Inviscid Disturbance Equations in Linear Stability Theory in Fluid Dynamics and Geophysics

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(Received October 17, 1983)

Analysis is given on the inviscid limit of the disturbance equations of linear stability theory of parallel flows or of linear barotropic instability equations of zonal flows of atmosphere. Existence is shown of the generators of inviscid limit dynamics. Some of the properties of generators are discussed in the context of the (Rossby) wave scattering in geophysical cases.

§ 1. Introduction

Let the cartesian xy-plane simulate the surface of the earth, x-axis be the equator and y-axis be directed north. Let there be given a two-dimensional zonal flow $U(y)$ of atmosphere in the $x$-direction, and let $\Phi(x, y, t)$ be the stream function of small, incompressible disturbance ($u_x, u_y$) superimposed on $U(y)$,

$$u_x(x, y, t) = \partial \Phi(x, y, t) / \partial y, \quad u_y(x, y, t) = - \partial \Phi(x, y, t) / \partial x.$$  \hspace{1cm} (1)

The disturbance obeys the linearized equation of motion \(^{1,2}\)

$$[\partial/\partial t + U(y)\partial/\partial x]A\Phi + [\beta - U''(y)]\partial \Phi / \partial x = \nu A^2 \Phi,$$  \hspace{1cm} (2)

Here $\nu > 0$ is the viscosity of atmosphere, and the constant $\beta > 0$ (in Northern Hemisphere) approximates the effect of Coriolis' force ($\beta$-plane approximation\(^2\)). The case $\beta = 0$ of (2) is also the basic equation in the linear stability theory of parallel flows.\(^3-8\)

For mathematical simplicity, the present note takes\(^*) Eq. (2) on $(x, y) \in [0, X] \times \mathbb{R}, R \equiv (-\infty, \infty)$, with periodic boundary condition posed in the variable $x$. In the inviscid limit $\nu \downarrow 0$ of the case $\beta \neq 0$, Eq. (2) may admit a type of sinusoidal motion called Rossby waves. To see this point, assume the form

$$\Phi(x, y, t) = \Phi(y) \exp[i\alpha(x - \omega t)] = \Phi(y) \exp(i\alpha x - i\omega t),$$

$$\omega = \alpha c, \quad \alpha = 2\pi n / X, \quad n = \pm 1, \pm 2, \ldots.$$  \hspace{1cm} (3)**

Then (2) is reduced to a Schrödinger's type of equation

$$-d^2 \Phi / dy^2 + [(U''(y) - \beta) / (U(y) - c)] \Phi = -a^2 \Phi.$$  \hspace{1cm} (4)

If the limiting values $U \equiv U(\pm \infty)$ exist, the following wave form solution of (2) is obtained for $\beta \neq 0$ under suitable configurations of $\{U, \alpha, \beta, c\}$.

\(*\) In the geophysical context, $\beta$-plane approximation is valid only for a finite interval of $y$ that represents middle latitude zone.

\(**\) The case $n = \alpha = 0$ of (3) yields by (2) the diffusion equation $\partial \Phi / \partial t = \nu \Delta \Phi, \quad \phi = - \Delta \Phi$, and convergence is manifest for $\nu \downarrow 0$. We omit the case $\alpha = 0$ from consideration.
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\[ \Phi(y) = \begin{cases} e^{iky} + Re^{iky}, & y > 1, \quad k = [\beta/(U_+ - c) - a^2]^{1/2}, \\ Te^{-ik'y}, & y < -1, \quad k' = [\beta/(U_- - c) - a^2]^{1/2}. \end{cases} \]  

Equation (4) should be defined as the limit \( \nu \downarrow 0 \) of the non-conservative equation (2). Thus (4) does not conserve energy \( \int (u_x^2 + u_y^2) \, dx \, dy / 2 \) notwithstanding its resemblance to Schrödinger's equation. There exist in (2) mechanisms of interchange of energy between \( U \) and the disturbance; the energy of disturbance is also lost by \( \nu \Delta \Phi \). In the limit \( \nu \downarrow 0 \) both effects survive in general. The peculiarities of the problem are that \( |\Phi|^2 \) is not proportional to the physical energy density of the flow field, and also that in (4) the effects of \( \nu \sim 0 \) concentrate on one mathematical point, the definition of (4) at the critical point \( y = y_c \),

\[ U(y_c) = c, \quad c : \text{real}. \]  

These peculiarities result in the enhanced scattering \( |R| > 1^{7,8} \) or \( |T| > 1^9 \) in (5) under some circumstances.

The aim of the present note is, first of all, to show the existence of well-defined inviscid dynamics of disturbance as the limit \( \nu \downarrow 0 \) of (2). The physical phenomena of wave scattering call naturally for the notion of wave packets. The notion is embodied mathematically \(^{10}\) by taking \( \Phi \) in some \( L^2 \)-type function spaces. The dynamics is the time evolution of \( \Phi \) in such a space, and it is characterized by the generator (Hamiltonian) of evolution semigroup. This note will prove the existence of a suitable function space for \( \Phi \), and also prove the convergence of the generator of motion as \( \nu \downarrow 0 \). The phenomena of Rossby waves will thus be shown to have a formalism completely parallel to quantum mechanical scattering. \(^{10}\)

Another aim of the note is to discuss the mentioned novelties and related subjects on a firm footing provided by this formalism. We shall have an insight into some problems of the inviscid limit that arose decades ago with the linear stability theory of parallel flows \(^{3,11,12}\) at \( \beta = 0 \). A natural proposal will be made on analytical means to evaluate the scattering data \( R \) and \( T \) approximately. \(^*\) Last but not least, we shall have prospects on problems for the future.

§ 2. Preliminaries

The stream function is not directly observable. By (1) it is appropriate to introduce the following type of norm (the measure of magnitude) for \( \Phi = \Phi(x, y, t) \),

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [u_x^2(x, y, t) + u_y^2(x, y, t)] \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x, y, t) (-\Delta) \Phi(x, y, t) \, dx \, dy. \]

The following conventions, notations and simplifying assumptions [1] – [6] will be adopted throughout the present note for compactness of descriptions.

\(^*\) The phenomenon \( |T| > 1 \) was found with a realistic \( U(y) \) by Yamada and Okamura, oral communication. Both of \( |R| > 1 \) and \( |T| > 1 \) may typically be seen with square-well potential approximation to (4); see §4. As to the first half of 8) see also Yamada and Gotoh.\(^{9}\)
\begin{equation}
\Phi(x, y, t) = \Phi(y, t)e^{ix}
\end{equation}

with some fixed \( a \in \mathbb{R}, a \neq 0 \). The part \( \Phi(y, t) \) will be called the stream function, and denoted exclusively by a capital Greek letter \( \Phi \) or \( \Psi \).

[2] The classes \( \mathcal{S}, L^2(d\mu) \) and \( H^1 \) denote, respectively, the collection of infinitely differentiable complex functions on \( \mathbb{R} \) with rapid decrease, the \( L^2 \)-functions on \( \mathbb{R} \) w. r. t. the measure \( d\mu \), and the Sobolev space of functions of \( L^2(dy) \) that have first generalized derivatives in \( L^2(dy) \).

[3] The vorticity associated with the stream function \( \Psi(y, t) \) (for example in \( \mathcal{S} \) as a function of \( y \) at a fixed \( t \)) is defined by

\begin{equation}
\phi(y, t) \equiv (-\Delta)\Psi(y, t) \equiv (a^2 - \partial^2/\partial y^2)\Psi(y, t),
\end{equation}

and denoted by the corresponding lower case Greek letters. It is the physical vorticity modulo the factor \( e^{ixa} \). Once \( \Phi \) or \( \phi \) is introduced, the corresponding \( \Phi \) or \( \Psi \) will be used without comments.

[4] Fourier transforms in \( y \) are defined and denoted as follows, abiding the rules of notation in [1] and [3].

\begin{equation}
\hat{\phi}(k, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-iky}\phi(y, t)dy \equiv \mathcal{F}\phi,
\end{equation}

\begin{equation}
\hat{\Psi}(k, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-iky}\Psi(y, t)dy \equiv \mathcal{F}\Psi.
\end{equation}

[5] For stream functions \( \Phi, \Psi \in \mathcal{S} \) their inner product is defined by

\begin{equation}
\int_{-\infty}^{\infty} \Phi^*(y)(-\Delta)\Psi(y)dy \equiv \int_{-\infty}^{\infty} \hat{\Phi}^*(k)(a^2 + k^2)\hat{\Psi}(k)dk.
\end{equation}

For the corresponding vorticities \( \phi(y), \psi(y) \in \mathcal{S} \) the inner product is defined by

\begin{equation}
\langle \phi, \psi \rangle \equiv \int_{-\infty}^{\infty} \phi^*(y)(-\Delta)^{-1}\psi(y)dy \equiv \int_{-\infty}^{\infty} \hat{\phi}^*(k)(a^2 + k^2)^{-1}\hat{\psi}(k)dk,
\end{equation}

and \( \langle \phi, \psi \rangle / 2 \) will be called the energy of disturbance. For vorticities we also introduce the usual \( L^2(dy) \)-inner product

\begin{equation}
\langle \phi, \psi \rangle \equiv \int_{-\infty}^{\infty} \phi^*(y)\psi(y)dy \equiv \int_{-\infty}^{\infty} \hat{\phi}^*(k)\hat{\psi}(k)dk,
\end{equation}

and call \( \langle \phi, \psi \rangle / 2 \) the enstrophy.

[6] The Hilbert space \( \mathcal{H} \) is defined to be the completion of \( \mathcal{S} \), \( \Phi \in \mathcal{S} \) w.r.t. the inner product \( \langle \phi, \psi \rangle \) or the norm \( \| \phi \| \equiv \langle \phi, \psi \rangle^{1/2} \). The \( L^2 \)-norm is denoted as \( \| \phi \| \equiv \langle \phi, \phi \rangle^{1/2} \).

A few comments are now in order.

[Note 1] An element \( \phi(y) \in \mathcal{H} \) is characterized by \( \hat{\phi}(k) \in L^2(dk/(a^2 + k^2)) = (a^2 + k^2)^{1/2}L^2(dk) \). Typically, \( \hat{\phi}(k) = (2\pi)^{-1/2} \) or \( \phi(y) = \delta(y) \) is in the class \( \mathcal{H} \). Given a vorticity \( \phi(y) \in \mathcal{S} \subset \mathcal{H} \), the ordinary differential equation \( -\Delta \Phi(y) = (a^2 - d^2/\partial y^2)\Phi(y) = \phi(y) \) has a unique solution in \( L^2(dy) \),

\begin{equation}
\Phi(y) = (-\Delta)^{-1}\phi(y) = (2a)^{-1}\int_{-\infty}^{\infty} e^{-a|y-y'|}\phi(y')dy' \in \mathcal{S},
\end{equation}

\begin{equation}
\Phi(k) = \hat{\phi}(k)/(a^2 + k^2).
\end{equation}
Therefore, the correspondence \(-\Delta : \Phi(y) \mapsto \phi \equiv (-\Delta)\Phi\) can be inverted on \(\mathcal{S}\), and extends to an isomorphism between the Sobolev space \(H^1\) for the stream function \(\Phi(y)\) with the inner product

\[
\int_{-\infty}^{\infty} [a^2 \Phi^*(y) \Psi(y) + \Phi''(y) \Psi'(y)]dy,
\]

and the space \(\mathcal{K}\) for vorticity. In fact, we have for \(\Phi, \phi, \Psi, \phi \in \mathcal{S}\),

\[
(14)= \langle \phi, \psi \rangle = \int_{-\infty}^{\infty} \Phi^*(y)(-\Delta)\Psi(y)dy.
\]

[Note 2] In order to see the mode of convergence in the vorticity space \(\mathcal{K}\), consider \(\phi(y) = \delta(y-z)\). There hold

\[
\|\delta(y-z)\|^2 = \int_{-\infty}^{\infty} \delta(y-z)(2\pi)^{-1} e^{-\pi(y-y')}\delta(y'-z)dydy' = (2\pi)^{-1},
\]

\[
\lim_{z \to 0} \|\delta(y-z) - \delta(y)\|^2 = \lim_{z \to 0} (2\pi)^{-1} \int_{-\infty}^{\infty} |e^{ikz} - 1|^2 dk/ (a^2 + k^2) = 0,
\]

where the dominated convergence theorem is used in the second line.

[Note 3] Consider a function \(f(y)\) as a multiplication operator on \(\phi(y) \in \mathcal{K}\). Even if \(f(y)\) is uniformly bounded, the operator

\[
f(y): \phi(y) \in \mathcal{K} \rightarrow f(y)\phi(y)
\]

is not bounded (or continuous); if \(f(y)\) has a discontinuity of the first kind at \(y=0\), for example, we have

\[
\lim_{z \to 0} \|f(y)\delta(y-z) - f(y)\delta(y+z)\| = 0, \quad \lim_{z \to 0} \|\delta(y-z) - \delta(y+z)\| = 0.
\]

A necessary and sufficient condition for an operator \(A\) on \(\phi(y) \in \mathcal{K}\) to be bounded is by \(\mathcal{K} \equiv (a^2 + k^2)^{1/2}L^2(dk)\):
(a) \((-\Delta)^{-1/2}A(-\Delta)^{1/2}\) is bounded on \(f(y) \in L^2(dk)\) or,
(b) \((a^2 + k^2)^{-1/2}A(a^2 + k^2)^{1/2}\) is bounded on \(f(k) \in L^2(dk)\) where \(A = \mathcal{F} A \mathcal{F}^{-1}\).

§ 3. Dynamics

In this section \(U(y), U'(y), U''(y)\) are assumed to be (measurable and) uniformly bounded; we denote such a class of \(U(y)\) as \(U\). If \(\Phi(x, y, t) = \Phi(y, t)e^{iat}\) has all necessary derivatives and satisfies (2), the Cauchy problem reads

\[
\partial \phi(y, t)/\partial t = \nu \Delta \phi + A \phi , \quad \phi(y, 0) = \phi_0(y) = \text{given}, \quad \nu \geq 0 ,\]

\[
A \equiv A(U) = -iaU(y) - i\lambda(U''(y) - \beta)(-\Delta)^{-1}, \quad t \in \mathbb{R}^+ = [0, \infty). \quad (16)
\]

Consider more abstractly that \(\phi(y, t)\) is an element in \(L^2(dy)\) parametrized by \(t\), and define the operator \(\Delta = -a^2 + \delta^2/\partial y^2\) on \(\phi(y) \in L^2(dy)\) by a self-adjoint, negative unbounded multiplication operator \(-a^2 + k^2\phi(k)\) in Fourier space, \(\mathcal{F}(\phi) = -(a^2 + k^2)\phi(k)\). See Chap. 5, §5.2 of Kato,\(^{11}\) for example. The domain of \(\Delta\) consists of such \(\phi(y) \in L^2(dy)\) that \((a^2 + k^2)\phi(k) \in L^2(dk)\) holds. The operator \(A(U)\) is bounded on \(L^2(dy)\): This is manifest for \(U(y)\) and \(U''(y)\) as multiplication operators; \((-\Delta)^{-1}\) is a bounded
multiplication operator \((\alpha^2 + k^2)^{-1} \leq \alpha^{-2}\) on \(L^2(\Omega)\).

These facts provide us with the following.

**Proposition 1.** There exists a family of strongly continuous semigroup of bounded operators on \(L^2(\Omega)\),

\[
T_{t}^{(t)} = T_{t}(\nu, U) = \exp\{t[\nu A + A(U)]\}, \quad t \in \mathbb{R}^+, \ \nu \geq 0,
\]

\[
\|T_{t}(\nu, U)\| \leq e^{\nu t}, \quad \forall \nu > \|A(U)\|.
\]  

(17)

Here \(\|T_t\|\) or \(\|A\|\) is the \(L^2(\Omega)\) operator norm. Let there be \(\bar{U}(y) \in \mathcal{U}\), and let \(t\) be restricted on \([0, T]\), \(T < \infty\). There holds

\[
\|T_{t}(\nu, U) - T_{t}(\nu, \bar{U})\| \leq C\|A(U) - A(\bar{U})\|, \quad 0 \leq \nu \leq T,
\]

(18)

where \(C\) depends on \(T\) and \(\max\{\|A(U)\|, \|A(\bar{U})\|\}\), but not on \(\nu \geq 0\).

**Proof** Existence of the semigroup \(e^{tu}A\) or \(T_{t}^{(t)} = e^{tu}A^{(t)}\) is wellknown; see Kato,\(^{13}\) Chap. 9, §§1.1 and 1.2. Existence of the perturbed semigroup (17) is Theorem 2.1, §2.1, Chap. 9 of \(^{13}\). In this latter theorem we take \(T_{t}(\nu, U)\) to be the unperturbed semigroup, while \(A(\bar{U}) - A(U)\) is the perturbation on the generator \(\nu A + A(U)\). The proof of the theorem gives (18) at once; \(C\) is independent of \(\nu\) because of the \(\nu\)-independent estimate in (17). \(\square\)

In passing we note the following. If \(U(y) \in \mathcal{U}\) has a uniformly Hölder continuous \(U''(y)\), a well-behaved initial data \(\phi_{0}(y)\) (for example, \(\phi_{0}(y) \in \mathcal{F}\)) assures the existence of a classical solution of (16) for \(\forall \nu > 0\) that has all the derivatives figuring in (16). In fact, (16) may be rewritten for \(\nu > 0\),

\[
\phi(y, t) = \int_{-\infty}^{\infty} G(y - z, \nu t)\phi_{0}(z)dz + \int_0^t ds \int_{-\infty}^{\infty} dz G[y - z, \nu(t - s)]A(U)\phi(z, s),
\]

\[
G(y, \nu t) = (4\pi \nu t)^{-1/2}\exp[-y^2/(4\nu t)].
\]

Though \(A(U)\) includes nonlocal integral operator \((-\Delta)^{-1}\), the construction of the fundamental solutions and volume potentials may be discussed as Friedman,\(^{14}\) Chap. 1, §§2~7. Since this is not of use and the result is manifest to intuition, we omit the details.

The space \(L^2(\Omega)\) for vorticity must be extended to \(\mathcal{H}\). To this end we use the following lemma as the basic tool. The lemma is a version of Ascoli-Arzelà theorem and wellknown; we give a proof for clarity.

**Lemma 2.** Let \(Q = \{x\} \) be a normed space with norm \(\omega(x)\), and \(B, B_{n}(n = 1, 2, \cdots)\) be a uniformly bounded operators on \(Q\),

\[
\omega(B) = \sup_{x \in B} \omega(Bx)/\omega(x) \leq M, \quad \omega(B_{n}) \leq M.
\]

Suppose \(Q_{0} \subseteq Q\) is dense, and on \(\forall \chi \in Q_{0}\) the vectors \(B_{n}\chi\) converge to \(B\chi\) as \(n \to \infty\). Then \(B_{n}\) converges strongly (i.e., on every \(\chi \in Q\) ) to \(B\).

**Proof** Let \(\chi \in Q\) and \(\varepsilon > 0\) be arbitrary. There exists \(\chi' \in Q_{0}\) such that \(\omega(\chi - \chi') < \varepsilon/(3M)\) holds. By triangle inequality we have

\[
\omega(B_{n}\chi - B\chi) \leq \omega(B_{n}\chi - B_{n}\chi') + \omega(B_{n}\chi' - B\chi') + \omega(B\chi' - B\chi)
\]

\[
\leq 2\varepsilon/3 + \omega(B_{n}\chi' - B\chi')
\]

By \(\chi' \in Q_{0}\) there exists \(N\) that gives for \(\forall \ n > N\) the inequality \(\omega(B_{n}\chi' - B\chi') < \varepsilon/3\). This
proves $\omega(B_nX-BX)<\varepsilon$ for $n>N$. □

The resolvent of the semigroup $T_t^{(\nu)} = T_t(\nu, U)$ on $L^2(dy)$ is defined by

$$R_\nu(\lambda) = \left[\lambda - \nu A - A(U)\right]^{-1} = \int_0^\infty e^{-\mu t} T_t(\nu, U) dt.$$  \hspace{1cm} (19)

By (17) $R_\nu(\lambda)$ is welldefined for $\lambda$ with $\Re(\lambda) > \frac{\nu}{\lambda} L > \|A(U)\|$ for any $\nu > 0$.

Lemma 3. Let $\Lambda_\nu \equiv \Re(\lambda) > L$ hold. The resolvent $R_\nu(\lambda)$ converges in the strong operator topology to $R_\nu(\lambda)$ as $\nu \downarrow 0$ on $L^2(dy)$.

Proof. Take first an element $\phi(y) \in \mathcal{P}$ that is in the domain of $A$. We have $\|\left(\lambda - \nu A - A\right)^{-1}(\lambda - A)\phi - \phi\| = \nu\|R_\nu(\lambda)A\phi\|$. By $\|R_\nu(\lambda)\| < (\lambda_\nu - L)^{-1} < \infty$, we obtain

$$s-l\lim_{\nu \downarrow 0} R_\nu(\lambda)R_\nu(\lambda)^{-1}\phi = \phi, \quad \|R_\nu(\lambda)R_\nu(\lambda)^{-1}\| \leq (\lambda_\nu - L)^{-1}(\lambda + L).$$

Take $B = 1$, $B_\nu = R_{\nu n}(\lambda)R_\nu(\lambda)^{-1}$, $M = \max\left[1, (\lambda_\nu - L)^{-1}(\lambda + L)\right]$, $\Omega = L^2(dy)$ and $\Omega_\nu = \mathcal{P}$ in Lemma 2; we have the convergence of $R_\nu(\lambda)R_\nu(\lambda)^{-1}$ to 1 in the strong operator topology of $L^2(dy)$. Thus $R_\nu(\lambda)R_\nu(\lambda)^{-1}R_\nu(\lambda) - R_\nu(\lambda)R_\nu(\lambda)^{-1}$ holds as in the lemma. □

Define that $U_\nu(y) \in \Omega U$ $U^2$-converges to $U(y) \in \Omega$ as $n \to \infty$, if there exists $M < \infty$ that gives $\|U_\nu^{(k)}(y)\|_\Omega < M$ for all $k = 0, 1$ and 2 irrespective of $n$, and also if $U_\nu^{(k)}(y)$ converges at almost all $y \in \Omega$ to $U^{(k)}(y)$ for all $k = 0, 1$ and 2. Here $\|f(y)\|_\Omega$ is defined by

$$\|f(y)\|_\Omega = \text{ess. sup}_{y \in \Omega}|f(y)|.$$  

For any $U(y) \in \Omega$ such a $U^2$-convergent sequence $U_\nu(y)$ may be constructed as follows. Let $\rho(y) \geq 0$ be an infinitely differentiable function with support in $|y| \leq 1$, together with the property $\int_{-\infty}^{\infty} \rho(y) dy = 1$. If we define

$$U_\nu(y) = \int_{-\infty}^{\infty} dz \rho[n(y-z)] U(z), \quad n = 1, 2, \ldots,$$

$U_\nu(y)$ $U^2$-converges to $U(y)$ as $n \to \infty$.

Corollary 4. Let $U(y) \in \Omega$ be given, and $\{U_\nu(y) \in \Omega\}$ $U^2$-converge to $U(y)$. The family of semigroups $\{T_t(\nu, U_n) = \exp[t(\nu A + A(U_n))]\; \nu > 0, n = 1, 2, \ldots\}$ converges in the strong operator topology of $L^2(dy)$ to $T_t^{(0)} = T_t(0, U)$ as $\nu \downarrow 0$ and $n \to \infty$. The convergence is uniform in $t$ on a compact interval, and the order of $\nu \downarrow 0$ and $n \to \infty$ is immaterial.

Proof. By (18) we have for $\psi(y) \in L^2(dy)$,

$$\|T_t(\nu, U_n)\psi - T_t(0, U)\psi\| \leq \|T_t(\nu, U_n) - T_t(\nu, U)\|\psi\| + \|T_t(\nu, U) - T_t(0, U)\|\psi\| \leq C\|A(U_n) - A(U)\|\psi\| + \|T_t(\nu, U) - T_t(0, U)\|\psi\|,$$

where the constant $C$ does not depend on $\nu \geq 0$ nor on $0 \leq t \leq T < \infty$. The first term on the r.h.s. can be made arbitrarily small by taking $n$ large irrespective of $\nu \geq 0$ or $t$. As to the second term $\|T_t(\nu, U) - T_t(0, U)\|$ we have a theorem of Trotter and Kato, Theorem 2.16, §2.5, Chap. 9 of Kato.\(^{13}\) It proves that $T_t(\nu, U)$ converges strongly to $T_t(0, U)$ by Lemma 3, uniformly in $t$ on a compact interval, as $\nu \downarrow 0$. □

The final key for the extension of $T_t(\nu, U)$ onto $\mathcal{H}$ is the following estimate.
Lemma 5. Let $\phi_0(y) \in \mathcal{S}$ hold. There holds a $v$-independent estimate

$$E(t) = \| T_t(v) \phi_0(y) \|^2 / 2 \leq E(0) \exp(\| U' \|_\infty t), \quad v \geq 0. \quad (20)$$

Proof. The sesquilinear form $\langle \xi, \eta \rangle$ on $\xi, \eta \in L^2(dy)$ is bi-continuous by $\| \langle \xi, \eta \rangle \| \leq \int |\xi^*(k) \eta(k)/(k^2 + a^2)| dk \leq a^{-2} \| \xi \| \cdot \| \eta \|$. Take $\psi(y) \in \mathcal{S}$. By partial integration we have

$$\int_{-\infty}^{\infty} \psi(y)^2 dy + (ia/2) \int_{-\infty}^{\infty} U(y)(\Delta \psi^*) \psi^* - \psi^* (\Delta \psi^*) dy$$

$$\leq u' \| U' \|_\infty / 2 \int_{-\infty}^{\infty} [a^2 \psi^* (y) \psi^* (y) - \psi^* (y) a \psi (y)] dy$$

$$\leq (\| U' \|_\infty / 2) \int_{-\infty}^{\infty} [\| \psi (y) \|^2 + \| \psi^* (y) \|^2] dy = (\| U' \|_\infty / 2) \| \psi (y) \|^2.$$

By $\phi_0(y) \in \mathcal{S}$ the function $\phi(y, t) = e^{(\nu A + A)} \phi_0(y)$ stays in the $L^2(dy)$-domain of $\nu A + A$ for any $t \geq 0$. Since $\mathcal{S}$ is dense in $L^2(dy)$, $\phi(y, t)$ may be approximated by $\{ \phi_n(y) \in \mathcal{S}; n = 1, 2, \ldots \}$ that converges in $L^2(dy)$ to $\phi(y, t)$ an $n \to \infty$. Further, $\nu A + A$ is closed in $L^2 (dy)$. We have

$$dE(t)/dt = (1/2) d\langle \phi(y, t), \phi(y, t) \rangle / dt$$

$$= \langle \psi(y), (\nu A + A) \phi(y) \rangle + \langle (\nu A + A) \phi(y), \psi(y) \rangle / 2$$

$$= \lim_{n \to \infty} \langle \psi_n, (\nu A + A) \phi_n \rangle + \langle (\nu A + A) \phi_n, \psi_n \rangle / 2$$

$$\leq (\| U' \|_\infty / 2) \lim_{n \to \infty} \| \psi_n(y) \|^2 = \| U' \|_\infty E(t).$$

This gives Gronwall's inequality

$$0 \leq E(t) \leq E(0) + \| U' \|_\infty \int_0^t E(s) ds,$$

which proves (20). \qed

Theorem 6. Let $U(y) \in \mathcal{U}$ be given. There exists a family of $\mathcal{A}$-strongly continuous semigroups (dynamics) of operators on $\mathcal{A}$, denoted $\{ T_t(\nu, U) = \exp_t(\nu A + A)) \} \nu \geq 0, U \in \mathcal{U}, t \in \mathbb{R}_+ \}$, with the following properties.

(a) On $L^2(dy)$, $T_t(v) = T_t(\nu, U)$ is the semigroup of Proposition 1 generated by (16) for $\nu \geq 0$. (b) If $\{ U_n(y) \in \mathcal{U} \} U_2$-converges to $U(y) \in \mathcal{U}$ as $n \to \infty$, then $T_t(\nu, U_n)$ converges in the strong operator topology of $\mathcal{A}$ to $T_t(0, U)$ as $\nu \downarrow 0$ and $n \to \infty$, uniformly in $t$ on any finite interval.

Proof. Let $\{ \phi_0 \} \phi(m) \in \mathcal{S}$ converge in $\mathcal{H}$ to $\phi(y) \in \mathcal{H}$. By (20) we have for $\{ T_t(\nu, U); \nu \geq 0, t \in \mathbb{R}_+, n = 1, 2, \ldots \}$ of Proposition 1,

$$\| T_t(\nu, U)[\phi_0(l) - \phi_0(m)] \| \leq e^{M t} \| \phi_0(l) - \phi_0(m) \|,$$

$$0 \leq \sqrt{t} \leq T < \infty, \quad M \equiv \| U' \|_\infty / 2. \quad (21)$$

Define $T_t(\nu, U) \phi_0(y) = \lim_{n \to \infty} T_t(\nu, U) \phi_0(m)(y)$, where the limit is in the sense of $\mathcal{A}$-norm; by (21) this limit exists uniquely, and defines a bounded operator $T_t(\nu, U)$ on $\mathcal{A}$ with the operator norm $\| T_t(\nu, U) \| \leq e^{M t}$. Semigroup property $T_s T_t = T_{s+t}$ with $T_0(\nu, U) = 1$ is obvious. Strong continuity in $t$ is seen on $\phi(y) \in \mathcal{A}$ by taking $Q = \mathcal{A}$, $Q_0 = \mathcal{S}$ or
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$L^2(dy), B \equiv T_t(\nu, U)\text{ and } B_n \equiv T_{t+1/n}(\nu, U)$ in Lemma 2 with $\|\phi\| \leq \|\phi\|/\alpha$ for $\phi \in L^2(dy)$. Thus the semigroup $\{T_t(\nu, U)\}$ exists as in Theorem 6 for $\nu \geq 0$. The property (a) is obvious and (b) is proved by Corollary 4 and Lemma 2. 

§ 4. Remarks

Let there be given $U(y) \in \mathcal{U}$. If an initial data $\phi_0(y) \in L^2(dy)$ is given (i.e., if the vorticity field has a finite total enstrophy to start with), then $\phi(y, t) = T_t(\nu, U)\phi_0(y)$ retains finite enstrophy at all $t$ for $\nu \geq 0$. The function $\phi(y, t)$ need not have derivatives in the usual sense and the differentiability problem becomes more intricate for $\phi_0(y)$ in $\mathcal{H}$ but not in $L^2(dy)$. Nevertheless Theorem 6(b) states that the time evolution of $\phi(y, t)$ with a finite total energy at the start is well-defined, and also that $T_t(0, U)\phi_0(y)$ is the limit of the classical solutions of (16) with all partial derivatives for well-behaved (say, infinitely differentiable) $U_n(y)$ and $\phi_0^{(m)}(y), \phi_m(y, t) = T_t(\nu, U_n)\phi_0^{(m)}(y), \nu \downarrow \nu, m \to \infty$ and $n \to \infty$.

Assume (3) again, and call $\Phi(y)$ the stream function. For $\nu > 0$ we should have

$$[U(y) - c]\Phi''(y) - \alpha^2 \Phi(y) + [\beta - U''(y)]\Phi(y) = -(iv/\alpha)(d^2/\alpha^2 - d^2)\Phi(y). \quad (22)$$

The formal limit $\nu \downarrow 0$ of (22) is (4), but a problem of definition arises with (4) at $y = y_c$ of (6). This was solved by Wasow$^6$ and Lin.$^3$ The critical point $y_c$ is a regular singular point$^{18}$ of (4), and the two fundamental solutions are locally as follows:

$$\Phi_1(y) = (y - y_c) + (\chi_0/2)(y - y_c)^2 + \ldots, \quad \chi_0 = [U''(y_c) - \beta]/U'(y_c),$$

$$\Phi_2(y) = 1/\chi_0 + \Phi_1(y) \log(y - y_c) + \ldots, \quad (23)$$

where $\chi_0 \neq 0$ is assumed, and $\Phi_1$ and $\Phi_2 - \Phi_1 \log(y - y_c)$ are regular functions. Wasow showed that $\Phi_1(y)$ is, locally for $y \sim y_c$, always the limit $\nu \downarrow 0$ of a solution of (22), while there exists only one (physical) branch of $\log(y - y_c)$ of $\Phi_2(y)$ that forms a limit of a solution of (22) in a complex $y$-domain that includes both of real $y < y_c$ and $y > y_c$. A prescription$^9$ to pick up the correct branch is to solve (4) with $c$ replaced by $(c + i\varepsilon/\alpha)$ for $\varepsilon > 0$ and then let $\varepsilon \downarrow 0$ in the solution. The procedure may be summarized in a disguised form.

Proposition 7.$^3$ Let $y = y_c$ be a simple zero of regular $U(y)$ for a real $c$. The physical branch of the solution $\Phi(y)$ of (4), as the inviscid limit of (22), is obtained locally for $y \sim y_c$, by matching the pieces of solutions for $y < y_c$ and $y > y_c$, so that (a) $\Phi(y)$ is continuous at $y = y_c$, and (b) the following holds.

$$\lim_{\varepsilon \downarrow 0} \Phi'(y_c + \varepsilon) - \Phi'(y_c - \varepsilon) = i\pi \chi_0 \sigma \Phi(y_c), \quad \sigma \equiv \text{sign}[U'(y_c)]. \quad (24)$$

By its way of construction Proposition 7 gives the means to construct eigenmodes of the generator $A$ in (16) of the inviscid time evolution $T_t(0, U)$ for real eigenvalue $c$ in a continuous range.

$^*)$ In other words, we need to consider an equation $\partial \phi/\partial t = (-\varepsilon + A)\phi$ in place of the correct $\partial \phi/\partial t = (\nu A + A)\phi$, and assume (3) on $\Phi$. 

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This eigenmode $\Phi(y)$, in known cases, is not in the class $H^1$ (or $\phi \in \mathcal{H}$ does not hold); it has the sense of a generalized eigenfunction or an extended mode. The rules (a) and (b) give the definition of (4),

$$
-d^2\Phi/\,dy^2 + V(y)\Phi(y) = -\alpha^2 \Phi,
$$

$$
V(y) \equiv \chi_0 \{ \mathcal{P} [1/(y-y_c)] + i\pi \delta(y-y_c) \}, \quad y \approx y_c.
$$

This exact definition shows that a vorticity mode $\phi(y)$ with time evolution $\propto e^{-i\omega t}$ should have the local form for $y \sim y_c$,

$$
\phi(y) = (a^2 - d^2/\,dy^2)\Phi(y) = -V(y)\Phi(y)
\simeq -\Phi(y_c) \chi_0 \{ \mathcal{P} [1/(y-y_c)] + i\pi \delta(y-y_c) \}.
$$

The real and imaginary parts of this $\phi(y)$, $\mathcal{P}[1/(y-y_c)]$ and $\delta(y-y_c)$ whose Fourier transforms are essentially $\exp(-iky_c)$ sign$(k)$ and $\exp(-iky_c)$, are surely in $\mathcal{H} \equiv L^2(\mathbb{R}^2/(a^2+k^2))$, but not in $L^2(\mathbb{R})$ for vorticities with finite enstrophy. We should expect for vorticity $\phi(y, t) \in \mathcal{H}$ these singularities in the inviscid limit.

On a class of initial-boundary value problems on $0 \leq y \leq y_1 < \infty$ with $\nu = 0$ Case$^{11}$ discussed the convergence of Laplace transforms of $T_t^{(\nu)} \phi_0(y)$ for $\nu \downarrow 0$. His arguments point exactly to the convergence of resolvents of the generators, as discussed in §3. Theorem 6 of the present note confirms the conclusions of Case on a general, though somewhat different, setting. An issue in the conclusions of Case$^{11}$ and Lin$^{22}$ was concerned with the following. Suppose (4) has a solution $\{\Phi(y), c\}$ with a complex $c$ (say, a stable mode $\text{Im}(c) < 0$). Equation (4) is invariant under the transformation $I: \{\Phi, c\} \rightarrow \{\Phi^*, c^*\}$, so that another (unstable) solution $\{\Phi^*, c^*\}$ exists. Invariance in $I$ is lacking in (22) with $\nu > 0$: There cannot be complex conjugate solutions that converge to $\{\Phi, c\}$ and $\{\Phi^*, c^*\}$ as $\nu \downarrow 0$. However, Cauchy problems may be posed with $\phi_0(y) = -\Delta \Phi(y) \in \mathcal{H}$ and $\phi_0^*(y)$. The time evolution $T_t^{(\nu)} \phi_0^*(y)$ should be as close to $T_t^{(0)} \phi_0^*(y)$ in $\mathcal{H}$-norm as we wish at any finite $t$, if $\nu$ is taken to be small enough.

As resolved by Lin,$^{22}$ this is not a discrepancy. Absence of a solution $\Phi(y) \exp[i\alpha(x-cvt)]$ of (22) convergent to $\Phi(y) \exp[i\alpha(x-ct)]$ as $\nu \downarrow 0$ can be one fact irrespective of the nearness of $T_t^{(\nu)} \phi_0(y)$ to $T_t^{(0)} \phi_0(y)$. The conclusion on stability of the given flow $U(y)$ is independent of this point.$^{30}$ We may add another fact that an exponential type of decay may be approximated in a variety of ways by other functions. See specifically Tatsumi and Gotoh$^{16}$ for the behavior of decaying disturbance modes as $\nu \downarrow 0$.

Finally we consider the problems of Rossby waves for $\nu = +0$ and $\beta > 0$. We start with the case of packets. An initial data $\phi_0(y) \in L^2(dy)$ has the inviscid time evolution $\phi(y, t) = T_t^{(\nu)} \phi_0(y)$ defined by

$$
i \partial \phi/\partial t = H(U)\phi, \quad H(U) = iA(U) = aU(y) + a [U''(y) - \beta] (-\Delta)^{-1}.
$$

The operator $a [U''(y) - \beta] (-\Delta)^{-1}$ is symmetric on $\mathcal{H}$. The part $aU(y)$ is nonsymmetric in general. Assume the existence of $U_\pm \equiv U(\pm \infty)$. At $y \sim \pm \infty$ we have the free dynamics
(29)

where \( \omega_{\pm}(k) \) give the dispersion relations of Rossby waves \( \phi(y, t) \propto \exp(\pm i k y - i \omega_{\pm} t) \) at \( y \sim \pm \infty \), respectively. The free Hamiltonian \( H_{\pm} \) is self-adjoint but bounded. The remainder of \( H(U) \), call it the interaction Hamiltonian \( H_{int}^{(\pm)} \) relative to \( H_{\pm} \),

\[
H_{int}^{(\pm)} = a[U(y) - U_{\pm}] + aU''(y)(-\Delta)^{-1},
\]

is generally unbounded\(^*\) and not small relative to \( H_{\pm} \). The free Rossby waves are feeble; their mode of existence depends strongly on respective forms of \( U(y) \).\(^*\)

Consider now a mode, including the extended case. It should obey (26), the exact inviscid limit of (4). This special version of one-dimensional Schrödinger equation with complex potential enables us to see the mentioned diversity — according to the cases of \( U(y) — \) of the wave phenomena with ease. We do not go into respective details\(^8,9\) and note the following three special cases which may be seen readily with (26).

(A) Positive high frequency, \( \omega > \text{amix}(U_{+}, U_{-}) \); no wave exists. For \( \omega > \sup_{y} a U(y) \), bound states with real \( \omega \) may exist.

(B) Negative high frequency, \( \omega < \text{amin}(U_{+}, U_{-}) - \beta/\alpha \), no wave exists. For \( \omega < \inf_{y} a U(y) \), bound states may exist.

(C) Waves can exist at \( y \sim \pm \infty \) if \( a U_{\pm} - \beta/\alpha < \omega < a U_{\pm} \) holds, respectively.

The most distinctive feature of the problem appears with critical points, which arise with \( c \) in the range (25) or \( \inf_{y} a U(y) < \omega < \sup_{y} a U(y) \). The potential \( V(y) \) of (26) is now complex, and provides an example of optical potential, though the Hamiltonian \( H(U) \) is itself non-symmetric in the space \( \mathcal{H} \). We note that the matching conditions (a) and (b) of Lin in Proposition 7 are identical with those of real \( \delta \)-well potentials or Kronig-Penney model. This suggests to approximate \( V(y) \) of (26) by simple forms such as broken line profiles or square wells.\(^**\) As we have seen at the end of §2, discontinuities in \( U(y) \) introduce unbounded perturbation on the generator \( A(U) \) in \( H \). The approximation is rough. However, the results reproduce some aspects of the scattering data \( R \) and \( T \) nicely, including the enhanced cases. This will be presented elsewhere; the point to be noted here is that the class \( \mathcal{U} \) for \( U(y) \) seems to be too narrow. The dynamics seem to be extended continuously for \( U(y) \) outside of \( \mathcal{U} \).

The central problem in the stability theory is to seek for the eigenvalues and eigenmodes of the generator of disturbance dynamics. Also, the completeness of such (generalized) eigenmodes has been a subject of keen interest.\(^8,9,17\)\textendash19\) The present note has shown that the dynamics for small \( \nu > 0 \) is approximated in the sense of resolvent by the inviscid dynamics on the function space \( \mathcal{H} \) with a well-defined generator or resolvent. An immediate problem will be to question on the eigenvalues and eigenmodes of this generator in typical cases. Realistic, two-dimensional problems should naturally be pursued and, hopefully, more general classes for the main flows than, for example, \( \mathcal{U} \) considered here, should also be taken into account, as the recent work of Ruelle\(^20\) on turbulence suggests strongly.

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\(^*\) If we step out of the class \( \mathcal{U} \), the case \( U(y) \propto y \) of Couette flow gives the case that waves are completely extinguished.

\(^**\) See Betchov and Criminale\(^*\) or Drazin and Reid\(^9\) for such approximations on flow profiles \( U(y) \) in the stability theory.
References

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