§ 1. Introduction

One of the most important astrophysical and cosmological problems is to explain the structure and the distribution of galaxies in the present universe within the scenario of the evolutionary cosmology. Many theories have so far been presented on this problem:1) The gravitational instability theory,2,3 the cosmic turbulence theory,4 the thermal instability theory,5 the theory appealing to successive explosion of small objects,6 etc. The most important common feature of these theories, at least of the theories which have survived until now,7 is that some seed density fluctuations on large scale with small amplitudes are required to exist in the early stage of the universe.

Unfortunately, in most of the theories so far presented the origin of these seed density fluctuations have not been explained and often attributed to the initial condition of the universe at the Big-bang point or at the Planck time before which the classical description of the geometry itself breaks down. Of course we cannot exclude the possibility that the present structure was essentially built-in in the initial condition of the universe. Especially the idea is very fascinating that the quantum fluctuation of the metric itself around the Planck time might have played important roles to generate the seed fluctuations.8 However, there is also the alternative possibility that the dominant part of the seed fluctuations are generated from some spontaneous perturbations in the course of the cosmic evolution.9 This latter possibility has recently become more and more important by the new development in the cosmology, namely the application of the grand unified theories to the early universe.10 This new cosmology (GUT cosmology) not only has resolved various long-standing cosmological problems, but also has suggested the occurrence of various strong and weak transient phenomena in the very early stage of the universe.11,12 These transient phenomena are generally associated with spontaneous fluctuations (mostly of stochastic nature) and may generate the seed density fluctuations.13

In this paper we investigate in a general way the possibility that the seed density
fluctuations, which can grow to the presently observed density contrast on the large scale, were generated by some spontaneous perturbations associated with weak transient phenomena in the early universe. One important point to be noted concerned with this problem is that the seed density fluctuations corresponding to the present galaxies or larger scale structures had scales far greater than the horizon size (or exactly speaking, the effective horizon size defined by the cosmic scale factor $a$ as $a/ (da/dt)$, where the unit $c$ (the light velocity) = 1 is used) in the early stage. As a consequence, spontaneous perturbations as a trigger for the seed density fluctuations cannot be associated with the perturbation of energy density, hence must be pure stress perturbation.

This type of possibility has been already studied by Press-Vishniac and Bardeen. Their result was rather negative: The density fluctuations generated by stress perturbations have, when they come within the horizon, amplitudes of the same order as that of the original stress perturbation. However, one important assumption was implicitly made in their analysis: There is no change in the equation of state of the background cosmic matter (which will be abbreviated as B-EOS from now on) throughout the whole stage. The main purpose of the present paper is to estimate the effect of the small change in B-EOS, which is generally associated with transient phenomena.

As the main tool of the analysis we utilize the gauge invariant formulation for the perturbations in the spatially homogeneous and isotropic universe developed by Bardeen recently. In §2 this formulation is reviewed briefly. Then first in §3 the lowest-order effect of the change of B-EOS is examined, and next in §4 the higher-order effect is estimated. Section 5 is devoted to discussion. Throughout this paper the units $c = 8\pi G$ = 1 are used.

§ 2. Formulation

In this section we briefly review Bardeen's gauge invariant treatment of the perturbed Einstein equation in a spatially homogeneous and isotropic background. The background spacetime is described by some version of the Robertson-Walker metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + \gamma_{ij}dx^i dx^j),$$

where the tensor $\gamma_{ij}$ is the metric tensor for a three-space of uniform spatial curvature $K$ which is independent of time. We represent the covariant derivative of a three-tensor with respect to $\gamma_{ij}$ by a slash and the covariant derivative of a four-tensor with respect to $g_{\mu\nu}$ by a semicolon. The unperturbed energy-momentum tensor must be formally that of a perfect fluid at rest relative to the above coordinates. The only non-zero components are

$$T^0_0 = -\rho, \quad T^i_j = p\delta^i_j,$$

where $\rho$ and $p$ are the energy density and the pressure of the background cosmic matter. The time evolution of the background is governed by the equations

$$\frac{1}{a^2} \left( \frac{da}{d\eta} \right)^2 = \frac{1}{3} \rho a^2 - K,$$

$$\frac{d\rho}{d\eta} = -\frac{1}{a} \frac{da}{d\eta} (\rho + p).$$
In a spatially homogeneous and isotropic background the time dependence and the spatial dependence of perturbation in various quantities can be separated by the expansion in harmonic functions. In this paper we restrict ourselves to the scalar perturbations since the vector and the tensor perturbations decouple from the density perturbation. \(^{15}\)

The scalar perturbation is expanded by scalar harmonics which are solutions of the Helmholtz equation

\[
Q^i_{\cdot i} + k^2 Q = 0,
\]

where \(k\) represents the wave number of the perturbation with respect to the comoving background coordinates. We use the divergence-less vector

\[
Q_i = - k^{-1} Q_{\cdot i}
\]

and the traceless, symmetric second-rank tensor

\[
Q_{i\cdot k} = k^{-2} Q_{\cdot i\cdot k} + \frac{1}{3} \gamma_{i\cdot k} Q
\]

to expand perturbations of vector or tensor quantities associated with the scalar perturbations.

With these harmonics the perturbed metric \(\tilde{g}_{\mu\nu}\) is written as

\[
\begin{align*}
\tilde{g}_{00} &= - a^2 (1 + 2 AQ), \\
\tilde{g}_{0i} &= - a^2 B Q_i, \\
\tilde{g}_{ij} &= a^2 \left( [1 + 2 H_L Q] \gamma_{ij} + 2 H_T Q_{ij} \right).
\end{align*}
\]

The perturbed energy-momentum tensor \(\tilde{T}_{\mu\nu}\) is characterized by the time-like unit eigenvector \(\tilde{u}^\mu\), the proper energy density \(\tilde{\rho}\) and the stress tensor \(\tilde{T}^i_{\cdot j}\) with respect to \(\tilde{u}^\mu\):

\[
\begin{align*}
\tilde{u}^0 &= a^{-1} [1 - A Q], \\
\tilde{u}^i &= \tilde{u}^0 / \tilde{u}^0 = v Q^i, \\
\tilde{\rho} &= - \tilde{T}^0_{\cdot 0} = \rho [1 + \delta Q], \\
\tilde{T}^i_{\cdot j} &= p \left( [1 + \pi_L Q] \delta^i_{\cdot j} + \pi_T Q^i_{\cdot j} \right), \\
\tilde{T}^0_{\cdot j} &= (p + \rho) (v - B) Q_{ij}, \\
\tilde{T}^i_{\cdot 0} &= -(p + \rho) v Q^i.
\end{align*}
\]

Thus the scalar perturbation is described by the eight quantities \(A, B, H_L, H_T, v, \delta, \pi_L\) and \(\pi_T\).

Following Bardeen we construct quantities which are invariant under the infinitesimal coordinate transformation

\[
\bar{\eta} = \eta + T(\eta) Q, \quad \bar{x}^i = x^i + L(\eta) Q^i.
\]

Since this gauge transformation has two-dimensional freedom, we obtain six gauge invariant quantities. With the aid of the variable \(x\) defined by

\[
dx = k d\eta = k dt / a
\]

\[
(2.12)
\]
which is invariant under the constant rescaling of the cosmic scale factor $a$, these gauge invariant quantities are expressed as

$$
\Phi = H_L + \frac{1}{3} H_T - a \frac{dH_T}{dx} + aB , \tag{2.13}
$$

$$
\Psi = - \frac{d^2 H_T}{dx^2} - a \frac{dH_T}{dx} + A + \frac{dB}{dx} + aB , \tag{2.14}
$$

$$
V = v - \frac{dH_T}{dx} , \tag{2.15}
$$

$$
\Delta = \delta + 3(1+w)a(v-B) , \tag{2.16}
$$

$$
\Sigma = \pi_L - \frac{c_s^2}{w} \delta , \tag{2.17}
$$

$$
\Pi = \pi_T , \tag{2.17}'
$$

where

$$
w = \frac{b}{\rho} , \quad c_s^2 = \frac{dp}{d\rho} \tag{2.18}
$$

and

$$
a = \frac{d \ln a}{dx} = \frac{\dot{a}}{a} \quad \frac{k}{a} = \text{Proper wave length} \quad \text{Effective horizon size} . \tag{2.19}
$$

The Einstein equations can be written only by these gauge invariant quantities and are classified into two groups. The first group consists of algebraic relations between $\Phi$, $\Psi$, $\Delta$ and $\Pi$:

$$
2\left(1 - \frac{3K}{k^2}\right) \Phi = 3 \left( a^2 + \frac{K}{k^2} \right) \Delta , \tag{2.20}
$$

$$
\Phi + \Psi = -3w\left( a^2 + \frac{K}{k^2} \right) \Pi . \tag{2.21}
$$

The second group gives time evolution equations for $\Delta$ and $V$:

$$
\frac{dV}{dx} + aV = \left( -\frac{3}{2} a^2 + \frac{c_s^2}{1+w} \right) \Delta + \frac{w}{1+w}\left( \Sigma \left[ 3(1+w)a^2 + \frac{2}{3} \right] \Pi \right) , \tag{2.22}
$$

$$
\frac{d\Delta}{dx} = 3w a \Delta = -(1+w)V - 2wa \Pi . \tag{2.23}
$$

From Eqs. (2.3) and (2.9) the constant ratio $K/k^2$ is expressed as

$$
K/k^2 = (\mathcal{Q} - 1)a^2 , \tag{2.24}
$$

where $\mathcal{Q}$ is the ratio of the proper density $\rho$ to the critical density $\rho_{cr} = 3((da/dt)/a)^2$. Equation (2.24) means that $K/k^2 < 1$ for perturbations with sizes which are sufficiently smaller than the horizon size of the present universe since $\mathcal{Q} = O(1)$ at present from observation. Hence we put $K/k^2 = 0$ from now on. Then if $w$ and $c_s^2$ are smooth functions of $x$, the Einstein equations (2.20)~(2.23) are reduced to the following second-
order ordinary differential equation for the gauge invariant density contrast $\Delta$:

$$
\frac{d^2 \Delta}{dx^2} + (1 + 3c_s^2 - 6w)a^2 \frac{d\Delta}{dx} + a^2 \left\{ \frac{9}{2}w^2 - 12w - \frac{3}{2} + 9c_s^2 + \frac{3w}{a^2} \right\} \Delta \\
= -w\Sigma + 2a^2 \left\{ 3w^2 + 3c_s^2 - 2w + \frac{w}{3a^2} \right\} \Pi - 2wa^2 \frac{d\Pi}{dx}.
$$

(2.25)

In order to analyze Eq. (2.25), it is more convenient to use the variable $\xi$ which is proportional to the scale factor $a$ as the time variable instead of $x$:

$$
\frac{d\xi}{dx} = a \xi.
$$

(2.26)

We normalize $\xi$ by the condition that $\xi = 1$ when $a = 1$, namely when the perturbations come within the horizon. With $\xi$, Eq. (2.25) is rewritten as

$$
\frac{d^2 \Delta}{d\xi^2} - \frac{\mu}{\xi} \frac{d\Delta}{d\xi} + \left( \frac{2 + \nu + c_s^2}{\xi^2} \right) \Delta \\
= -\frac{w}{f^3} \left( \Sigma - \frac{2}{3} \Pi \right) + \frac{2}{\xi^2} (3w^2 + 3c_s^2 - 2w) \Pi - \frac{2w}{\xi} \frac{d\Pi}{d\xi} \\
= S,
$$

(2.27)

where

$$
\mu = \frac{5}{2}(1 - 3w) + 1 - 3c_s^2,
$$

(2.28)

$$
\nu = \frac{1}{2}(1 - 3w)(7 - 3w) + 3(1 - 3c_s^2)
$$

(2.29)

and

$$
f = a\xi.
$$

(2.30)

From Eqs. (2.3), (2.4), (2.19) and (2.26), $f$ is expressed by $w$ as

$$
\frac{f}{f_*} = \exp \left[ \frac{1}{2} \int_{f_*} f \left( 1 - 3w \right) \frac{d\xi}{\xi} \right],
$$

(2.31)

where $f_*=f(\xi_*)$ and $\xi_*$ is some reference time.

For the adiabatic perturbations ($\Sigma = \Pi = 0$), Eq. (2.27) yields a homogeneous evolution equation for $\Delta$. Especially in the stage when the proper wave length of the perturbation is much larger than the effective horizon size ($a \gg 1$), $c_s^2/f^2$ can be neglected and Eq. (2.27) is written as

$$
\frac{d^2 \Delta}{d\xi^2} - \frac{\mu}{\xi} \frac{d\Delta}{d\xi} - \frac{2 + \nu}{\xi^2} \Delta = 0.
$$

(2.32)

If $c_s^2$ and $w$ are independent of time, Eq. (2.32) has two solutions expressed as $\xi^{1+}$ and $\xi^{1-}$ where
Generation and Evolution of the Density Fluctuation

Fig. 1. Diagram showing the signs of $A_+$ and $A_-$ in the $X \cdot Y$ plane where $X = 1 - 3w$ and $Y = 1 - 3c_s^2$. The part shaded by vertical lines depicts the regions where $D < 0$, i.e., $A_+$ and $A_-$ have imaginary parts, and the part shaded by dots the region where $A_+ > 0$ and $A_- < 0$. In the lower-left blank part which is bounded by the boundaries of the dotted region and the vertically shaded region, $A_+ > 0$ and $A_- > 0$. In the rest blank region $A_+ < 0$ and $A_- < 0$. Finally, the dot and dash line in the $D < 0$ region shows the line on which the real part of $A_+$, $\text{Re} A_+$, vanishes and above this line $\text{Re} A_+ > 0$. The dashed line shows the line on which $w = c_s^2$.

\[ \lambda_\pm = \frac{1}{2} (1 + \mu \pm D^{1/2}); \quad D = 9 + 2\mu + \mu^2 + 4\nu. \]  
(2.23)

Especially for the case

\[ 1 - 3c_s^2 > \frac{1}{6} (1 - 3w)^2 + 1 - 3w - \frac{2}{3}, \]  
(2.34)

$\xi^{A_+}$ yields a pure growing mode and $\xi^{A_-}$ a pure decaying mode. The signs of $\lambda_\pm$ for the general case are shown in Fig. 1. In most period of the cosmic evolution in which the matter is effectively described by a single component structureless fluid, Eq. (2.27) is reduced to a simple homogeneous equation for $A$ since $\Sigma$ and $\Pi$ are proportional to $A$ even if they do not vanish. In contrast, in transient stages such as the stages when massive particles decouple from radiation and become non-relativistic and the stages when various phase transitions occur, the matter behaves as multi-component fluid or acquires inhomogeneous structures. In these stages, $\Sigma$ and/or $\Pi$ can take non-zero value independent of $A$, and act as source for the density fluctuations through Eq. (2.27). Hence new density fluctuations can be generated through these transient phenomena. In the following sections, we investigate this possibility.

§ 3. Lowest-order effect

Let us consider the following situation. First, until some time $t_1$ the B-EOS is described by a simple relation $w = c_s^2 = \text{constant} (= \gamma)$ and there exists no perturbation at all ($\Delta = \Sigma = \Pi = 0$). Then at some time after $t_1$ stress perturbations are provoked by some mechanism, which at the same time makes $w$ and/or $c_s^2$ deviate from $\gamma$. The stress perturbations and the deviation of $w$ and $c_s^2$ continue to exist for a finite time and then vanishes before a time $t_2$. After $t_2$ the B-EOS is again given by $w = c_s^2 = \gamma$. In such a situation some density fluctuations are expected to be generated during the transient stage $t_1 < t < t_2$ as was noted in §2. For the case in which the strength of the stress perturbations and the deviation of the B-EOS are small ($|\Sigma| \ll 1$, $|\Pi| \ll 1$, $|w - \gamma| \ll 1$ and $|c_s^2 - \gamma| \ll 1$), the amplitude of the generated density fluctuations can be estimated by an iterative method.

For that purpose we rewrite Eq. (2.27) as

\[ \frac{d^2 \Delta}{d\xi^2} + \frac{\mu_0}{\xi} \frac{d\Delta}{d\xi} + \frac{2 + \nu_0}{\xi^2} \Delta = S \frac{d\mu}{d\xi} + \frac{\delta \nu - c_s^2 \alpha^2}{\xi^2} \Delta, \]  
(3.1)
where
\[ \mu = \mu_0 + \delta \mu; \quad \mu_0 = -\frac{3}{2}(1-3\gamma), \]
\[ \nu = \nu_0 + \delta \nu; \quad \nu_0 = -\frac{1}{2}(1-3\gamma)^2. \]  
(3.2)

Equation (3.1) can be transformed into an integral equation with the aid of the Green function for the differential operator on the left-hand side of Eq. (3.1). The Green function \( G(\xi, \xi') \) is given by
\[ G(\xi, \xi') = D(\xi')^{-1}[U_+(\xi)U_-(\xi') - U_-(\xi)U_+(\xi')]\theta(\xi - \xi'), \]  
(3.3)
where \( \theta(\xi) \) is the Heaviside function and
\[ U_\pm(\xi) = \xi^{\pm\lambda}; \quad \lambda_+ = 1+3\gamma, \quad \lambda_- = -\frac{3}{2}(1-\gamma), \]  
(3.4)
\[ D(\xi) = U_-' \frac{dU_+}{d\xi} - U_+ \frac{dU_-}{d\xi} = \frac{3\gamma+5}{2} \xi^{-3(1-3\gamma)/2} \]  
(3.5)

Applying the Green function (3.3) to Eq. (3.1), we obtain for \( \xi > \xi_1 \)
\[ \Delta(\xi) = \Delta_0(\xi) + L * \Delta(\xi), \]  
(3.6)
\[ \Delta_0(\xi) = P_0(\xi)U_+(\xi) + Q_0(\xi)U_-(\xi), \]  
(3.7)
\[ P_0(\xi) = \int_{\xi_1}^{\xi} \frac{d\xi'}{D(\xi')} U_-(\xi')S(\xi'), \]  
(3.8)
\[ Q_0(\xi) = -\int_{\xi_1}^{\xi} \frac{d\xi'}{D(\xi')} U_+(\xi')S(\xi'), \]  
(3.9)
\[ L * \Delta(\xi) = U_+(\xi) \int_{\xi_1}^{\xi} \frac{d\xi'}{D(\xi')} U_-(\xi') \left( \frac{\delta \mu}{\xi'} \frac{dA}{d\xi'} + \frac{\delta \nu - c_s^2 a^{-2}}{\xi'^2} \right) \]  
\[ - U_-(\xi) \int_{\xi_1}^{\xi} \frac{d\xi'}{D(\xi')} U_+(\xi') \left( \frac{\delta \mu}{\xi'} \frac{dA}{d\xi'} + \frac{\delta \nu - c_s^2 a^{-2}}{\xi'^2} \right). \]  
(3.10)

Here and hereafter the indices 1 and 2 denote the values at \( t = t_1 \) and \( t = t_2 \), respectively. Since \( S \) vanishes after \( t = t_2 \), \( P_0(\xi) \) and \( Q_0(\xi) \) become constant for \( t > t_2 \). Hence in Eq. (3.7) the first term represents the pure growing mode and the second term the pure decaying mode for \( t > t_2 \).

Since we are interested in the generation of large scale density fluctuations in an early stage of the universe, \( \alpha \) can be considered extremely large in the concerned stage \( t_1 < t < t_2 \). Hence in the discussion of the generation of density fluctuations the terms proportional to \( c_s^2 \alpha^{-2} \) in Eq. (3.10) can be neglected. Then Eq. (3.10) shows that the second term on the right-hand side of Eq. (3.6) is of a higher order than the first term with respect to the deviation of the B-EOS. In this section we estimate the lowest-order term \( \Delta_0(\xi) \). The higher-order correction will be discussed in the next section.

What we are interested in is the amplitude of the density fluctuations at the time \( t_H \) when the fluctuations come within the horizon. Since the contributions of the isotropic stress perturbation and the anisotropic one to this amplitude are quite different, we discuss...
them separately. First, let us assume that there exists only the isotropic stress perturbation: 
\[ S = -\omega \Sigma / f^3. \]
Since \( \alpha = \xi = 1 \) at \( t = t_H \), \( f(t_H) = 1 \) from Eq. (2·30), hence from Eq. (2·31) \( f \) is expressed as
\[ f = \xi^{(1-3\gamma)/2} \exp \left[ \frac{3}{2} \int_{\xi}^{1} \delta w \frac{d\xi'}{\xi'} \right]. \] (3·11)

Hence \( P_0(\xi) \) and \( Q_0(\xi) \) are written as
\[ P_0(\xi) = \frac{2}{3\gamma + 5} \int_{\xi_1}^{\xi} \frac{d\xi'}{\xi'} \Sigma_1(\xi'), \] (3·12)
\[ Q_0(\xi) = -\frac{2}{3\gamma + 5} \int_{\xi_1}^{\xi} \frac{d\xi'}{\xi'} (\xi')^{3(1+\gamma)/2} \Sigma_1(\xi'), \] (3·13)
where
\[ \Sigma_1 = \xi^{(1-3\gamma)/2} \Sigma = -\omega \Sigma \exp \left[ -3 \int_{\xi}^{1} \delta w \frac{d\xi'}{\xi'} \right]. \] (3·14)

From these equations the following estimates are obtained:
\[ |P_0(\xi)| \leq \frac{2}{3\gamma + 5} \| \Sigma_1 \| \ln \left( \frac{\xi_2}{\xi_1} \right), \] (3·15)
\[ |Q_0(\xi)| \leq \left( \frac{2}{3\gamma + 5} \right)^2 \| \Sigma_1 \| (\xi \wedge \xi_2)^{(3\gamma + 5)/2}, \] (3·16)
where \( \| Z \| \) denotes the maximum value of \( |Z| \) during \( t_1 < t < t_2 \) and \( \xi \wedge \xi_2 \) represents the smaller one between \( \xi \) and \( \xi_2 \). Especially it follows that
\[ |\Delta_0(t_H)| \approx O(\| \Sigma \|). \] (3·17)

Equation (3·17) shows that the amplitude of a density fluctuation produced by an isotropic stress perturbation, when it comes within the horizon, is of the same order as the strength of the original stress perturbation if the deviation of the B-EOS is neglected. This confirms in a little more general way the conclusion obtained by Press and Vishniac through a delicate argument based on the synchronous gauge.\(^{14}\) Since \( \Sigma \) is extremely small in general, this result implies that the isotropic stress perturbation which is not associated with a large change of the B-EOS is not important in the discussion of the origin of the large scale structure in the present universe. For example, let us consider an isotropic stress perturbation given rise to by a statistical fluctuation of the distribution of some objects or components which have a B-EOS different from that of the rest of the cosmic matter. Let \( \rho_1 \) and \( c_{s1} \) be the energy density and the sound velocity of this component and \( \rho_2 \) and \( c_{s2} \) be those of the rest of the cosmic matter. Then if the average energy of each object is represented by \( E \), the statistical fluctuation of \( \rho_1 \) in a scale \( L \) is given by
\[ \delta \rho_1 \sim L^{-3} \times E(\rho_1 L^3 / E)^{1/2}. \] (3·18)
Since there exists essentially no fluctuation in the total energy density when the stress perturbation is provoked, \( \Sigma \) is expressed as
\[ \Sigma = (c_{s1}^2 - c_{s2}^2) \{(\rho_2 + p_2) \delta \rho_1 - (\rho_1 + p_1) \delta \rho_2 \} / \rho (\rho + p) \]
\[
\frac{c_{51}^2 - c_{52}^2}{w} \left( \frac{\rho_1 L^3}{E} \right)^{1/2} - \frac{\epsilon}{\rho L^3/E} = \left( \frac{\rho L^3/E}{\rho L^3} \right)^{-1/2},
\]
(3·19)

where \( \epsilon = \rho_1/\rho \). Hence in the standard hot Big-Bang universe model \( \Delta_0(t_H) \) is given by
\[
|\Delta_0(t_H)| \sim 3 \times 10^{-39} \epsilon^{1/2} \left( \frac{E}{T} \right)^{1/2} \left( \frac{M_5}{10^{12} M_6} \right)^{-1/2} \left( \frac{T_{BB}/2.7 K}{\Omega_0 h_0^3} \right)^{-1/2},
\]
(3·20)

where \( T \) is a typical temperature when the stress perturbation is working, \( M_5 \) is the baryon mass contained in the perturbation scale, \( T_{BB} \) is the present photon temperature of the universe, \( h_0 \) is the Hubble constant normalized by 100 km/s/Mpc, and \( \Omega_0 \) is the density parameter of the present universe. In general, \( E \) cannot be larger than the energy contained within the horizon scale, which is expressed in the radiation dominant stage as
\[
E_H \sim \left( \frac{T_{pl}}{T} \right)^2 T_{pl},
\]
(3·21)

where \( T_{pl} \) is the Planck temperature \( \sim 10^{19} \text{GeV} \). Hence from Eq. (3·20) it follows
\[
|\Delta_0(t_H)| \sim 3 \times 10^{-39} \epsilon^{1/2} \left( \frac{T}{T_{pl}} \right)^{-3/2} \left( \frac{M_5}{10^{12} M_6} \right)^{-1/2}.
\]
(3·22)

This shows that isotropic pressure perturbations associated with transition phenomena which occurred while \( T \geq 10 \text{keV} \) produce density fluctuations with amplitudes much smaller than \( O(10^{-3}) \), which is required to produce the present large scale structure of the universe, hence have no importance.

Next let us study the effect of anisotropic stress perturbations. The contribution of an anisotropic stress perturbation to \( \Delta_0 \) is obtained by putting \( S = 2\xi^{-2} \left[ 3(1 + \omega) c_s^2 - 2\omega \right] \Pi - 2w \xi^{-1} d\Pi/d\xi \) in Eqs. (3·10) and (3·11). Partial integrations yield
\[
P_0 = \int_{\xi_1}^{\xi_2} \frac{d\xi'}{D(\xi')} \Pi_U_1(\xi')(\xi')^{-2} 6\omega(2w - c_s^2 - \gamma) \Pi,
\]
(3·23)
\[
Q_0 = -\int_{\xi_1}^{\xi_2} \frac{d\xi'}{D(\xi')} \Pi_U_2(\xi')(\xi')^{-2} 6\omega(2w - c_s^2 - 1/2 \gamma + 5/6) \Pi.
\]
(3·24)

Note that the growing mode component of \( \Delta_0(\hat{\xi}) \), the first term, is of first order in the deviation of B-EOS, \( \delta w = w - \gamma \) and \( \delta c_s^2 = c_s^2 - r \), in contrast to the isotropic perturbation case. Replacing the terms containing \( w, c_s^2 \) and \( \Pi \) by their average values and performing the integration, we obtain the following estimate of \( \Delta_0(\hat{\xi}) \):
\[
\Delta_0(\hat{\xi}) \approx \frac{12\gamma}{(3\gamma + 1)(3\gamma + 5)} \left( 2w - \gamma - c_s^2 \right) \Pi \frac{U_1(\hat{\xi})}{U_1(\xi_1)} - \frac{4\gamma}{3(1 - \gamma)} \Pi \frac{U_2(\hat{\xi})}{U_2(\xi_2)},
\]
(3·25)

where \( \langle Q \rangle \) denotes the average of \( Q \) during \( t_1 \) and \( t_2 \). Equation (3·25) apparently shows that the anisotropic perturbations produce growing density fluctuations with amplitudes \( \sim \delta w \Pi \) just after the perturbations vanish, hence is much more effective than the isotropic stress perturbation in generating density fluctuations. Unfortunately, however, this conclusion is not correct since the correction term in Eq. (1·8), \( L \ast \Delta(\hat{\xi}) \) is also of first order with respect to the deviation of the B-EOS. In order to get the correct conclusion we must add the contribution from \( L \ast \Delta(\hat{\xi}) \) to \( \Delta_0 \). Surprisingly enough, as
will be shown in the next section, these two contributions cancel exactly at least up to the first order in the deviation of the B-EOS. Therefore the amplitudes of density fluctuations generated from the anisotropic stress perturbation is of the same order as the isotropic one in this order, hence has no importance.

§ 4. Higher-order effect

In order to estimate the higher-order effect with respect to the deviation of B-EOS, we must solve the integral equation (3.6) iteratively. For that purpose we first perform partial integration and eliminate \( \frac{dL_l}{d\xi} \) in Eq. (3.10). Then it follows that

\[
L \ast \Delta(\xi) = U_+(\xi) \int_{\xi_1}^\xi d\xi' \left( \xi_1 - \frac{2c_s^2}{3\gamma + 5} \alpha^{-2} \xi' \right)^{(2+3\gamma)} \Delta(\xi') d\xi' 
+ U_-(\xi) \int_{\xi_1}^\xi d\xi' \left( \xi_2 + \frac{2c_s^2}{3\gamma + 5} \alpha^{-2} \xi' \right)^{(1-3\gamma)} \Delta(\xi') d\xi', \tag{4.1}
\]

where

\[
\begin{align*}
\xi_1 &= \frac{4}{3\gamma + 5} \left\{ (1+3\gamma) \delta \mu + \delta \nu + \frac{45}{2} (1+w)(c_s^2-w) - 9x(1+w) \right\}, \tag{4.2} \\
\xi_2 &= \frac{3}{3\gamma + 5} \left\{ \frac{2}{3} (1-\gamma) \delta \mu - \delta \nu - \frac{45}{2} (1+w)(c_s^2-w) + 9x(1+w) \right\}, \tag{4.3}
\end{align*}
\]

in which

\[
x = \mu \frac{d^2 p}{dp^2}. \tag{4.4}
\]

Note that \( \xi_1 \) and \( \xi_2 \) vanish outside the period \( t_1 < t < t_2 \) and their moduli are of the same order as \( \delta w \) and \( \delta c_s^2 \).

Now the solution of Eq. (3.6) is expressed as the formal series

\[
\Delta(\xi) = \sum_{n=0}^\infty \Delta_n(\xi), \quad \Delta_{n+1}(\xi) = L \ast \Delta_n(\xi). \tag{4.5}
\]

As before let us express \( \Delta_n(\xi) \) as

\[
\Delta_n(\xi) = G_n(\xi) U_+(\xi) + D_n(\xi) U_-(\xi). \tag{4.6}
\]

Then \( G_n \) and \( D_n \) satisfy the recurrence formulae

\[
\begin{align*}
G_{n+1}(\xi) &= \int_{\xi_1}^\xi d\xi' \left( \xi_1 - \frac{2c_s^2}{3\gamma + 5} \alpha^{-2} \xi' \right)^{-1} \Delta_n(\xi') \nonumber \\
&\times \left[ G_n(\xi') + (\xi')^{-1} (5+3\gamma)/2 D_n(\xi') \right], \tag{4.7} \\
D_{n+1}(\xi) &= \int_{\xi_1}^\xi d\xi' \left( \xi_2 + \frac{2c_s^2}{3\gamma + 5} \alpha^{-2} \xi' \right)^{3(1+\gamma)/2} \Delta_n(\xi') \\
&\times \left[ G_n(\xi') + (\xi')^{-1} (5+3\gamma)/2 D_n(\xi') \right]. \tag{4.8}
\end{align*}
\]

From Eqs. (3.15), (3.16), (3.23) and (3.24), \( G_0(\xi) \) and \( D_0(\xi) \) satisfy the inequalities

\[
|G_0(\xi)| \leq G_0^*, \quad |D_0(\xi)| \leq D_0^*(\xi \wedge \xi_2)^{(3\gamma+5)/2}, \tag{4.9}
\]
where

$$G_0^* = \frac{2}{3\gamma + 5} \left( \ln \frac{\xi_2}{\xi_1} \right) \Sigma_1 + \frac{2}{(3\gamma + 5)(3\gamma + 1)} \xi_1^{-1-3\gamma} \left\| 6w(2w - c_s^2 - \gamma) \Pi \right\|,$$

$$D_0^* = \left( \frac{2}{3\gamma + 5} \right)^3 \left\| \Sigma_1 + \frac{24}{3(3\gamma + 5)(1 - \gamma)} \xi_2^{-1-3\gamma} \right\| w \left( 2w - c_s^2 - \frac{1}{2} \gamma + \frac{5}{6} \right) \Pi \right\|.$$  

Hence noting that from Eq. (3·11) $\alpha$ is expressed as

$$\alpha^{-1} = \frac{\xi}{f} = \xi^{1+3\gamma/2} \quad \text{for } \xi > \xi_2,$$

one obtains the following estimate for $G_1(\xi)$ and $D_1(\xi)$:

$$|G_1(\xi)| \leq (G_0^* + D_0^* p_0^2/\rho_0^2) p_0,$$

$$|D_1(\xi)| \leq (G_0^* + D_0^*) \rho_0 (\xi \wedge \xi_2)^{(5+3\gamma)/2},$$

where

$$\rho_0 = \left\| \xi_1 - \frac{2c_s^2}{3\gamma + 5} \alpha^{-2} \right\| \ln \frac{\xi_2}{\xi_1} + \frac{2\gamma}{(3\gamma + 5)(3\gamma + 1)} \xi_1^{1+3\gamma},$$

$$\rho_0^2 = \frac{2}{3\gamma + 5} \left\| \xi_2 + \frac{2c_s^2}{3\gamma + 5} \alpha^{-2} \right\| + \frac{4\gamma}{(3\gamma + 5)(9\gamma + 7)} \xi_2^{1+3\gamma},$$

and $\rho_{02} = \rho_0(\xi_2)$. Repeating this procedure one obtains

$$|G_n(\xi)| \leq (G_0^* + D_0^* p_0^2/\rho_0^2) p_0 (\xi \wedge \xi_2)^{n-1},$$

$$|D_n(\xi)| \leq (G_0^* + D_0^*) \rho_0 (\xi \wedge \xi_2)^{(5+3\gamma)/2}$$

for $n \geq 1$.

Equations (4·17) and (4·18) show that the formal series (4·5) converges absolutely if $\| \epsilon_1 \| \ln (\xi_2/\xi_1) < 1/2, \| \epsilon_3 \| < 1, 0 \leq \gamma \leq 1/3$ and $\xi \leq 1$. In this case the higher-order terms are estimated as

$$\left| \sum_{n=1}^{\infty} A_n(\xi) \right| \leq \frac{1}{1 - (\rho_0 + \rho_0^2)} (\rho_0 G_0^* + \rho_0^2 D_0^*) U_1(\xi) \times \left[ 1 + \frac{\rho_0^2 (G_0^* + D_0^*) (\xi \wedge \xi_2)^{3+5\gamma/2}}{\rho_0 G_0^* + \rho_0^2 D_0^*} \right].$$

Since $\rho_0$ and $\rho_0^2$ are of the same order and $\xi_2 \ll 1$ in general, the second term in the square bracket can be neglected. Since $|A_0(\xi)| \sim G_0^* U_1(\xi)$ for $\xi \sim \xi_{H} \sim 1$ and $\xi_2 \ll 1$, Eq. (4·19) shows that the higher-order terms are at most of the same order as $A_0(\xi)$. Hence the higher-order contributions do not play an essential role in general. Furthermore, since the order-zero terms in $\rho_0$ and $\rho_0^2$, namely the second terms on the right sides of Eqs. (4·15) and (4·16), come from the term $c_s^2 \alpha^{-2} \xi^{-2} A$ on the right side of Eq. (3·1), the correction $\sum_{n=1}^{\infty} A_n(\xi)$ is made genuinely higher order than $A_0(\xi)$ with respect to the deviation of B-EOS if one uses the Bessel functions

$$\xi^{(3+5\gamma)/2} J_\nu(2\gamma^{1/2}(1+3\gamma)^{-1} \xi^{1+3\gamma/2})$$

and
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\[ \xi^{(\gamma-1)/4} N_\nu (2\gamma^{1/2}(1+3\gamma)^{-1} \xi^{(1+3\gamma)/2}) \]

instead of \( \xi^{+\gamma} \) and \( \xi^{-\gamma} \) as \( U_+ (\xi) \) and \( U_- (\xi) \), where \( \nu = (5+3\gamma)/2(1+3\gamma) \).

Finally we show that \( G_0 \) and \( G_1 \), which are the only terms of first order with respect to the deviation of B-EOS, exactly cancel and the growing density fluctuations are not generated in this order by the anisotropic stress perturbation. For the above choice of \( U_+ \) and \( U_- \), \( G_0 \) and \( D_0 \) are expressed as

\[ G_0 (\xi) = \int_0^\xi d\xi' \frac{U_- (\xi')}{(\xi')^2 D} \frac{2\nu}{3(2w-c_s^2-\gamma)+\beta_- - \lambda_-} II, \tag{4.20} \]

\[ D_0 (\xi) = -\int_0^\xi d\xi' \frac{U_+ (\xi')}{(\xi')^2 D} \frac{2\nu}{3(2w-c_s^2-\gamma)+\beta_+ - \lambda_-} II, \tag{4.21} \]

where

\[ D = U_- \frac{dU_+}{d\xi} - U_+ \frac{dU_-}{d\xi} = \frac{3\gamma + 5}{2} \xi^{-3(1-3\gamma)/2}, \tag{4.22} \]

\[ \beta_\pm = \frac{\xi}{U_\pm} \left( \frac{dU_\pm}{d\xi} \right), \tag{4.23} \]

and the recurrence formulae for \( G_n \) and \( D_n \) are given by

\[ G_{n+1} = \int_0^\xi d\xi' \frac{U_- (\xi')}{(\xi')^2 D} (G_n U_+ + D_n U_-) \times \left[ \delta \nu - \delta (c_s^2 a^2) - \frac{1}{2} \left( -\frac{9\gamma}{2} + \beta_- \right) \delta \mu - \xi \frac{d\delta \mu}{d\xi} \right], \tag{4.24} \]

\[ D_{n+1} = \int_0^\xi d\xi' \frac{U_+ (\xi')}{(\xi')^2 D} (G_n U_+ + D_n U_-) \times \left[ \delta \nu - \delta (c_s^2 a^2) - \frac{1}{2} \left( -\frac{9\gamma}{2} + \beta_+ \right) \delta \mu - \xi \frac{d\delta \mu}{d\xi} \right]. \tag{4.25} \]

Especially the dominant part of \( G_1 \) with respect to the deviation of B-EOS is expressed as

\[ G_1 \approx -\int_0^\xi d\xi' \frac{U_+ (\xi')}{(\xi')^2 D} 2\nu [\beta_- - \beta_-] II \int_0^\xi d\xi'' \frac{\epsilon_3}{(\xi'')^2 D} U_-^2. \tag{4.26} \]

where

\[ \epsilon_3 = -(3+9\gamma)(w-r) + \left( \frac{21}{2} + \frac{27\gamma}{2} \right)(c_s^2 - \gamma) \]

\[ + 3\xi \frac{dc_s^2}{d\xi} \delta (c_s^2 a^2) + \text{higher order terms}. \tag{4.27} \]

From Eqs. (4.20) and (4.26) it follows that the ratio

\[ \frac{6w - 3c_s^2 - 3\gamma + \beta_- - \lambda_- U_-}{\beta_+ - \beta_-} : \int_0^\xi d\xi' \frac{\epsilon_3}{(\xi')^2 D} U_-^2 \tag{4.28} \]

should be unity for \( \xi_1 < \xi < \xi_2 \), in order that \( G_0 \) and \( G_1 \) cancel up to the first order in the deviation of B-EOS for an arbitrary \( II \). If one writes the first term of Eq. (4.28) as \( X \), then this condition is equivalent to the condition that the ratio
should be unity. The explicit calculation yields

\[
\frac{dX}{d\xi} : -\frac{\varepsilon_3}{\xi^2 D} U^{-2}
\]

which coincides with $-\varepsilon_3$ given in Eq. (4.27) except for the extremely small terms of order $O(\alpha^{-2})$. Therefore the first order parts of $G_0$ and $G_1$ with respect to the deviation of B-EOS cancels exactly.

§ 5. Conclusion and discussion

In this paper we have extended the analysis on the generation of density fluctuations from stress perturbations by Press-Vishniac to the case that the change in the equation of state of the background cosmic matter coexists with stress perturbations. We have found that the generated density fluctuations have, when they enter within the particle horizon, amplitudes of the same order as those of the seed stress perturbations, irrespective of the type of stress perturbations, which are negligibly small in realistic situations to explain the presently observed large scale structure of the universe.

As for the anisotropic stress perturbation we have considered in this paper up to the first order with respect to the deviation of the B-EOS. However, the same conclusion seems to hold even if the consideration is extended to the full order because the explicit calculation in the case that the cosmic matter is described by the mixture of pure radiation and pressure-free particles while the stress perturbations are provoked shows that the amplitudes of density fluctuations generated from anisotropic stress perturbations are exactly of the same order as those from isotropic ones.

Another point to be mentioned is the assumption that the change in the B-EOS is small. Especially the expression (4.19) apparently suggests the possibility that the stress perturbation may generate density fluctuations of large amplitudes if the B-EOS changes largely while it is provoked, since then $\rho_n$ and $\rho_0$ become larger than unity and the iteration argument in § 4 breaks down. However, the explicit calculation in the special case that the universe is described by a mixture of radiation and pressure-free particles and undergoes smooth change from radiation dominance to matter dominance while the stress perturbations are provoked shows that the same conclusion as is the weak change can hold. Of course it is not known whether it is the case in general.

Thus we are in a quite discouraging situation about the generation of density fluctuations as the seed for the large scale structure of the present universe in the course of the cosmic evolution. The remaining possibility is to appeal to the existence of some exotic transition phenomena in the early universe, such as the amplification and the enlargement of quantum fluctuations in the inflationary universe, and the bubble and string formation by the GUT phase transitions, unless we attribute the present universe structure to the initial condition or consider a rather inhomogeneous initial state of the universe and its subsequent smoothing by some unknown mechanism.
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References

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