Hidden Mattis Phase in the Sherrington-Kirkpatrick Model

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(Received December 26, 1983)

The hidden Mattis phase, the spin ordering phase in the frustrationless state suggested previously in the annealed Ising system with competing interactions, is investigated by the replica method in the Sherrington-Kirkpatrick model. An expression for the order parameter of Edwards-Anderson type is obtained for the hidden Mattis phase.

The random bond mixture with competing interactions in the quenched system has drawn much attention as a spin-glass problem. It is, however, inconclusive on the thermodynamic properties of the quenched system in spite of many efforts since the Edwards-Anderson theory appeared. On the other hand, the annealed system is well understood owing to the existence of exact solutions.

In a viewpoint of the n-replica method introduced by Edwards and Anderson, the quenched system is described as the limit \( n \to 0 \) and the annealed one corresponds to the case \( n = 1 \). In addition, Suzuki offered the real n-replica method and emphasized the importance of \( n \geq 2 \) replica systems; because these systems have the Edwards-Anderson type order parameter.

By an analytic continuation of the real n-replica system to the non-integer one, it was suggested that a certain kind of ordered phase may exist over the intermediate concentration region even in the \( n = 1 \) replica system (i.e., the annealed system). It was called the hidden Mattis phase, which has been, of course, recognized as the paramagnetic phase in the exact solutions. The transition temperature of the hidden Mattis phase was interpreted as a singular point of the following free energy \( F(n) \) defined by

\[
F(n) = -kT \ln \langle Z_n \rangle_{av},
\]

where \( Z_n \) denotes the partition function for a given random bond configuration \( \theta \) and \( \langle \cdots \rangle_{av} \) means the average over all possible bond configurations in a fixed concentration. Then the hidden Mattis phase appears in \( F(n) \) explicitly. These situations are similar to the formulation of the percolation problem in terms of \( s \to 1 \) limit of the s-state Potts model.

In this note we examine above statements by using the Sherrington-Kirkpatrick (SK) model which allows the exact calculations. The SK model has the infinite-ranged interactions contrary to the Ising model with the short-ranged interactions discussed previously. Nevertheless, we consider that the essential points of the hidden Mattis phase can be investigated in the SK model sufficiently.

We consider \( N \) Ising spins with the infinite-ranged interactions. The Hamiltonian is

\[
H = -\sum_{<ij>} J_{ij} S_i S_j, \quad S_i = \pm 1,
\]

where \( J_{ij} \) is the random variable distributed independently with the probability \( P(J_{ij}) \),

\[
P(J_{ij}) = \frac{1}{\sqrt{2\pi}J^2} e^{-J_{ij}^2/J^2}, \quad J = \sqrt{\frac{\beta J}{N}},
\]

and the summation for \( \langle ij \rangle \) is taken over all interacting pairs. From expressions (3) and (4), we may write

\[
\langle Z_n \rangle_{av} = \int_0^\infty \left( \prod_{<ij>} P(J_{ij}) dJ_{ij} \right) [Z(J_{ij})]^n,
\]

where \( \beta = (kT)^{-1} \) and the summation for \( \{S_i\} \) is taken over all spin configurations. With the use of the saddle point method, \( F(n) \) is calculated in the thermodynamic limit as

\[
-\beta F(n) = -\frac{1}{4} \langle \beta a \rangle^2 \left[ n - 2nq - n(n-1)q^2 \right] + \ln \left\{ \frac{1}{\sqrt{2\pi N}} \int_0^\infty dz e^{-z/2} \right\} 
\times [2 \cosh(\beta a \sqrt{q} z)]^n,
\]

where \( q \) is the order parameter generally defined.
in the real $n$-replica system and is written as

$$q = \langle \sum_{\{\sigma_i\}_{i=1}^{n}} S_i^x S_i^y \rangle^\phi \times \exp[\beta \sum_{\{\sigma_i\}_{i=1}^{n}} J_i \langle \sum_{i=1}^{n} S_i^x S_i^y \rangle]_{\text{av}} / \langle Z_s \rangle_{\text{av}}. \quad (8)$$

Here, regarding $n$ as a continuous variable and performing the procedure given in expression (1), we can obtain

$$-\beta \tilde{F}(1) = \ln 2 + \frac{1}{4} (\beta \tilde{J})^2 (1 - 2q - a^2)$$

$$+ \frac{1}{\sqrt{2\pi}} e^{-(\beta \tilde{J})^2 q^2} \int_{-\infty}^{\infty} dz \times e^{-\frac{z^2}{2}} \text{sech}(\beta \tilde{J} \sqrt{q} z) \ln[\cosh(\beta \tilde{J} \sqrt{q} z)]. \quad (9)$$

The self-consistent equation can be derived by minimization of $\tilde{F}(1)$ with respect to $q$,

$$\frac{\partial \tilde{F}(1)}{\partial q} = 0,$$  \hspace{1cm} (10)

which yields

$$q = 1 - \frac{1}{\sqrt{2\pi}} e^{-(\beta \tilde{J})^2 q^2} \times \int_{-\infty}^{\infty} dz e^{-\frac{z^2}{2}} \text{sech}(\beta \tilde{J} \sqrt{q} z).$$  \hspace{1cm} (11)

Equation (11) has the solution $q \neq 0$ for $T < T_c$ and $q = 0$ for $T > T_c$ with $\tilde{J}/kT_c = 1$, where $T_c$ can be considered to be the transition temperature of the hidden Mattis phase. While, from (7), we can obtain

$$-\beta F(1) = \frac{1}{4} (\beta \tilde{J})^2 + \ln 2.$$  \hspace{1cm} (12)

The above free energy $F(1)$ has no singularity. This means that the hidden Mattis phase does not appear in $F(1)$ but appears in $\tilde{F}(1)$.

Now we discuss general aspects of the hidden Mattis phase. From (1) and (2), $\tilde{F}(1)$ can be written in general form as

$$\tilde{F}(1) = \langle Z_s F_s \rangle_{\text{av}} / \langle Z_s \rangle_{\text{av}}, \quad Z_s = \text{Tr}[e^{-\beta H_s}], \quad (13)$$

where $H_s$ denotes the Hamiltonian for a given bond configuration and $F_s = -kT \ln Z_s$. The average over the random bond configurations is similar to that of the quenched system except for the factor of $Z_s / \langle Z_s \rangle_{\text{av}}$. Since there exists the factor $Z_s / \langle Z_s \rangle_{\text{av}}$ in (13), this system can be considered to be the annealed one. Then we can believe that $\tilde{F}(1)$ is the natural extension of $F(1)$.

On the other hand, in order to derive the expression of the order parameter of the hidden Mattis phase, it is convenient to rewrite Eq. (8) as

$$q = \langle \langle S_i \rangle^2 Z_s \rangle_{\text{av}} / \langle Z_s \rangle_{\text{av}}, \quad (14)$$

where $\langle S_i \rangle$ denotes the thermal average of $S_i$ for a fixed random bond configuration $\theta$,

$$\langle S_i \rangle = \text{Tr}[S_i e^{-\beta H_s}] / \text{Tr}[e^{-\beta H_s}].$$  \hspace{1cm} (15)

Setting $n = 1$ in Eq. (14), we obtain the expression for the order parameter in this system as follows,

$$q = \langle \langle S_i \rangle^2 Z_s \rangle_{\text{av}} / \langle Z_s \rangle_{\text{av}}.$$  \hspace{1cm} (16)

Apparently this is a natural extension of the Edwards-Anderson order parameter to the annealed system. At the ground state, the frustrationless configurations only contribute to $q$ owing to the factor of $Z_s / \langle Z_s \rangle_{\text{av}}$, therefore spins are completely frozen and $q$ becomes one. As $T$ is increased, the frozen moment is destroyed both by the thermal agitation and by the frustrated configurations. Then we can expect the phase transition to occur.

Generally speaking, the origin of the Mattis phase appearing in the general $n$-replica system is essentially the spin ordering in the frustrationless state. For the case $n \geq 2$, the replicating procedure mainly plays a role of a detector of this spin ordering. The same spin ordering can be expected in the case $n = 1$ merely missing such a detector. That is, the original annealed system has the Mattis phase with the Edwards-Anderson type order parameter $q$ in (16), which extracts the critical property from the annealed system. Furthermore we can consider that this order parameter is related to a cluster structure constructed from the spin ordering in the frustrationless state in the original annealed system.