found by Frolov is \(-0.125\). These all appear to offer modest agreement, considering the nature of the approximations. An infinite series solution of the square cantilever plate problem has been offered by Mirsky,\(^4\) and he has shown that the first two terms of the solution appear to give reasonable accuracy. Unfortunately, he does not offer any numerical values for any other aspect-ratio plates.

**Author's Closure**

The writer would like to thank Professor Nash for his remarks on the paper, and for bringing to his attention the work of Frolov.\(^6\)

The values of the spanwise moments \(M_y\) at the mid-point of the cantilever root lie within 1–2 percent for all three methods, and on that basis must be judged reasonably accurate. However, agreement between chordwise moments \(M_z\) is much poorer, and further investigations appear necessary.

Along the root, the deflections, and hence the curvatures \(\partial^2 u / \partial y^2\), are zero everywhere. In that case, from the usual moment-curvature relationship for an isotropic plate,

\[
M_z = -D \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right),
\]

it is seen that \(M_y = \rho M_z\), where \(\rho\) is Poisson's ratio for the material.

If the mean value of \(M_y\) from the three methods is accepted, i.e., \(-0.504 \rho a^2\), the value of \(M_y\) corresponding to a Poisson's ratio of 0.3 then becomes \(-0.1512 \rho a^2\). This value appears to indicate that Nash's results are the most accurate from the point of view of self-consistency.

The discrepancy in the writer's results is probably due to the nature of the solution; since bending moments are approximated directly, the true strain conditions at the cantilever root cannot be specified exactly across the chord of the plate. The greatest errors in the solution would then be expected to occur in the region of the clamped edge.

Experimental confirmation of the various results is not yet available owing to the difficulty of producing a true clamped-edge condition and of measuring accurately the stresses at the root section.

\(^4\) I. Mirsky, “The Deflection of a Thin Flat Cantilever Square Plate Subjected to Uniform Normal Loading,” Report S.M. 177, Aeronautical Research Laboratories, Department of Supply, Melbourne, Australia, September, 1951.

\(^6\) V. M. Frolov, “Application of a Method of Correcting Functions in the Analysis of Deformations of Cantilever Plates,” Vypusk No. 705, Trudy Ts恩@иного Aero-Gidrodinamicheskого Instituta, Obozrev, Moscow, 1967.

**Laminar Radial Flow Between Parallel Plates**

J. D. JACKSON.\(^3\) In an earlier treatment of the problem of laminar radial flow between parallel plates, Hunt and Torbe\(^5\) employed an approach which was similar to the one used by Savage in the paper under discussion. The starting point was to assume a series expansion for \(u\) of the form

\[
u = \frac{1}{r} g_1(z) + \frac{1}{r^2} g_2(z) + \frac{1}{r^3} g_3(z) + \ldots
\]

A further related series for \(v\) resulted as a consequence of continuity considerations. The Navier-Stokes equations for the \(r\) and \(\theta\) directions were combined to eliminate \(\rho\) and the series for \(\rho\) and \(v\) were substituted into the resulting equation. In comparing the coefficients of powers of \(1/r\) and introducing the appropriate boundary conditions the functions \(g(z), g_2(z), g_3(z), \ldots\), were established. At that stage Hunt and Torbe simply used the functions to demonstrate that inertia effects were not important in the experimental work on hydrostatic thrust bearings which they had carried out. The present writer has since continued the analysis to obtain the pressure distribution along the plates from the equation of motion for the \(r\) direction which, when \(\theta = \pm \theta_0\), reduces to

\[
\frac{\partial p}{\partial r} = \frac{\rho}{\rho_0} \frac{\partial^2 u}{\partial r^2}
\]

The resulting equation for pressure distribution is identical with that obtained by Savage [equation (20)]. It is interesting that the solution just outlined was obtained without assuming a combined logarithmic and series expression for \(p\) as a starting point.

A very simple analysis which ignores \(v\) components of velocity also leads to the result given by Savage was developed by the writer several years ago. First, an approximation for the velocity profile

\[
\frac{Q}{8\pi h^2} (R^2 - z^2)
\]

is obtained by neglecting the inertia term in the simplified equation of motion for the \(r\) direction

\[
\frac{\partial^2 u}{\partial r^2} = \frac{dp}{dr} - \frac{\rho}{\rho_0} u^2
\]

and integrating twice with respect to \(r\) (alternatively the same result can be obtained by repeating the above procedure with the inertia term formed from the mean velocity \(Q/(\pi r h))\). Next, an improved velocity profile is obtained by repeating the procedure with the inertia term in (4) formed using (3). Then, on introducing the overall continuity condition

\[
Q = 2 \int_0^h 2\pi \rho \bar{u} d\theta
\]

the result given as equation (20) by Savage is obtained. The process can be continued to give higher order terms in the pressure distribution expression and the result obtained at the end of the next stage is

\[
\rho - \rho_0 = \frac{3aQ}{4\pi h^2} \ln \frac{R}{r} = \frac{27pQ^2}{560\pi h^2} \left[ \frac{1}{r^2} - \frac{1}{R^2} \right] - \frac{0.0004353Q^2}{2.1^4} \left[ \frac{1}{r^2} - \frac{1}{R^2} \right] - \frac{0.00000114Q^2}{\pi^4} \left[ \frac{1}{r^2} - \frac{1}{R^2} \right]
\]

Of course, if the series expansion procedures adopted by Hunt and Torbe and by Savage were to be taken further, then one would expect to obtain better results than those given by the present simplified approach.

B. G. NEWMAN.\(^4\) Savage has obtained an elegant solution for laminar incompressible flow. It is estimated that the second approximation is sufficiently accurate in the absence of entry effects if \(Re = Re \left( \frac{R}{h} \right) < 2\). Moller [1] has demonstrated that the

\(^4\) Canadian Professor of Aerodynamics, McGill University, Montreal, P. Q., Canada.

\(^5\) Numbers in brackets indicate References at end of this Discussion.
flow will be laminar if the local channel Reynolds number 
\( \text{Re} \left( \frac{R}{\lambda} \right) < 500 \). These values, combined with a restriction on the 
magnitude of the inlet pressure to avoid the effects of compressibility [2], define the limits of application of Savage's theory.

Following an earlier suggestion by Savage, Moller has modified Livesey's theory by assuming a velocity profile which is slightly 
perturbed from the parabolic profile particularly near the wall. The magnitude of the perturbation term is determined by satisfy-
ing the first compatibility condition at the wall. This integral 
method has been extended to deal with isothermal compressible 
flow, and therefore it is of interest to compare the incompressible 
form of this solution with that of Savage. The pressure distribution 
may be written as:

\[
\frac{3}{6\pi^2 \text{Re}} \ln \left( \frac{\hat{R}}{\lambda} \right) \left( \frac{p - p_0}{\rho Q^2} \right) = \frac{2\pi}{\lambda^2} \left( \frac{h}{R} \right)^2 \left( \left( \frac{\hat{R}}{\hat{r}} \right)^2 - 1 \right)
\]

where

\[
\begin{align*}
\lambda &= 720 \text{ Livesey} \\
&= 600 \text{ Moller} \\
&= 560 \text{ Savage (second approximation).}
\end{align*}
\]

The results are compared with one set of Moller's experimental 
data for both a sharp inlet corner and for a rounded inlet corner; 
see Fig. 1. It is interesting to note that the effects of the inlet 
geometry extend well downstream into the channel and that 
within the scatter of results there is little to choose between the 
two theories of Moller and Savage. Reattachment of the sep-
ated flow at the sharp inlet corner will have occurred very close 
to the corner [1] \( \left( \frac{R}{\lambda}^2 - 1 > 30 \right) \) so the difference between the 
two sets of results is not due to any gross effects of separation 
there; indeed, the influence of the inlet geometry is probably less 
for the sharp corner than for the rounded corner. It would be 
interesting to extend Savage's solution to the third approxima-
tion and to compare this with more accurate measurements near the 
corner.

Radial channel flow is of practical interest in the design of air 
bearings. In such cases the air is usually supplied at high pres-
\( \text{ures and the subsequent flow is compressible [2] and may also be supersonic [3]. In other practical situations the flow may be turbu}
\]

tent [1, 2, 3]; the inertia term then frequently overrides the 
friction term and the pressure within the channel rises in the 
downstream direction. This is the situation which exists when a 
VTOL aircraft with centrally located, downward-pointing jets hovers close to the ground and experiences a loss of lift: the 
average pressure underneath the wings and fuselage is less than 
that of the surroundings. If care is taken to eliminate separation 
at the inlet corner, the radial channel configuration also makes an 
efficient subsonic diffuser with an efficiency close to that of a 
conical diffuser with the same inlet conditions [4].

References
1 P. S. Moller, "Radial Flow Without Swirl Between Parallel 
2 P. S. Moller, "Radial Flow Without Swirl Between Parallel 
Discs When the Flow Is Compressible and Subsonic," Report 63-7, 
published in Aeronautical Quarterly.
3 P. S. Moller, "Radial Flow Without Swirl Between Parallel 
Discs Having Both Supersonic and Subsonic Regions." Report 63-10, 
4 P. S. Moller, "An Investigation of a Radial Diffuser Using 

Author's Closure

The author wishes to thank Professors Jackson and Newman 
for their thoughtful discussions. Professor Jackson's extension 
of Hunt and Torbe's analysis is equivalent to and yields the same 
results as the author's treatment, although it proceeds in a some-
what different manner. The expansion for the pressure distribution 
used by the author is obvious after consideration of the 
creeping flow solution; however, the method of Hunt and Torbe 
might be preferred since it has a neater formulation.

Professor Jackson has presented an interesting iterative 
analysis which agrees with the author's expression for the pres-
sure to order \( (1/\theta) \). As pointed out in the author's analysis, 
departures from the simplified equations of motion used by Jack-
son will occur in the third approximation. For this reason, as 
suggested by Jackson, terms of order higher than \( (1/\theta) \) given 
by the simplified approach may be inaccurate.

Professor Newman has carefully defined the limits of applica-
bility of the author's treatment and has discussed the effects of 
inlet geometry. The problem of inlet geometry requires further 
study since, under certain conditions, entry effects play a domi-
nant role in determining the flow conditions.