - \theta_{\text{in}} \). The pion degrees of freedom are treated by imposing the restriction

\[
\sigma^2 + \pi^2 = f_{\pi}^2.
\] (2.14)

As in the studies by Brown, Rho, Vento and coworkers,\(^9\) we represent the scalar and pion field by \( \phi^i (i = 1 \sim 3) \) as follows:

\[
\sigma = f_{\pi} (1 + \phi^2/f_{\pi}^2)^{-1/2}, \quad \pi^i = \phi^i (1 + \phi^2/f_{\pi}^2)^{-1/2}.
\] (2.15)

The field equations and boundary conditions are derived from (2.13) by the action principle. Inside the bag we find the free Dirac equation (2.1). Outside the bag the field equation for \( \phi^i \) is given as
Quark Spin-Flavor Layered Structure of High Density Matter

\[ \partial_\mu D_\mu \phi^i = m_\pi^2 \phi^i (1 + \phi^2 / f_\pi^2)^{-1/2}, \]
\[ D_\mu = (1 + \phi^2 / f_\pi^2)^{-1} \partial_\mu. \] (2.16)

By the use of the relation given by\(^1\)

\[ n_\mu \lambda_\mu = -1 / 2 f_\pi, \]
we obtain the boundary condition for the quark field

\[ n_\mu \gamma_\mu \psi = (1 + \phi^2 / f_\pi^2)^{-1/2} (1 + i \tau \cdot \gamma^5 / f_\pi) \psi \] (2.17)

and that for the pion field \( \phi^i \)

\[ n_\mu \partial_\mu \phi^i = -(1 + \phi^2 / f_\pi^2)^{1/2} \bar{\phi} (\phi^i / f_\pi - i \tau^i \gamma_5) \phi / f_\pi. \] (2.18)

Also we get an additional boundary condition from the property that the axial vector current

\[ A_\mu^i = i \bar{\phi} \gamma_5 \gamma_\mu \frac{1}{2} \tau^i \phi \theta_\text{in} + f_\pi D_\mu \phi^i \theta_\text{out} \] (2.19)

is partially conserved, i.e.,

\[ \partial_\mu A_\mu^i = m_\pi^2 f_\pi \tau^i \theta_\text{out} - im_\pi \bar{\phi} \gamma_5 \tau^i \phi \theta_\text{in}. \] (2.20)

At the boundary the condition is

\[ f_\pi n_\mu D_\mu \phi^i = i \bar{\phi} \gamma_5 \gamma_\mu \frac{1}{2} \tau^i n_\mu \gamma_\mu \phi \] (2.21)

which gives the derivative of \( \phi^i \) in terms of \( \phi \) at the bag surface. The boundary conditions (2.18) and (2.21) are not independent under the quark boundary condition (2.17), and we can use either (2.18) or (2.21).

Now we show that the quark ALSF structure is obtained as a solution of these equations. We set \( \phi^1 = \phi^2 = 0 \) because only the condensed \( \pi^0 \) field is considered and we write \( \pi^0 (\phi^0) \) for \( \pi^3 (\phi^3) \). From now on we treat this field as a classical field. Therefore, in Eqs. (2.18) and (2.21), the expectation value with respect to the ground state of the quark system is to be taken. The condensed \( \pi^0 \) field is static because of its vanishing chemical potential and depends only on \( z \). Hereafter we write \( \phi^0 (r) \) as \( \phi^0 (z) \). Here \( \theta_\text{in} = 1(0) \) at \( z \) (|\( z \rangle \langle z |) b \rangle \) where \( \tilde{z} = z - ld \). We consider the symmetry property of \( \phi^0 (z) \) in the ALS phase in the hadronic level;

\[ \phi^0 (\tilde{z}) = -\phi^0 (-\tilde{z}) \] (2.22)

about the \( l \)-th layer center \( z = ld \). This property persists also in the quark ALSF structure. Actually we shall see later that the periodic nature with the property (2.22), as shown in Fig. 1, is constructed in our model. To solve the field equations, it is also sufficient to treat only the region \( b \leq z \leq d - b \). We parametrize \( \phi^0 (z) \) as

\[ \phi^0 (z) = f_\pi \tan \theta (z) \] (2.23)

with

\[ \theta (z) = -\theta (-z). \] (2.23')
Single-quark wave functions \( \varphi_a(x) \) (\( a \) stands for the set of \( q_{\perp} \), spin, flavor, and color) satisfying (2.1) and (2.17) can be written in the same form as (2.3). Since \( \varphi_a(x) \) obeys the free Dirac equation (2.1), the \( x \)-th component of \( u(z) \) (\( x = 1 \sim 4 \)) has the same form as (2.4). The boundary condition can be written in terms of \( \theta(z) \) as follows; from the upper component

\[
\begin{align*}
  u_3(\pm b) &= \pm i \cos \theta(\pm b) u_1(\pm b) \mp \tau_3 \sin \theta(\pm b) u_3(\pm b), \\
  u_4(\pm b) &= \mp i \cos \theta(\pm b) u_2(\pm b) \pm \tau_3 \sin \theta(\pm b) u_4(\pm b),
\end{align*}
\]

and from the lower one

\[
\begin{align*}
  u_1(\pm b) &= \mp i \cos \theta(\pm b) u_3(\pm b) \pm \tau_3 \sin \theta(\pm b) u_1(\pm b), \\
  u_2(\pm b) &= \pm i \cos \theta(\pm b) u_4(\pm b) \mp \tau_3 \sin \theta(\pm b) u_2(\pm b),
\end{align*}
\]

where \( \tau_3 = +1 (-1) \) for \( u(d) \) quark. Two sets of the equations are the same, and we can use either of them. Of course in the case without pion field \( \theta(z) = 0 \) Eq. (2.24) is reduced to Eq. (2.5).

When we postulate (2.6a) as the first type (c-type) solutions, using (2.23'), we obtain

\[
\begin{align*}
  u^{(s_0=1)}(z) &= A_1 \begin{bmatrix} 
  \cos q_{z2} \\
  -i q_{z2} \sin q_{z2}/X W_+ \\
  il \sin q_{z2}/X \\
  q \cos q_{z2}/W_+ 
  \end{bmatrix}, \\
  u^{(s_0=2)}(z) &= A_2 \begin{bmatrix} 
  -i q_{z2} \sin q_{z2}/X W_- \\
  \cos q_{z2} \\
  q \cos q_{z2}/W_- \\
  -i l \sin q_{z2}/X 
  \end{bmatrix},
\end{align*}
\]

\( I_{z^a} = (1 \mp \tau_3 \sin \theta(b)) / \cos \theta(b), \)

\( W_{\pm} = \omega + m + q I_{z^a}/X, \)

\( X = \tan q_z b. \)

Note that the effect of the \( \pi^0 \) field lies in \( I_{z^a} \) and

\( I_{z^a} I_{-z^a} = 1. \)

\( A_1 \) and \( A_2 \) are the normalization factors which are determined by the condition

\[
\int_{-b}^{b} u^{(s_0)}(z)^\dagger u^{(s_0)}(z) \, dz = 1;
\]

\[
|A_1|^2 = b(1 + q_{z0}^2/W_{z0}^2)((1+I_{z0}^2/X^2)+(1-I_{z0}^2/X^2)\sin 2q_z b/2q_z b),
\]

and \( A_2 \) is given by using \( I_- \) and \( W_- \) in place of \( I_+ \) and \( W_+ \) in \( A_1 \), respectively. As in §2.1 \( s_0 = 1 \) (\( s_0 = 2 \)) corresponds to spin up (down) quark in \( |q_{\perp}| \rightarrow 0 \) limit. We determine \( q_{z2} \) by the eigenvalue equations,

\[
\begin{align*}
  q_z(I_{z} X - I_{z}^0/X) &= 2m \quad \text{for } s_0 = 1, \\
  q_z(I_{-z} X - I_{-z}^0/X) &= 2m \quad \text{for } s_0 = 2,
\end{align*}
\]
with the restriction $X > 0$. Although there also exist the solutions corresponding to $X < 0$ as $(2\cdot8)$ (s-type), we do not write down them here because we restrict ourselves to the states with the lowest eigenvalue which belongs to the solutions with $X > 0$. The effect of the $\pi^0$ field is to deviate $I_+^a$ from unity.

Now we proceed to the problem which type of quark configurations is favorable in energy; Configuration I (Fig. 3a) or II (Fig. 3b). We are concerned with $u$ and $d$ quarks whose current masses are less than 10 MeV. For simplicity we set $m=0$. In this case, by using Eq. (2-29), Eqs. (2-30a) and (2-30b) give $X = I_+^a (L^a)$ for $s_a = 1 (2)$; that is,

$$X = (1 + \sigma_a \tau_a \sin \theta(b)) / \cos \theta(b),$$

(2-31)

where we define $\sigma_a = +1 (-1)$ for $s_a = 1 (2)$ and symbolically denote it by $\uparrow (\downarrow)$. These equations can be solved; with $n = 0, 1, 2, \ldots$, $q_{zb} = \pi/4 + \sigma_a \tau_a \theta(b)/2 + n\pi$

(2-32)

among which we take the lowest one ($n = 0$). In order that the $\pi^0$ field brings about the attractive effect to reduce the eigenvalue $q_z$, we take the spin-flavor combination with $\sigma_a \tau_a \theta(b) < 0$, as shown in the upper half of Fig. 4. Therefore the most favorable configuration is that in the layer with $\theta(b) > 0 (<0)$ there are $u \uparrow$ and $d \downarrow$ only ($u \uparrow$ and $d \downarrow$ only). (Of course $b$ should be replaced by $b + ld$ for $l = 0, \pm 1, \pm 2, \ldots$.) According to the above consideration we expect the energy for Configuration I to be lower than that for Configuration II, because in the former all the quarks utilize the attractive pionic effect. Therefore we restrict ourselves to Configuration I which we denote by $|\phi_{\text{ALSF}}>$ hereafter.

For the quark states corresponding to the solutions with $X = -I_+^a < 0$, the eigenvalues are given by $q_{zb} = 3\pi/4 - \sigma_a \tau_a \theta(b)/2 + n\pi$, as shown in the lower half of Fig. 4. The next excited states (the $n = 0$ case in the above) are higher than the lowest ones by $\pi/2b$, the same as in §2.1 without $\pi^0$ field. The excited states firstly appearing with the same $\sigma_a$ and $\tau_a$ with those of the lowest ones are higher by $\pi/b$ for a given $\theta(b)$ (about 1.2 GeV for $b \sim 0.5$ fm). Therefore the restriction to the lowest states is justified also in the presence of the condensed $\pi^0$ field.

2.4. Condensed $\pi^0$ field between adjacent layers

With a given quark state we can solve the $\pi^0$ field equation between adjacent layers as a classical static field, which is expressed in terms of $\theta(z)$ by (2-23). Substitution of
(2·23) into (2·16) leads to the sine-Gordon equation,
\[ d^2 \theta / dz^2 - m_\pi^2 \sin \theta = 0. \]  \hfill (2·33)

The condition of the axial vector current (2·21) is written in terms of \( \theta(z) \), by taking the expectation value in its r.h.s. with respect to \( |\Phi_{\text{ALSF}}\rangle \),
\[ \pm f_\pi^2 d\theta / dz |_{\tilde{z}=\pm b} = \pm \langle \Phi_{\text{ALSF}} | \frac{1}{2} i \bar{\psi} \gamma_5 \gamma_5 \tau_3 \psi | \Phi_{\text{ALSF}} \rangle |_{\tilde{z}=\pm b}. \] \hfill (2·34)

Direct calculation for the massless quarks leads to
\[ f_\pi^2 d\theta / dz |_{\tilde{z}=\pm b} = \frac{1}{2} \sum_{\alpha} \sum_{|q_0|<q_0(\alpha)} \sum_{\alpha} U^\alpha(q_L, q_x) / Q_L, \]
\[ U^\alpha(q_L, q_x) = |A_\alpha|^2 (1 - |q_L|^2 / (q_L + \omega)^2), \] \hfill (2·35)

where the sum over \( \alpha \) is taken for the color, \( s_\alpha \) and \( \tau_\alpha \) of the occupied states. The derivatives of \( \theta(z) \) at \( \tilde{z} = \pm b \) has the same sign depending only on \( l \). When we take one of \((u \uparrow, d \uparrow)\) layers as the \( l=0 \) layer, the spin-flavor combination is ordered as \( \sigma_\alpha \tau_\alpha = (-)^{l+1} \) for the \( l \)-th layer. Owing to this order and the symmetry property (2·23') the condensed \( \pi^0 \) field corresponding the quark ALSF structure results as shown in Fig. 1(c).

In addition to these equations there is the so-called pressure balance equation, which is derived from Lagrangian (2·13) by the variation with respect to the volume. Instead of using this equation we examine various changes in energy properties with respect to the parameters specifying the quark ALSF structure.

§ 3. Results and discussion

For a given density \( \rho \) and a given layer distance \( d \), the Fermi momentum of each two-dimensional quark sea \( (q_{LF}(\alpha)) \) is given by \( \rho_L = \rho d \) and Eq. (2·12). Layer thickness \( 2b \) is treated as a parameter to be determined variationally. Calculations of energy of the system are carried out with use of the quark eigenstates and the outside \( \pi^0 \) field which are obtained as the solutions of the coupled equations derived in the previous section. Owing to the symmetry properties of the solutions given in Eq. (2·23') we need to solve the equations only in the region, \( \tilde{z} = 0 \sim d/2 \), for the \( l=0 \) layer (\( \tilde{z} = z \)).

Calculational procedure is to be done self-consistently as follows:

(i) At first we set a value of the \( \pi^0 \) field at the middle point of the outside region, namely \( \theta_0 = \theta(z = d/2) \) and set its derivative equal to zero because of the symmetry of \( \theta(z) \) about \( z = d/2 \). With this starting condition, the sine-Gordon equation for \( \theta(z) \), Eq. (2·33), is solved to the layer surface. The solution is expressed by the use of the Jacobian elliptic function \( cd(x) \); \( \theta = 2 \cos^{-1}[k cd(m_\pi(z - d/2))] \) with modulus \( k = \cos(\theta_0/2) \). Then we have the boundary values, \( \theta_0 = \theta(z = d/2 - b) \) and \( (d\theta / dz)_b = d\theta / dz|_{z=d/2-b} \).

(ii) For this \( \theta_0 \), we get the quark eigenvalues \( q_x \) as a function of \( b \) through Eq. (2·30) (Eq. (2·32) for \( m=0 \)), and then \( (d\theta / dz)_b \) is calculated from the quark eigenstates by using Eq. (2·35).

(iii) A starting value of \( \theta_0 \) is searched so that the two values of \( (d\theta / dz)_b \), coincide, that is, the one from the sine-Gordon equation and the other from the axial vector current conservation become equal.
3.1. Eigenvalues of quarks and $\pi^0$ field

A typical example of the solutions $u(z)$ and $\theta(z)$ is shown in Figs. 2 and 5, respectively for $d=1.2$ fm at $\rho=5\rho_0$. The $z$-dependence of $\theta(z)$ is rather weak and less steep than $\theta(z)$ of the Klein-Gordon equation, $\theta(z)=\theta_0 \cosh [m\pi(z-d/2)]$, as shown in Fig. 5. Generally $\theta(z)$ is close to the fully developed $\pi^0$ condensation ($\sigma=0$ and $\pi^0=\pi\pi$ corresponding to $\theta(z)=\pi/2$). This leads to drastic reduction of the quark eigenvalues $q_z$ from $q_z=\pi/4b$, the value for no $\pi^0$ field; Fig. 6 shows the $b$-dependence of $q_z$. Such reduction of $q_z$ can be seen from the relation Eq. (2.32),

$$q_z b = \pi/4 - |\theta_b|/2$$  \hspace{1cm} (3.1)

for the configuration favorable in energy, if $|\theta_b|$ is close to $\pi/2$. This feature can be attributed to the PCAC in a sense that the situation is close to that for the massless pion.
(CAC). For $m_z=0$, Eq. (2.33) and the symmetry condition on $\theta(z)$ about $z=d/2$ lead to $\theta(z)=\text{const.}$ and thus $(d\theta/dz)_b=0$. The continuity condition of axial vector current demands that the r.h.s. of Eq. (2.35) should vanish; this can be satisfied by taking $q_z=0$ because $U(q, q_z=0)=0$. In this case the fully developed $\pi^0$ condensation plays only the role to lower $q_z$, but gives no energy contribution in the outside as seen in Eq. (3.8c). To get further insight on such aspects characterized by large $|\theta_a|$ and small $q_z$, for comparison we quote the results of $q_z$ obtained in one-layer model in which the quark matter exists only in the central layer ($|z|\leq b$) surrounded by the $\pi^0$ field asymptotically tending to zero. Details of this model will be described elsewhere. The dot-dashed curve in Fig. 6 shows $q_z$ in one layer model. Comparison of this curve with the line for no $\pi^0$ field and the one for the ALSF case indicates the following aspects; the reduction of $q_z$ for small $b$ comes from the effect of the $\pi^0$ field existent even in one-layer case and the further reduction of $q_z$ for large $b$ comes from the symmetry condition about the middle point imposed on the $\pi^0$ field for the repeated ALSF structure of many layers. The feature of deviating from $\pi/4$ for the one-layer case is somewhat analogous to the reduction of the quark eigenenergy in one-baryon problem obtained in the hedgehog solution.\(^{1,4c}\)

On the basis of the aspects of the quark eigenvalues we can see that Configuration II is much higher in energy than Conf. I. If Conf. II shown in Fig. 3(b) is supposed to persist even in the presence of the $\pi^0$ field, $q_z(u^+)=q_z(d^-)\approx 0$ as in Conf. I but $q_z(d^+)=\pi/2b$ result for the $\theta(z)$ whose sign is reversed from that for Conf. I, and the energy per baryon of Conf. II becomes higher than that of Conf. I by about 200 MeV at $\rho \sim 5\rho_0$. In such a situation, however, because the number of $d^+$ being unfavorable in energy decrease, the problem should be reexamined from the outset by taking into account the charge neutrality and the $\beta$ equilibrium.

3.2. Energy properties

Energy of the system is composed of three terms: the energy of quarks $E_q$, the volume energy $E_B$ and the energy of pion field $E_{\pi}$. The energy of quarks includes the strong effect of the quark-pion interaction at the layer (bag) surface in terms of the eigenvalue $q_z$. The rest of the interaction effect appears through the change of the outside field.

Energy of the system is given by

$$
E = -\int_{\text{lat}} T_{\mu\nu}d^3x = -\int_{\text{bag}} T_{\mu\nu}d^3x + \int_{\text{bag}} B_d^3x - \int_{\text{out}} T_{\mu\nu}d^3x
$$

$$
= E_q + E_B + E_{\pi},
$$

where $T_{\mu\nu}$ is the energy-momentum tensor

$$
T_{\mu\nu} = (T_{\mu\nu}^0 - Bg_{\mu\nu})\theta(b-|\vec{z}|) + (\theta(\vec{z}) - b). \quad (3.3)
$$

Let $N_z$ be the total number of layers. $E_q$ is given by the expression,

$$
E_q = \langle \Phi_{\text{ALS}}| - T_{\mu\nu}^0| \Phi_{\text{ALS}} \rangle = \sum_{a} \frac{1}{2} (\psi_a^* i\partial \psi_a / \partial t - i\partial \psi_a^* / \partial t \psi_a)
$$

$$
= N_z \sum_{a} \omega_a \quad (3.4)
$$

because of $i\partial \psi_a / \partial t = \omega_a \psi_a$, where (OCC) means the sum over the occupied states. The
volume energy is

$$E_B = N_B B_{\Omega} \cdot 2b.$$  \hspace{1cm} (3.5)

The $\pi^0$ energy is calculated by using $\mathcal{L}_\phi$, the second term of (2.13)

$$E_\phi = \int_{\text{out}} (-\mathcal{L}_\phi) \theta(|\vec{z}| - b) d^3x.$$  \hspace{1cm} (3.6)

Energy per baryon $\mathcal{E}$ is expressed as follows:

$$\mathcal{E} = E / N_B = \mathcal{E}_q + \mathcal{E}_b + \mathcal{E}_\phi,$$  \hspace{1cm} (3.7)

where $N_B$ is the baryon number of the system and

$$\mathcal{E}_q = E_q / N_B = 3/\rho_\perp \sum_{z_a, s_a} \int_0^{q_\perp(z_a)} dq_\perp \sqrt{m^2 + q_\perp^2 + q_z^2(x_a, s_a, b)},$$  \hspace{1cm} (3.8a)

$$\mathcal{E}_b = E_b / N_B = 2B_b / \rho_\perp,$$  \hspace{1cm} (3.8b)

$$\mathcal{E}_\phi = E_\phi / N_B = 2f_\pi^2 / \rho_\perp \left[ \int_0^{d/2} \left\{ \frac{1}{2} \left( \frac{d\theta}{dz} \right)^2 - m_\pi^2 \cos \theta(z) \right\} dz + m_\pi^2 \left( d / 2 - b \right) \right].$$  \hspace{1cm} (3.8c)

To see various aspects of the quark ALSF structure, $\mathcal{E}$ and the three components are shown versus $b$ in Fig. 7 at $\rho = \rho_0$ and $5\rho_0$. We use $f_\pi = 93$ MeV and $B^{1/4} = 150$ MeV as standard. The density $\rho = 5\rho_0 = 0.85 \text{ fm}^{-3}$ is a typical high density such that the hadronic ALS structure is well-developed and $d = 1.2 \text{ fm}$ is the optimum layer distance. At the density not far from $5\rho_0$, structure change to the quark ALSF one is expected to possibly occur. Hereafter we consider the $5\rho_0$ case for the most part and the $\rho_0$ case for comparison.

Dependence of $\mathcal{E}_q$ on $b$ appears through $q_z$, and it is very weak because of the
smallness of $q_s^2 \ll q_F^2(a)$. Therefore the main contribution to $E_q$ comes from the kinetic energy of the two-dimensional quark Fermi gas. The volume term proportional to $b$ is most influential with respect to $b$-dependence, especially at not so high density. In $E_q$ the term involving the integral over $z$ is smaller than the rest, and $E_q$ decreases as $b$ increases. The opposite $b$-dependences of $E_q$ and $E_s$ cancel each other, and generally the $b$-dependence of $E$ is weak as shown in Fig. 7. As $\rho$ increases, owing to the factor $1/\rho$, $E_q$ and $E_s$ becomes smaller and smaller. Thus as a result of the strong $r^0$ condensation leading to the reduction of $q_s$, gross aspects of $E$ are brought about by $E_q$. Although the $b$-dependence of $E$ is weak, the energy minimum always occurs at $b=0$ even at $5\rho_0$. To prevent the system from such collapsing of the layers, we need some additional effect which favors the structure of a larger $b$. Repulsive nature of this effect is required, as inferred from the following arguments.

The ALSF structure of $d=1.2$ fm at $\rho=5\rho_0$ is close to the one obtained by Kunihiro and Tatsumi\(^9\) and by Takatsuka et al.\(^11\) in neutron matter. The energies including the neutron rest mass obtained by them are about 1 GeV for $d=1.2$ fm and for $d=1.2\sim1.3$ fm, respectively, and these are lower by about 10~50 MeV than that in the Fermi gas. The energy of the quark ALSF phase $E$ lies around 1100 MeV at $5\rho_0$, which is higher by about 100 MeV than the hadronic case. Such energy property seems reasonable at a glance. At $\rho=\rho_0$, however, $E$ becomes lower than the nucleon rest mass in apparent contradiction with reality, as can be seen in Fig. 7. In addition it is to be noted that the quark configuration treated here corresponds to the case implying the omission of the mass difference between nucleon and $\Delta(1232)$. It is reasonable to expect $E$ at the energy region higher than the nucleon rest mass by about 100~200 MeV at $\rho=\rho_0$. Therefore we need some repulsive effect which acts more significantly at low density.

Returning to the case of $\rho=5\rho_0$, we notice that this density region is just near the critical densities $\rho_c$ which have been reported in connection with phase transition to quark matter from the normal symmetric nuclear matter; Baym\(^12\) suggested $\rho_c=5\rho_0$ for a small scale parameter $\Lambda=180$ MeV/c entering into the QCD running coupling constant and Miyazawa\(^13\) also suggested $\rho_c\approx4.9\rho_0$ in his string model approach. In both models the energies of the system at these $\rho_c$ are 400~500 MeV higher than the nucleon rest mass. At least this shows the importance of the gluon effects being repulsive in high density matter, although it is not adequate to take seriously the quantitative comparison of these values with our energy ($E\approx1100$ MeV at $5\rho_0$) in neutron matter.

The arguments mentioned above indicate that our results are to be supplemented by the gluon effects which are not absorbed into the volume term. The first candidate is the one-gluon exchange (OGE) contribution. A preliminary calculation has shown that the OGE effect brings about a repulsive contribution to $E$ so as to prevent the system from collapsing, and to give an energy minimum at a reasonable value of $b$ if we take a running QCD coupling constant.\(^7b\) The details of this study will be reported in a succeeding paper.

In principle energy minima with respect to $d$ (layer distance) are to be searched. In our present model where only the degrees of freedom of quarks and pion field are included, we cannot have energy minima with respect to $b$ (layer thickness) as mentioned in §3.2. Therefore what we can do here is only to see dependences of $E$ on $d$. As energy of the quark ALSF phase $E$ is mainly determined by the kinetic energy of two-dimensional quark Fermi gas, $E$ becomes lower for smaller $q_F(a)$ corresponding to smaller $d$ because
of $q_{LF}(a) \propto q_{a}^{1/2} = (\rho d)^{1/2}$. When $d$ becomes equal to $2b$, this model loses applicability, as it stands. In order to get a reasonable value of $d$ indicating the persistence of the quark ALSF structure, we need further study including the gluon effects.

§ 4. Concluding remarks

From the viewpoint of the chiral bag model, we have pointed out a possible phase of the quark ALSF structure with the condensed $\pi^0$ field as the Nambu-Goldstone mode, which is expected to arise after the full growth of the hadronic ALS structure in neutron matter at high density. This means that pion condensation does not necessarily compete with phase transition to quark matter. Rather, prior to the transition there possibly appear intermediate stages of structure change where pion condensation and partial deconfinement of baryon bags coexist. Such coexistence comes from the coupling of quarks inside the layers with the outside $\pi^0$ field at the layer (bag) surface, which selects the particular spin-flavor combination in each layer, namely the ALSF order.

The quark-pion coupling at the surface acts so strongly that it gives rise to drastic decrease of the quark energy associated with the direction of the condensed-$\pi^0$ momentum. In addition to this, at the high density where such structure change is expected, the volume energy and the pion kinetic energy becomes small, and the main aspects of total energy is attributed to the kinetic energy of the two-dimensional quark Fermi gas. In the present model, energy minima occur at zero layer thickness. To restore stability of the layer structure from collapsing, we need to introduce some additional repulsive effect which becomes more efficient at lower densities and favors a larger layer thickness. A candidate to play such a role seems to be the OGE contribution. This problem is a subject of further investigation. It is an open problem whether or not such an intermediate stage of matter is actually realized, because comparison of energies in different phases is one of essentially quantitative problems and we have no direct experimental information concerning this point. We intend to try this comparison after introducing the OGE effect.

The results of energy calculations imply that the system is soft against variation of layer thickness if energy minimum be obtained due to some additional repulsive effect mentioned above. Therefore the layer thickness is to be treated as a dynamical variable in principle, such as in the framework of the generator coordinate method. If such a treatment be possible, the aspects of the layer boundary become natural as in the hadronic ALS structure. Therefore we may say that we can describe the well-developed ALS structure accompanying the $\pi^0$ condensation from the quark point of view, although there is a difference in quark states inside the layers, that is, the one in which quarks are confined in the trios and the one in which they are deconfined.

Finally it is to be noted that this model provides us with an example in the chiral bag model, which is exactly solved for massless quarks in nonperturbative way.

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