Momentum Distribution Condition
in the Cluster Variation Method

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The distribution function of the particle occupation probability in the momentum space is investigated by the cluster variation method. The distribution function calculated in the two-body cluster approximation often gives unreasonably too many particles jumping out of the degenerate state. A new condition is introduced to suppress the excitation of too many particles. The condition is applied to liquid helium-4 and neutron matter. Numerical results show that the condition regulates the behavior of the momentum distribution function, and is useful to avoid an unnatural kinetic energy.

§ 1. Introduction

In a previous paper we reported on a study of neutron matter by the cluster variation method. In that study we used the two-body approximation with a subsidiary condition related to the numbers of spin-singlet and triplet pairs. Since then, we have found another type of condition related to the momentum distribution of particles. In this paper, we report on our study of liquid helium-4 and neutron matter with use of this condition in the two-body approximation. It is hoped that these conditions improve the quality of the low-order approximation (typically the two-body cluster approximation), although confirmation of this statement will only be made by higher-order calculations.

Many of the conditions proposed so far are directly related to the spatial pair distribution function. By imposing these conditions kinematically or rather geometrically, unreasonable pair distribution function can often be eliminated. The potential energy is expressed as the integral of the product of the potential and the pair distribution function. Therefore, the above conditions serve to eliminate unnatural potential energies. On the other hand, we can avoid the occurrence of unnatural kinetic energy only by giving due consideration to the degrees of freedom other than the spatial distribution of the constituents. In the following, we discuss a new condition related to the occupation probability of particles in the momentum space. Ristig and Lam pointed out that in the electron plasma the particle distribution function in the momentum space has a logarithmic singularity at the Fermi momentum. This singularity occurs in any system as long as the pair correlation function has a long-range tail. As shown in §3, the solution of the unconstrained Euler-Lagrange equation in the two-body approximation has a long-range tail, corresponding to an unphysical situation that an infinite number of particles jump out of the degenerate state. In §2 we derive a new condition which eliminates such an unphysical situation. We apply this condition to liquid helium-4 in §3 and to neutron matter in §4. In these sections, we shall show that the long-range tail of the pair

1) In some extreme cases, even the positivity of kinetic energy is violated.
correlation function is eliminated and a natural behavior of the momentum distribution function can be obtained by imposing the condition. Section 5 is devoted to discussion.

§ 2. Momentum distribution condition

We consider the single-particle distribution function in the momentum space \( n(q) \) which is defined as the occupation probability of the single-particle state with momentum \( \hbar q \). For simplicity, we use the distribution function which is averaged over intrinsic degrees of freedom. Let us calculate the expectation value of an operator

\[
\hat{O} = \sum_{i=1}^{N} \omega(p_i),
\]

where \( \omega(p_i) \) can be any function of the single-particle momentum operator \( p_i \), and \( N \) is the total number of particles. This expectation value can be expressed with use of the single-particle distribution function \( n(q) \). On the assumption that the volume of the system (\( \Omega \)) is very large and the system is spatially uniform, we have

\[
\langle \sum_{i=1}^{N} \omega(p_i) \rangle = \int \omega(\hbar q) n(q) \frac{w \Omega}{(2\pi)^3} dq,
\]

where \( w \) is the multiplicity of the single-particle states due to intrinsic degrees of freedom. For example, \( w = 4 \) for nuclear matter, \( w = 2 \) for neutron matter, and conventionally \( w = 1 \) for the zero-spin boson system. Equation (2) may be used for the calculation of \( n(q) \).

The function \( n(q) \) must satisfy some general conditions. In any case,

\[
n(q) \geq 0 \quad \text{for all} \quad q.
\]

In the case of a fermion system, \( n(q) \) has an upper limit:

\[
n(q) \leq 1 \quad \text{for all} \quad q.
\]

In order to discuss conditions further, we first derive an expression for \( n(q) \). We evaluate the left-hand side of Eq. (2) by the cluster variation method, by using the wave function (\( \Psi \)) of Jastrow type:

\[
\Psi = \begin{cases} 
\prod_{i<j} f(|r_i-r_j|) \Omega^{-N/2} & \text{for boson system,} \\
\otimes \prod_{i<j} f_{ij}(|r_i-r_j|) \Phi & \text{for fermion system,}
\end{cases}
\]

where \( \Phi \) is the Slater determinant, and the functions \( f(|r_i-r_j|) \) and \( f_{ij}(|r_i-r_j|) \) are two-body correlation functions which reflect the correlation of the pair of particles. The operator \( \otimes \) is the symmetrizer with respect to the order of the product. It is necessary when the functions \( f_{ij} (|r_i-r_j|) \) do not commute with each other. Then, the expectation value of \( \hat{O} \) can be expanded into the cluster series\(^a\) as

\[
\langle \sum_{i=1}^{N} \omega(p_i) \rangle = X_1 + X_2 + \cdots.
\]

The expression for the \( m \)-body cluster term \( X_m \) depends on the properties of the constituents, especially on their statistics.
First, we consider a system consisting of one kind of bosons. We define $\omega(\mathbf{p}_i)$ as

$$\omega(\mathbf{p}_i) = \omega_0 + \omega_1(\mathbf{p}_i),$$

where $\omega_0$ is a constant, and

$$\omega_1(0) = 0.$$  

Then, the first two cluster terms in Eq. (6) are obtained as

$$X_1 = N\omega_0,$$

$$X_2 = \frac{N^2}{2Q^2} \int f(|\mathbf{r}_1 - \mathbf{r}_2|) \{\omega_1(\mathbf{p}_1) + \omega_1(\mathbf{p}_2)\} f(|\mathbf{r}_1 - \mathbf{r}_2|) d\mathbf{r}_1 d\mathbf{r}_2,$$

$$= \frac{N^2}{2Q^2} \int \{f(|\mathbf{r}_1 - \mathbf{r}_2|) - 1\} \{\omega_1(\mathbf{p}_1) + \omega_1(\mathbf{p}_2)\} f(|\mathbf{r}_1 - \mathbf{r}_2|) - 1 d\mathbf{r}_1 d\mathbf{r}_2.$$  

The second equality of Eq. (10) is due to the fact that the functions $f(|\mathbf{r}_1 - \mathbf{r}_2|)$ and $f(|\mathbf{r}_1 - \mathbf{r}_2|) - 1$ have the same Fourier transform $g(\mathbf{k})$ except for $\mathbf{k} = 0$. We take the ordinary boundary condition for $f(\mathbf{r})$. Namely,

$$f(\mathbf{r}) = 0 \quad \text{within the hard core, if any},$$

$$f(\mathbf{r}) \rightarrow 1 \quad \text{for } r \rightarrow \infty.$$  

Then, the Fourier transform of $f(\mathbf{r})$ has a singularity of $\delta$-function type, whereas that of $f(\mathbf{r}) - 1$ does not. Therefore, the second expression in Eq. (10) is much easier to handle. Now, we denote the Fourier transform of $f(\mathbf{r}) - 1$ by $g(\mathbf{k})$ for the whole $\mathbf{k}$-space. Namely,

$$f(\mathbf{r}) - 1 = \frac{1}{(2\pi)^3} \int g(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}.$$  

Substituting Eq. (12) into Eq. (10), we have

$$X_2 = \int \omega_1(\hbar \mathbf{q}) |g(\mathbf{q})|^2 \frac{N^2}{(2\pi)^3 Q} d\mathbf{q},$$  

and, in the two-body cluster approximation, we get

$$\langle \sum_{i=1}^{N} \omega(\mathbf{p}_i) \rangle = N\omega_0 + \int \omega_1(\hbar \mathbf{q}) |g(\mathbf{q})|^2 \frac{N^2}{(2\pi)^3 Q} d\mathbf{q}.$$  

On the other hand, from Eqs. (2) and (7) we obtain

$$\langle \sum_{i=1}^{N} \omega(\mathbf{p}_i) \rangle = N\omega_0 + \int \omega_1(\hbar \mathbf{q}) n_0(\mathbf{q}) \frac{Q}{(2\pi)^3} d\mathbf{q},$$

where the subscript $B$ means the boson system, and $\omega$ in Eq. (2) is put equal to unity. Comparing Eq. (14) with Eq. (15) (note that $\omega_1(\hbar \mathbf{q})$ can be any function with $\omega_1(0) = 0$), we get

$$n_0(\mathbf{q}) = \rho^2 |g(\mathbf{q})|^2 \quad \text{for } |\mathbf{q}| > 0$$

with $\rho$ representing the number density:
The value of $n_B(0)$ cannot be obtained from the comparison of Eqs. (14) and (15), because $\omega_0(0)=0$; it should rather be obtained as the difference between $N$ and the sum of $n_B(q)$ over all positive values of $|q|$. Thus, we have

$$n_B(q) = \rho^2 |g(q)|^2 + \left(2\pi^3 \rho - \int \rho^2 |g(q)|^2 d\mathbf{q} \right) \delta_3(q).$$

The positivity condition (3) is certainly satisfied by this distribution function at $|q|>0$, but at $|q|=0$ it requires the following inequality:

$$\eta_B = \frac{1}{(2\pi)^3} \int \rho |g(q)|^2 d\mathbf{q} \leq 1. \tag{19}$$

This quantity $\eta_B$ is the ratio of the excited-particle number to the total number, and is referred to hereafter as excitation ratio. It can also be expressed as

$$\eta_B = \rho \int (f(r) - 1)^2 dr. \tag{20}$$

In §3, we apply the condition (19) to the study of liquid helium-4.

Now, we turn to fermion systems and take up neutron matter as an example. Although the correlation function $f_{ij}(r_i-r_j)$ contains, in general, complicated terms reflecting the strong state dependence of the nuclear force, we assume a simple spin-dependent function of the following form:

$$f_{ij}(r) = \frac{1}{P_{ij}} f(r) + \frac{3}{P_{ij}} f(r), \tag{21}$$

where $\frac{1}{P_{ij}}$ and $\frac{3}{P_{ij}}$ stand for the projection operators for the spin singlet and triplet pairs respectively.\(*) The functions $\frac{1}{P_{ij}} f(r)$ and $\frac{3}{P_{ij}} f(r)$ are the correlation functions for these spin-multiplet pairs, and must satisfy the same boundary condition as Eqs. (11a) and (11b). Now, we can derive expressions for the cluster terms $X_1$ and $X_2$ by a straightforward calculation as

$$X_m = \int \omega(q) n_m(q) \frac{2\Omega}{(2\pi)^3} d\mathbf{q}, \quad (m=1, 2) \tag{22}$$

with

$$n_1(q) = H(k_F - |q|) \tag{23}$$

and

$$n_2(q) = \frac{3}{2 \pi^3} \frac{2S+1}{4} \left[ -\rho \int (f(r) - 1)^2 (1 + (-1)^3 J(k_F r) j_0(|q| r)) dr H(k_F - |q|) \right. \nonumber$$

$$+ \frac{\rho^2}{2 \Omega_F} \int \left\{ \int (f(r) - 1) j_0(|p| r) dr \right\}^2 H(k_F - |p + q|) dp \nonumber$$

$$\left. + (-1)^3 \frac{\rho^2}{2} \left[ \int (f(r) - 1) j_0(|q| r) dr \right] \right]^2. \tag{24}$$

\(*) In order to distinguish these two pair states, we use the spin multiplicity $\nu$ as a superscript.
Here, $k_F$ and $Q_F$ are the radius and volume of the Fermi sphere, respectively, $j_l(x)$ is the spherical Bessel function of order $l$, $J(x)$ is related to $j_l(x)$ as

$$J(x) = \frac{3j_l(x)}{x}, \quad (25)$$

and $H(x)$ is the Heaviside function:

$$H(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases} \quad (26)$$

Then, in the two-body cluster approximation, we have the distribution function of the fermion system (neutron matter) as

$$n_F(q) = n_1(q) + n_2(q). \quad (27)$$

The function $n_F(q)$ must obey the law of particle-number conservation:

$$\int_{\text{all } q} n_F(q) \frac{2Q}{(2\pi)^3} dq = N. \quad (28)$$

We can analytically show that this condition is satisfied in the two-body approximation.

Next, we define the excitation ratio for the fermion system by

$$\eta_F = \int |q| > k_F n_F(q) \frac{2Q}{(2\pi)^3} dq/N. \quad (29)$$

By using Eq. (24) we obtain the expression for $\eta_F$ in the two-body approximation as

$$\eta_F = \frac{1}{4} \sum_{s=0}^{2S+1} \left[ \rho \int \nu f(r) - 1 \right] \left[ (1 + (-1)^s J^2(k_F r)) dr \\
- \frac{\rho^2}{2} \int_0^{2k_F} \left\{ \int \nu f(r) - 1 \right\} j_0(pr) dr \right] \left[ (1 - \frac{3p^2}{4k_F^2} + \frac{p^4}{16k_F^4}) \right] \frac{3p^2}{2} dp \\
- (-1)^s \frac{\rho^2}{2} \int_0^{k_F} \left\{ \int \nu f(r) - 1 \right\} j_0(qr) dr \right] \frac{3q^2}{2} dq. \quad (30)$$

The quantity $\eta_F N$ represents the number of the particles excited above the Fermi sea. The ratio $\eta_F$ must satisfy

$$\eta_F \leq 1. \quad (31)$$

This inequality gives a rigorous upper bound for $\eta_F$. In many cases, however, we get more reasonable results by putting a smaller upper bound $\eta_0$ as

$$\eta_F \leq \eta_0, \quad (32)$$

where $\eta_0$ is a constant smaller than unity. There is no definite rule in determining the value $\eta_0$. However, when we study neutron matter in §4, we apply the inequality (32) by choosing a certain value for $\eta_0$ from a physical point of view.

§ 3. Liquid helium-4

In this section, we apply the condition (19) to the study of liquid helium-4, and
investigate how the condition affects variational calculations.

The energy per particle can be expressed in the two-body cluster approximation as

$$E_{B} = \frac{\rho}{2} \int f(r) \left\{ -\frac{\hbar^2}{M} \nabla^2 + v(r) \right\} f(r) \, dr,$$  \hspace{1cm} (33)

where \(v(r)\) stands for the potential. We search for the correlation function \(f(r)\) that minimizes the functional \(E_{B}\) while satisfying the inequality (19). However, before going into this problem, let us consider the case without the condition (19). In this case, the optimum function is obtained by solving the unconstrained Euler-Lagrange equation,

$$\left\{ -\frac{\hbar^2}{M} \nabla^2 + v(r) \right\} f(r) = 0,$$ \hspace{1cm} (34)

with the boundary condition (11). The solution of Eq. (34) has a long tail as

$$f(r) \sim 1 - a/r, \quad (r \gg 1)$$  \hspace{1cm} (35)

where \(a\) is the scattering length. Owing to this asymptotic form, the function \(n_{B}(q)\) badly diverges to infinity as \(|q| \rightarrow 0\), and hence the excitation ratio \(\eta_{B}\) also becomes infinity.*\(^{14}\) We can, therefore, conclude that the condition (19) is indispensable to discuss the momentum distribution in the two-body cluster approximation unless the scattering length is zero.

Now let us investigate the constrained variation. First we define the functional

$$I_{B} = E_{B} - \lambda_{B} \cdot \eta_{B},$$ \hspace{1cm} (36)

where \(\lambda_{B}\) is a Lagrange multiplier. By setting the first-order variation of \(I_{B}\) equal to zero, we get the constrained Euler-Lagrange equation,

$$\left\{ -\frac{\hbar^2}{M} \nabla^2 + v(r) \right\} f(r) = 2\lambda_{B} \{ f(r) - 1 \}. \hspace{1cm} (37)$$

The solution of Eq. (37) has a Yukawa-type tail as

$$f(r) \sim 1 + A \exp \{ -2(\lambda_{B}/M) h^2 r^2 \}/r, \quad (r \gg 1) \hspace{1cm} (38)$$

and, consequently, \(n_{B}(q)\) and \(\eta_{B}\) take finite values.

Now, We can summarize the procedure for numerical calculation as follows:

1) Assume a certain value for \(\lambda_{B}\).
2) Solve the Euler-Lagrange equation (37) under the boundary condition (11).
3) Calculate \(E_{B}\) and \(\eta_{B}\) by using expressions (33) and (20). Note, here, that there is a relationship between the quantities \(E_{B}\) and \(\eta_{B}\) through the parameter \(\lambda_{B}\). If \(E_{B}\) is a decreasing function of \(\eta_{B}\), the optimum value of \(\eta_{B}\) which gives the minimum of \(E_{B}\) within the range limited by the inequality (19) is \(\eta_{B} = 1\). It is numerically confirmed that the quantity \(E_{B}\) is certainly a decreasing function of \(\eta_{B}\) in the present case. Hence, the following process will conclude the whole procedure.
4) Adjust the value \(\lambda_{B}\) and repeat the above steps until the calculated value of \(\eta_{B}\) becomes unity.

We show the numerical results in Figs. 1~4. In the calculation we have used

\[*\) On the contrary, \(E_{B}\) has a finite lower bound if \(\rho\) is fixed.\(^{15}\)
the Lennard-Jones potential

\[ v(r) = V_0 \left\{ \left( \frac{\sigma}{r} \right)^{12} - 2 \left( \frac{\sigma}{r} \right)^{6} \right\} \]  

(39)

with the force parameters of de Boer and Michels,

\[ V_0 = 8.82 \times 10^{-4} \text{eV}, \]

\[ \sigma = 2.869 \text{Å}. \]  

(40)

The energy per particle is shown in Fig. 1 as a function of the parameter \( r_0 \) which is defined by

\[ -0.5 \]
\[ -1.0 \]

Fig. 1. Energy per particle of liquid helium-4. The solid line represents the calculation with the upper limit of the excitation ratio \( \eta_n = 1 \). We also show, for reference, the result obtained with the excitation ratio \( \eta_n = 0.9 \) by a dashed line.

\[ \lambda_0 \]

Fig. 2. Lagrange multiplier \( \lambda_0 \) for liquid helium-4. The negative of \( \lambda_0 \) is given in a logarithmic scale.

\[ n_s(q) \]

Fig. 3. Examples of the square of the pair correlation function \( f^2(r) \) for liquid helium-4. The numbers attached to the lines are \( r_0 \) (in units of \( \text{Å} \)).

\[ n_s(q) \]

Fig. 4. Single-particle distribution function \( n_s(q) \) for liquid helium-4. The numbers attached to the lines are \( r_0 \) (in units of \( \text{Å} \)).
Momentum Distribution Condition

\[ r_0 = (3/4\pi \rho)^{1/3}. \]  \hspace{1cm} (41)

The calculated minimum should be compared with the experimental data indicated by the cross. The Lagrange multiplier \( \lambda_B \) is shown in Fig. 2. The absolute value of \( \lambda_B \) rapidly increases as the system becomes dense. It follows from this and Eq. (38), that the function \( f(r) \) "heals" within a shorter distance as the density increases. The examples of \( f^2(r) \) given in Fig. 3 show this situation. In the denser case \((r_0 = 2.2\text{Å})\) the peak of the function \( f^2(r) \) is considerably suppressed. This seems to be favorable for the convergence of the cluster series. Finally, some examples of the distribution functions \( n_\theta(q) \) are shown in Fig. 4.

As stated above, we have attained the energy minimum at \( \eta_B = 1 \), which corresponds to the absence of the Bose condensation in the ordinary sense. However, we cannot insist on this result too strenuously, because we have only worked in the lowest-order approximation; we rather feel that our result is not inconsistent with the smallness of the condensate part (several percent) which was obtained by other authors. In Fig. 1 the energy per particle is also shown for \( \eta_B = 0.9 \). It should also be noted that the superfluidity seems to be more intimately connected with the independent-particle part of the wave function. This part in our wave function (5) is a constant \((Q^{-\kappa/2})\), which means that all particles are in the single-particle state of zero momentum corresponding to 100% condensation in a sense, irrespective of the value of \( \eta_B \).

§ 4. Neutron matter

The energy per particle in neutron matter is expressed in the two-body cluster approximation as

\[ E_F = \frac{3}{5} \frac{\hbar^2 k_F^2}{2M} + \sum_{s=0}^{1} \frac{2s+1}{4} \rho \int f(r)H_s f(r) dr \]  \hspace{1cm} (42)

with

\[ H_s = \frac{\hbar^2}{M} \frac{1}{r^2} \frac{d}{dr} r^2 L_s(k_Fr) \frac{d}{dr} + L_s(k_Fr) \nu(r). \]  \hspace{1cm} (43)

Here,

\[ L_s(x) = 1 + (-1)^s f^2(x), \]  \hspace{1cm} (44)

and \( \nu(r) \) and \( ^3\nu(r) \) are the nuclear potentials in the singlet-even \((\nu = 1)\) and triplet-odd \((\nu = 3)\) states respectively. We find the minimum value of the functional \( E_F \) under the condition (32). The Euler-Lagrange equation to be solved is

\[ H_s f(r) = 2\lambda_F L_s(k_Fr)\{f(r) - 1\} + \int U_s(r, r')\{f(r') - 1\} dr' \]  \hspace{1cm} (45)

with

\[ U_s(r, r') = -\lambda_\rho f(k_F|r-r'|)(J(k_F|r-r'|) + (-1)^s J(k_Fr)J(k_Fr'). \]  \hspace{1cm} (46)

As mentioned in §2, an important problem in the application of the inequality (32) is the choice of the constant \( \eta_0 \). In order to settle this problem, let us examine the general
behavior of the distribution function $n_F(q)$ as given by Eqs. (23)~(27). The function $n_F(q)$ is discontinuous, in general, at $|q|=k_F$. This is a consequence of the Heaviside functions in expressions (23) and (24). Actually, in the case of such long-range functions $\psi(r)$ as Eq. (35), the function $n_F(q)$ has an infinite discontinuity at $|q|=k_F$:

$$[n_F(q)]|_{q=k_F-0} \to -\infty,$$

$$[n_F(q)]|_{q=k_F+0} \to +\infty.$$  \hfill (47)

In this case, conditions (3), (4) and (31) are badly violated. Even if the function $\psi(r)$ are chosen so as to satisfy the condition (31), the function will remain discontinuous, and either or both of conditions (3) and (4) will be violated. This means that condition (31) is still too weak to regulate the function $n_F(q)$. One way to solve this situation is to strengthen the condition (31), which may be achieved by using the alternative inequality (32). The new upper limit $\eta_0$ should be low enough to make the function $n_F(q)$ behave reasonably, and as a natural choice, we take, in the present paper, the value of $\eta_0$ which makes the distribution function $n_F(q)$ satisfy the following inequality:

$$[n_F(q)]|_{q=k_F+0} - [n_F(q)]|_{q=k_F-0} \leq 0.$$  \hfill (48)

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**Fig. 5.** Energy per particle for neutron matter. Two groups of lines are distinguished by the attached numbers denoting the hard core radii of the potentials. The dashed lines are calculated with use of the momentum condition, while the solid lines are calculated with use of the momentum condition and the spin condition. The dash-dotted lines are taken from Ref. 1.

**Fig. 6.** Lagrange multiplier $\lambda_F$ for neutron matter. The negatives of $\lambda_F$ are given in a logarithmic scale. The numbers attached to the lines are the hard core radii of the potentials. The dashed lines are calculated with use of the momentum condition, while the solid lines are calculated with use of the momentum condition and the spin condition.
It turns out that the energy minimum under condition (48) is realized with the equality sign in this condition. Then we can summarize the procedure for numerical calculation as follows:

1) Assume a trial value for the upper bound $\eta_0 (\leq 1)$.
2) Solve the integro-differential equation (45) by assuming an appropriate value for $\lambda_F$, and calculate the excitation ratio $\eta_F$ (Eq. (30)). This step should be repeated by changing the value $\lambda_F$ until the calculated value $\eta_F$ becomes equal to $\eta_0$.
3) Calculate the discontinuity of the distribution function $n_F(q)$ (the left-hand side of Eq. (48)).
4) Reiterate the steps 1)~3) by varying the value of $\eta_0$ until the discontinuity vanishes.

It has been numerically confirmed that, as the value of $\eta_0$ decreases, the energy expectation value $E_F$ increases (the condition becomes severer), and the discontinuity of $n_F(q)$ decreases. For a certain value of $\eta_0$, this discontinuity vanishes, and the procedure is completed. Numerical calculations have been made for the semi-realistic OMY potential.\(^5\) Two sets of the force parameters corresponding to the hard core radii 0.4 and 0.6 fm have been used.

Figure 5 shows the energy per neutron as a function of $k_F$. The dashed lines represent the results of the above calculations, while the solid lines those of the calculations in which the spin-condition\(^1\) is imposed together with the present momentum condition. For comparison, we show in Fig. 5 the results obtained with the spin-condition only by dash-dotted lines. Figure 6 shows the Lagrange multiplier $\lambda_F$. It follows from

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**Fig. 7.** Excitation ratio in neutron matter. The numbers attached to the lines are the hard core radii of the potentials.

**Fig. 8.** Single particle distribution function $n_F(q)$ for neutron matter. (a) is for the hard core radius 0.4 fm, while (b) is for 0.6 fm. The numbers attached to the lines are the Fermi wave numbers $k_F$ (in units of fm\(^{-1}\)).
these figures that the effect of the momentum condition predominates over that of the spin condition in the region of high densities. The excitation ratio $\eta_F$ is shown in Fig. 7. The curves indicate that the excitation ratio lies in the range $0.2 \sim 0.3$ for $r_0 > 1.2$ fm. Incidentally, in the case of nuclear matter, the excitation ratio (which is often called "wound") is about 0.15 for $k_F = 1.5$ fm$^{-1}$. The momentum distribution functions $n_F(q)$ are shown in Figs. 8(a) and (b). Finally the pair correlation functions $v_f(r)$ are shown in Figs. 9(a) and (b).

§ 5. Discussion

In the case of liquid helium-4, the calculated energy minimum is 20 % deeper than the experimental value. Higher-order correlations may reduce this overbinding. It is noteworthy that in spite of its relative simplicity the present calculation explains the experimental results fairly well. This proves the effectiveness of the present momentum condition. As for the neutron matter calculation, we have used the semi-realistic potential and correlation function. In order to make a more quantitative argument, we have to use a more realistic nuclear potential. In such a case, the attractive component of the potential such as the spin-orbit force in the triplet-odd state will still be effective in the high-density region. Therefore, the predominance of the momentum condition in that region might not be so obvious as in §4.

In this work, we have limited our calculations to the lowest-order (two-body) approximation. However, the conditions developed in §2 are, in principle, extensible to calculations of arbitrarily higher order.

It should be noted that, if the potential has a hard core of radius $r_c$, the lower limit of $\eta_F$ in the two-body cluster approximation (Eq. (20)) is $(4\pi/3)r_c^3$ owing to the boundary condition (11). Hence, from inequality (19), we have the upper limit of the number density as

$$\rho \leq 3/4\pi r_c^3 \approx 0.2387/r_c^3.$$ 

This restriction is too severe because the real upper limit, which corresponds to the closest
packing, is $\sqrt{2}/\pi^3$. In the case of a soft-core potential, however, this inconsistency will be less severe. It is also expected that this inconsistency will be diminished if higher-order cluster terms are taken into account.

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