He proceeds to write down as necessary conditions for a minimum the equations (13) and (12) of the paper

\[ \frac{1}{2} L_2[F] = L_0(x, y) \]

\[ = -\rho A \xi + E l A [ (K \xi')' + (K y')'] = 0 \quad (c) \]

\[ \frac{1}{2} L_2[F] = L_0(x, y) \]

\[ = -\rho A \eta - E l A [ (K \eta')' + (K y')'] = 0 \quad (d) \]

where \( L_2 \) and \( L_0 \) are the Euler operators defined in (9) and (10). "To this set of two differential equations we must add the existing relation between the variables \( x \) and \( y \), which takes the form of a subsidiary condition . . ." namely, equation (b). Thus, the inextensibility condition and its derivatives are used to eliminate \( x', x'' \), etc., from equation (12) and equation (14) for the single unknown \( y \) is obtained.

While there may be several ways of taking subsidiary conditions into account, the above is not one of them. The simplest is perhaps to employ the technique of Lagrange multipliers. According to this method we may consider, instead of minimizing \( L \) subject to condition (b), the variational problem

\[ \delta \int_0^1 \int_0^l (F + QG)d\xi dt = 0 \quad (e) \]

where \( Q \) is, in general, a function of \( \xi \) and \( t \) is the Lagrange multiplier. The corresponding Euler equation will now be

\[ L_0(x, y) = (Qx)', \quad (f) \]

\[ L_0(x, y) = (Qy)' \quad (g) \]

which have to be supplemented with equation \( G(x', y') = 0 \). Thus, we have three equations and three unknowns \( x, y \), and \( Q \) in contrast to the author's three equations (b), (c), and (d) and the two unknowns \( x \) and \( y \).

The question arises: Is it possible that the two formulations are equivalent? To disprove this, we note that a necessary condition for equivalence is that \( (Qx)' = 0 \) and \( (Qy)' = 0 \). These equations may be integrated and, with the aid of the inextensibility condition (b), yield

\[ x = \frac{\alpha \xi}{(\alpha^2 + \beta^2)^{1/2}} + \alpha \xi(t) \]

\[ y = \frac{\beta \xi}{(\alpha^2 + \beta^2)^{1/2}} + \beta \xi(t) \]

\[ Q = (\alpha^2 + \beta^2)^{1/2}, \quad \alpha = \alpha(t), \quad \beta = \beta(t) \]

Thus, if our formulation is correct, the only acceptable solutions of the author's equations are the ones in which the beam remains a straight line. As a further simple test, consider the motion with constant tangential velocity \( v \) on a circle of radius \( l \) as given by

\[ x = l \cos \left( \frac{\xi + vt}{l} \right), \quad y = l \sin \left( \frac{\xi + vt}{l} \right) \]

which satisfy equations (b), (f), and (g) and yield

\[ Q = \frac{\rho A v^2}{l^2} - \frac{2El}{l} \]

but, when substituted into \( L_0(x, y) = 0, L_0(x, y) = 0, \) yield

\[ \left( \frac{\rho A v^2}{l^2} - \frac{2El}{l} \right) \cos \left( \frac{\xi + vt}{l} \right) = 0, \]

\[ \left( \frac{\rho A v^2}{l^2} - \frac{2El}{l} \right) \sin \left( \frac{\xi + vt}{l} \right) = 0 \]

These can only be true if \( \rho A v^2 = 2El/l^2 \). In fact, it is now easy to see what the role of \( Q \) is. It is related to the tension \( S \) in the beam, and one can show exactly that for the general problem

\[ Q = \frac{S}{2} - \frac{2EIk^3}{l^2} \]

The condition of inextensibility does not rule out the tension for the same reason that incompressibility in fluids does not rule out pressure. The correct differential equations of motion for inextensible beams arising also from a variational formulation are given in the literature and (except for the inertial terms \( \rho \) and \( \rho y \)) can be found also in Odeh and Tadjbakhsh.4


Author's Closure

Without any palliation or circumlocution, I am obliged to confess that the above objection to my paper is absolutely correct. I am, therefore, deeply grateful to Dr. Tadjbakhsh for pointing out this unfortunate misstep.

However, and I consider this an alibi of mine, I must have solved a similar problem, for my solution sounds very reasonable. Yet the only trouble is that I do not know what the physical meaning of my problem really is.

Finally, in this situation I have no other alternative but to quote Alexander Pope who (in his famous "Essay on Criticism") wrote in 1711:

"Content, if hence 'tis unlearn'd their wants may view,
   The learnt reflect on what before they knew."

An Integral Method of Liapunov Function Generation for Nonlinear Autonomous Systems

STEIN WEISSENBERGER.1 The authors present a method of generating Liapunov functions which they assert has "significant advantages over any other now in use." Unfortunately, they fail to describe or reference the principal existing techniques, so that it is difficult for the reader who has not carefully studied the literature to judge the validity of their assertion or the relationship between their technique and others. The purpose of this discussion is to compare their technique with existing ones, illustrating this comparison by means of examples in the literature.

The authors' method consists of writing a gradient of the \( V \)-function as the sum of two parts. The first part is defined by

\[ \frac{\partial V_1}{\partial x_i} = \int \frac{\partial g}{\partial x_i} dx_{n-1}, \quad i = 1, 2, \ldots, n - 2 \]

\[ \frac{\partial V_1}{\partial x_{n-1}} = g \quad (1) \]

\[ \frac{\partial V_1}{\partial x_n} = x_n + \int \frac{\partial g}{\partial x_n} dx_{n-1} \]

where the \( n \)-th order of differential equations has been expressed in phase coordinates with \( x_n = g(x_{n-1}, \ldots, x_n) \). The second part of the gradient is

\[ \frac{\partial V_2}{\partial x_i} = f_i, \quad i = 1, 2, \ldots, n \quad (2) \]

where the \( f_i \) must satisfy


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\[
\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_1} = f_i \tag{3}
\]

but are otherwise arbitrary. Part (1) has been defined to satisfy equation (3) so that the two parts added together (or used separately) form the gradient of a scalar function \(V\), found by the line integration

\[
V = \int_0^T (V_i \delta x^i) + \int_0^T (V_i \delta x^i) dx
\]

The method proceeds by forming \(V\) and choosing \(f_i\) [subject to equation (3)] to make \(V\) at least negative semidefinite in a neighborhood of the origin; finally, \(V\) itself must be checked for closedness.

The variable-gradient method of Gibson and Schultz \([1, 2]\), consists of the same process limited to the second part of the gradient \(\partial V/\partial x = f_i\). The authors have added terms to the gradient which contain the nonlinearity \(g\); their method thus contains the variable-gradient technique as a special case. Do these extra terms involving the nonlinearity, however, make their method more efficient? A heuristic argument is made in the paper for the wisdom of including these terms \(\delta f_i/\delta x\), by noting that \(V_i\) is a first integral for a nearby system, which seems to be simply a system defined such that \(V_i\) is indeed a first integral. A more logical heuristic test for the utility of these special terms would be to determine whether these terms by themselves generate a Liapunov function for a linear system. It may be demonstrated readily that they do not. In order to use this technique for purely linear systems, the functions \(f_i\) must, in every case, be carefully chosen. We would thus not expect Walker and Clark's method to be more "automatic" than the basic variable-gradient technique.

Of course, the real proof of a method is in the results. The identical "nonlinear feedback" example in the paper is worked by Gibson and Schultz \([2]\) with identical results. The second example in the paper is also worked in \([2]\) for the special case \(m = 3\), again with identical results. Finally, the problem of Barbasin is worked in \([2]\), with similar results.

Other examples in the paper invite comparison with the method of Ingwerson \([3, 4]\), which is more nearly an automatic process, although initial guesswork is involved and the method is not guaranteed to produce valid results. (Ingwerson's method does, however, automatically generate Liapunov functions for the linear part of the system.) The same relay-control example is worked by Ingwerson \([4]\) with identical results. The second example in the paper is also worked in \([4]\) for \(m = 3\), yielding a different Liapunov function but producing again the same results.

Based on a comparison of existing examples, then, it is difficult to find significant advantages in the authors' method. Their technique is, however, an interesting extension of the variable-gradient method and could well make the solution of some problems easier, by placing terms involving the nonlinearity in the gradient of \(V\). "Ease," of course, is a subjective measure which depends to a large extent on the experience and skill of the user.

The observation is made at the end of the paper that the primary difficulty in their method is in evaluating results; i.e., determining stability boundaries. This writer heartily agrees and adds that this is the central problem with any method. However, rather than awaiting "progress in the field of nonlinear algebraic equations" for solution, this problem has been largely solved by Rodden \([5]\) by applying existing numerical techniques and high-speed digital computation to the determination of the exact boundary estimate. This work has made possible the fully automatic generation of stability-boundary estimates in \([5]\) by means of the methods of Ingwerson and Zubov \([6]\), both of which have the considerable advantage of being suitable for machine computation. The numerical techniques in \([5]\) have also made possible a new purely numerical method of generating stability-boundary estimates by Liapunov functions \([7]\).

Finally, the authors erroneously apply the Poincaré-Bendixon theorem to a third-order system. Also, this theorem applies only to systems of second order \([8]\). The world would be a simpler place (and Liapunov functions easier to generate) if the properties of higher dimensional space were as simple as those of the plane.

**References**

2. J. E. Gibson and D. G. Schultz, "The Variable Gradient Method of Generating Lyapunov Functions with Applications to Automatic Control Systems," Control and Information Laboratory Report TREE 62-3, Purdue University, Lafayette, Ind., April 1962.
Liapunov function is developed automatically, and thus gives significant indications as to what further modifications should be made. The method is also sufficiently flexible to allow the simple introduction of these modifications. The proposed method thus seems to offer a much better balance between being sufficiently automatic and being sufficiently flexible than other methods now in use. Further, the only restriction on the system to be investigated is that it be autonomous. As the order of the system is increased, the method generally becomes more difficult but is still tractable after other methods have become completely unwieldy.

The generalizations stated here are supported by the body of the paper.

2. The discusser considers the authors' method to be a generalization of the Variable Gradient method [1, 2]. If anything, the opposite is true. The proposed method automatically generates about half, or more, of a final Liapunov function \( V \), with no restrictions upon the final form, and consequently gives significant indications as to how the remainder of \( V \) should be specified. On the other hand, the Variable Gradient method makes the very general statement that guessing the gradient of \( V \) correctly is equivalent to correctly guessing \( V \). The Variable Gradient method does, however, place certain restrictions on the form \( \nabla V \), but these restrictions are arbitrary and do little to reduce the guesswork. The only term in \( V \) which appears automatically due to these restrictions is \( \dot{x}_2^2 \). All other terms must be guessed with little indication as to how the guesswork should even be initiated. With these restrictions of form it is impossible to obtain either of the two \( V \)-functions generated for the fifth example, since their gradients are not of the required form. Therefore, if these restrictions of form are considered inherent to the Variable Gradient method, comparison is impossible between the two methods. If, however, the Variable Gradient method is considered to consist of only the very general statement mentioned above, then it is a generalization in the purest sense of not only the proposed method but of almost all existing methods.

3. The discusser feels that the Ingwerson method [3, 4] is not only more automatic than the proposed method (which generally is true), but is also easier to apply (which generally is not). The Ingwerson method actually generates \( n \) tentative \( V \)-functions automatically for an \( n \)-th order system. However, in general, no more than one particular and unspecified linear combination of these \( n \) functions is found to serve as a "good" Liapunov function, and often even this is possible only after nonautomatic modifications are made in the manner of choosing the \( V \)-function of the discusser's description. Determination of this particular combination is often far from simple; the computation and evaluation of the \( n \) functions are usually far from rapid, even though automatic; and the choice of possible needed modifying terms is no more automatic than it is in the proposed method.

The authors do not mean to unduly criticize the Ingwerson method, which is actually a technique of considerable effectiveness, but only to point out that the advantages and disadvantages of the two methods are quite different and quite significant.

4. Despite the computational capabilities of the modern digital computer, the authors are of the opinion that the term "solution" is best applied to an analytical result, given in terms of one or more unspecified design parameters. The authors are aware of no computer "solutions" of this type at this time.

5. Finally, the authors most definitely did not "apply the Poincaré-Bendixson Theorem to a third-order system." They did, however, suggest application to a closed \( M_t \) integral manifold of the system

\[
\dot{x}_1 = \theta_1 \frac{\partial V}{\partial x_2}, \\
\dot{x}_2 = \frac{\partial V}{\partial x_2} - \theta_1 \frac{\partial V}{\partial x_1}, \\
\dot{x}_3 = -\frac{\partial V}{\partial x_2}
\]  

(36)

where \( V \) is a function positive definite in a neighborhood of the origin. Such application was made previously by G. P. Szegö [11] and indicates the existence of at least one singularity of (36) on each closed surface \( V = \text{constant} \).

Alas, of truly valid criticism there appears to be a lack.

References


Stress Distribution in Bonded Dissimilar Materials Containing Circular or Ring-Shaped Cavities

T. LEKO. The writer would like to know the conditions of uniqueness of the solution of the simultaneous singular integral equations (19). It appears that there is more than one solution of the problem by referring to the Appendix and equations (47)-(54).

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The Complementary Energy Theorem in Finite Elasticity

CHARLES LIEBO. The writer believes that the factor one-half in Equation (13) should be minus one-half. With this correction made, it is interesting to note that the two expressions for \( J \) [Equations (13) and (19)] are equivalent to the \( V \)-functions obtained by the writer for a continuum using his apparently quite different approach (see Ref. [8] and its forthcoming closing discussion).

The writer wishes that Professor Levinson had stated the variational theorems for his two \( J \)'s more explicitly, giving precisely all the admissibility conditions on stress, strain, displacement, and displacement gradients and pointing out clearly which interior and boundary conditions need not be imposed a priori. Then a comparison of our variational theorems, not only our functionals, would be possible. In this connection, it would appear that moment-equilibrium conditions must be imposed on the \( \psi \) in addition to the force-equilibrium conditions of Equation (20).

The stress-strain relations of Equation (18), which also arose in the writer's work as a necessary and sufficient condition for the existence of the complementary energy, are rather unusual looking, and the writer believes that an investigation of the implications of these stress-strain relations would therefore be in order. It would appear, for example, that in the infinitesimal case, \( \psi \) would have to be a function not only of the \( \psi \) but also of the rotations.

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