Finite-Size Scaling for Transient Similarity and Fractals

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A "finite-size scaling law" is formulated for transient fractals. Transient Koch curve, the fractal structure of rias coastlines in Japan and transient fractals in chaos are discussed from the present general viewpoint of finite-size scaling.

Mandelbrot introduced the concept of fractals on the basis of the similarity of geometrical objects. As was discussed in previous papers, a formal description of fractals is given in the following. We denote an infinite set of figures by \( \{F_n\} \). We transform the initial figure \( F_0 \) to \( F_1 \) by a transformation \( F \), and so on. That is, we have

\[
F F_0 = F_1, \quad F F_1 = F_2, \ldots, \quad F F_n = F_{n+1}, \ldots. \quad (1)
\]

The fractal figure is defined by the limiting figure \( F^* \), namely, the following fixed point:

\[
\lim_{n \to \infty} F_n = F^*. \quad (2)
\]

The resolution of measurement of length for \( F_n \) is denoted by \( b_n \). Then Mandelbrot defined the fractal dimensionality \( D_{\text{frac}} \) of this limiting geometrical object by

\[
D_{\text{frac}} = \lim_{n \to \infty} \log V(F_n)/\log(1/b_n), \quad (3)
\]

where \( V(F_n) \) or \( L(F_n) \) denotes the volume (surface or length) of \( F_n \) in units of \( b_n \). This definition is based on the infinitely small-scale similarity of the system.

There are, however, many other interesting systems in which the similarity is valid only in some range of the scale factor \( b_n \), namely, we have "transient similarity". These systems may be called "transient fractals".

As was already used by many authors, a convenient definition of the transient fractal dimensionality is given by

\[
D_n = \log \frac{V(F_{n+1})}{V(F_n)} / \log(1/r_n), \quad (4)
\]

or equivalently the derivative of \( \log V(F_n) \) with respect to \( \log r_n \), where the relative scale factor \( r_n \) is defined by

\[
r_n = b_n/b_{n-1}, \quad (5)
\]

and consequently we have

\[
b_n = r_1 r_2 \cdots r_n \quad (6)
\]

with \( b_0 = 1 \). These transient fractal dimensionalities \( \{D_n\} \) are functions of scales \( \{b_n\} \) and a certain characteristic scale factor \( b^* \) as

\[
D_n = D(b_n, b^*). \quad (7)
\]

There are some possibilities:

a) \( D_n \) is independent of \( b_n \) and \( b^* \). This is an idealistic fractal dimensionality.

b) There occurs a crossover effect of fractals around the scale factor \( b_n = b^* \), as was already discussed by Mandelbrot and many other people.

c) There are several different ranges in which \( \{D_n\} \) are approximately constant. Namely, there exist several characteristic scales \( b_1^*, b_2^*, \ldots \).

d) \( D_n \) is such a continuous function of \( b_n \) as changes gradually.

e) More complicated structures might appear in nature.

In the present paper, we discuss mainly case b) in which there exists a unique characteristic scale factor \( b^* \).

The main purpose of the present paper is to give a speculation on the finite-size scaling of transient fractals in the above case b). For this purpose, it should be noted that there exists a close similarity between transient fractals and finite-size critical phenomena. The finite-size scaling law of critical phenomena claims that the order parameter \( Q(\epsilon, h, L) \) takes the following scaling form:

\[
Q(\epsilon, h, L) = L^{\delta} \phi(h \epsilon^{-\delta}, \epsilon L^{1/\nu}), \quad (8)
\]

where \( \epsilon = (T - T_c)/T_c \) with the critical temperature \( T_c \), \( h \) denotes a symmetry breaking field, and \( L \) is the system size. The scaling exponents \( \beta, \delta \) and \( \nu \) are ordinary critical exponents. In critical phenomena, the system size \( L \) is a
relevant parameter in the finite-size scaling law, while the lattice spacing \(a_0\) is fixed. However, one may change the lattice spacing \(a_0\) for \(L\) fixed. Then, the effective relative size \(L_{\text{eff}}\) defined by

\[
L_{\text{eff}} = \frac{L}{a_0}
\]

should be used in (8) instead of \(L\). If we let \(a_0\) go to zero, then \(L_{\text{eff}}\) goes to infinity. It is clear that the lattice spacing \(a_0\) in critical phenomena corresponds to the scale \(b_n\) of measurement in fractals.

From this correspondence or analogy, we may speculate the following finite-size scaling law of transient fractals. We consider many systems with different values of \(b^*\). We assume that these fractal systems (or fractals) are constructed by the same mechanism but with different characteristic scales \(b^*\). Then, it is now expected that a hyper-similarity of the whole set of these fractal systems with different values of \(b^*\) should hold in the sense that transient fractals have the following scaling form:

\[
V(t_n) = b_n^{-\phi} f^{(\text{sec})}(b^*/b_n);
\]

or equivalently

\[
\log V(t_n) = D \log (1/b_n) + \log f^{(\text{sec})}(b^*/b_n). 
\]

Consequently the transient fractal dimensionality \(D_{\text{tran}}\) is, through (4), given by

\[
D_{\text{tran}} = D + \log \frac{f^{(\text{sec})}(b^*/b_{n+1})}{f^{(\text{sec})}(b^*/b_n)} / \log \tau_{n+1}
\]

In the limit \(b_{n+1} \to b_n\) (i.e., \(\tau_{n+1} \to 1\)), we have

\[
D_{\text{tran}} = D + \phi^{(\text{sec})}(b^*/b_n) \equiv D^{(\text{sec})}(b^*/b_n),
\]

where

\[
\phi^{(\text{sec})}(x) = \partial \log f^{(\text{sec})}(x) / \partial \log x.
\]

Thus, the transient fractal dimensionality \(D_{\text{tran}}\) itself takes the scaling form (13). This scaling law for the fractal dimensionality can also be confirmed to hold even for a general finite ratio \(b_n/b_{n-1} = \tau_n\), when the scaling form (10) is valid.

The property \(\phi^{(\text{sec})}(0) = 0\) is easily found from \(f^{(\text{sec})}(0) = 0\) in (10) and from the analyticity of \(f^{(\text{sec})}(x)\) with respect to \(x\) and consequently we obtain \(D^{(\text{sec})}(0) = D\). On the other hand, it is expected that \(f^{(\text{sec})}(b^*/b_n)\) behaves like

\[
f^{(\text{sec})}(b^*/b_n) \approx (b^*/b_n)^\phi
\]

in the limit \(b_n \ll b^*\) with some appropriate exponent \(\phi\). Then, we have

\[
D_{\text{tran}} = D + \phi = D'
\]

for \(b_n \ll b^*\). Here, \(D\) and \(D'\) denote fractal or original dimensionalities of the relevant figures embedded into the \(d\)-dimensional real space. Thus, \(D\) or \(D'\) may or may not be equal to \(d\) (or \(d-1, d-2, \ldots\)).

These situations are shown schematically in Fig. 1.

In the following, we discuss some examples.

(i) Transient Koch curve

It is quite easy to construct mathematical (or artificial) systems which show the above "finite-size scaling property" of transient fractals. For example, the Koch curve \(^{1,2}\) may be extended as in Fig. 2. That is, the relative scale factor \(\tau_{n+1} = b_{n+1}/b_n\) of this extended Koch curve is assigned artificially as

\[
\tau_{n+1} = R(e/b_n) \quad \text{and} \quad b_n = r_1 r_2 \cdots r_n
\]

with some monotonically increasing function

\[
\frac{b_n}{b_{n-1}} \approx R(e/b_n)
\]

In the limit \(b_{n+1} \to b_n\) (i.e., \(\tau_{n+1} \to 1\)), we have

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\[
\frac{b_n}{b_{n-1}} \approx R(e/b_n)
\]
$R(x)$. Here, $\varepsilon$ denotes a certain smallness parameter (which corresponds to $b^*$). The fractal dimensionality of this transient Koch curve in Fig. 2 is given by
\[
D_n = \log \left( \frac{L_{n+1}}{L_n} \right) / \log \left( \frac{1}{b_{n+1}} \right)
\]

Thus, the above transient Koch curve satisfies exactly the finite-size scaling law (13).

Clearly, we have
\[
D = \log 4 / \log R(0) \quad \text{and} \quad D' = \log 4 / \log R(\infty).
\]

That is, the effective dimensionality $D_n$ changes from $D$ to $D'$ as $b_n$ decreases, and the crossover occurs around
\[
b_n \approx \varepsilon, \quad \text{namely,} \quad n \approx \log(1/\varepsilon).
\]

A specific form of the function $R(x)$ may be, for example, given by
\[
R(x) = 4 \left[ 1 + \frac{1}{3(x+1)} \right]^{-1},
\]

namely, $R(0) = 3$ and $R(\infty) = 4$. Then, the corresponding transient Koch curve has the fractal dimensions
\[
D = \log 4 / \log 3 = 1.26 \ldots
\]

for $n < \log(1/\varepsilon)$ and it has the ordinary dimension $d = 1$ for $n \gg \log(1/\varepsilon)$.

(ii) Rias coastlines in Japan

Quite recently T. Nakano\(^7\) studied the fractal structure of some rias in Japan and he found that there exist two different regimes of scale in which two fractal dimensions $D$ and $D'$ are defined for larger and smaller scales, respectively. Here $D \approx 1.35 \sim 1.40$ and $D' \approx 1.12 \sim 1.27$. It will be of great interest to study this transient fractals from our viewpoint. In fact, it is found from Nakano's data that the scaling law
\[
L(\Sigma_n) = b_n^{-D} f^{(se)}((\varepsilon - \varepsilon)\varepsilon / b_n)
\]

holds qualitatively, where $b^*$ is interpreted\(^7\) to depend on the geologic age and rock types. That is, Nakano speculated that $b^*$ is a function of time $t$ (or age), that $b^* = b^*(t)$ increases as $t$ increases, and consequently that $D_{trans}$ becomes finally $D' \approx 1.12 \sim 1.27$ in the whole region of scale.

It is expected in the future that the fractal structure of rias coastlines is studied in more detail from the present general viewpoint of finite-size scaling.

(iv) Transient fractals in chaos

Fractal structures are closely related to chaos, as was discussed by Mandelbrot\(^1\) and by Yamaguti and Hata\(^10\). For example, curves described by Weierstrass' functions have fractal dimensions\(^1\) and they are generating functions of exactly soluble chaos\(^10\) such as the Ulam-von Neumann model\(^11\)–\(^13\).

Quite recently Kaneko\(^14\) found numerically the fractalization of torus at the critical point $\varepsilon = \varepsilon_c$. From our general viewpoint of transient fractals, it is expected that fractals observed numerically are always transient, because it is impossible to study the fractal nature of the torus system numerically just at the critical point and because the torus system is not fractal for $\varepsilon \neq \varepsilon_c$. Therefore, it is the most reliable method to confirm the finite-size scaling law
\[
L(\Sigma_n) = b_n^{-D} f^{(se)}((\varepsilon - \varepsilon)\varepsilon / b_n)
\]
in order to conclude that the torus system is fractal just at the critical point $\varepsilon = \varepsilon_c$. It may also be possible to determine numerically the critical point $\varepsilon_c$. Critical exponent $\phi$ and scaling function $f^{(se)}(x)$, from which the fractal dimensionality $D'$ of the torus system at the critical point $\varepsilon = \varepsilon_c$ is determined by extrapolating the scaling function $f^{(se)}(x)$.

Sawada et al.\(^4\) have also found such an example of the random patterns produced by computer to simulate electric breakdown in an insulator, as has two regions of different fractal dimensions. That is, they have observed the existence of the crossover scale length $b^*$.

It will be interesting to study in the future whether the above finite-size scaling law of transient fractals is valid in other systems or not.

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\begin{enumerate}
\item M. Suzuki, "Applications of Fractal Analysis to Phase Transitions and Other Phenomena" in