
**p–x imaging by localized slant stacks of T–x data**

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Received 1982 June 16

**Summary.** An image can be produced directly in the ray parameter–distance (p–x) plane from data in the travel time–distance plane by computing an overlapping sequence of local slant stacks. The centre of each local stack in the series is slightly displaced in x from the previous one. Processing of three synthetic data profiles shows that p–x images produced by localized stacking correspond with analytically determined p–x curves. Apparent phase velocities measured from post-critical reflection branches with an array contain systematic deviations related to the presence of an x-dependent intra-wavelet phase shift.

**Introduction**

In seismology, many inversion schemes have been formulated directly as the mathematical inverse of the corresponding forward (modelling) problem. For example, in refraction seismology, the Wiechert-Herglotz integral is the dual of ray tracing in a laterally homogeneous medium and can be derived from the expressions for travel time and distance. In reflection seismology, finite difference modelling and migration are both wavefield continuations that differ only in the sign of the depth or time steps taken. Recent work of this type in refraction seismology involves the use of the forward Radon transform to produce a seismogram profile in the travel time–distance (T–x) plane from a ray parameter–time intercept (p–τ) curve and the inverse Radon transform to decompose a T–x profile into plane-wave elements displayed in the p–τ plane. Examples can be found in Chapman (1978), McMechan & Ottolini (1980), Stoffa et al. (1981), and Phinney, Chowdhury & Frazer (1981) among others. All these involve the p–τ plane as one of the two transform planes. It is also known (cf. Wiggins 1976) that a similar duality exists between the T–x and p–x planes. Synthetic seismogram profiles have been computed from p–x curves; however, it has not been shown how a complete p–x image may be directly extracted from the T–x plane. The reason for this is apparently that the inverse Radon transform (or 'slant stack') that is the process used for measuring p is defined as an integral over all x, so that information relevant to any particular x is not conveniently available. The purpose of this paper is to demonstrate that a p–x image can be constructed directly from a T–x wavefield by performing a sequence of...
overlapping localized slant stacks centred at each $x$ for which $p$ values are required. The result is an image in the $p-x$ plane that corresponds to the classical $p-x$ curve.

Theory

The concepts of velocity filtering and slant stacking have long been used to measure apparent phase velocities of mantle arrivals with finite aperture arrays (cf. Johnson 1967; Simpson, Mereu & King 1974; England, Worthington & King 1977; Burdick & Powell 1980 among many others). The results of these processes are generally displayed in the kinematic form of sloping line segments plotted through the appropriate $T-x$ points or as point $p-x$ measurements. The differences between these studies and what follows below are that the total aperture will be considered to be composed of an overlapping sequence of local apertures and that the results of phase velocity estimations in each local aperture will be displayed as a separate vector in a $p-x$ image.

Consider a controlled source refraction experiment with closely spaced observations. In this case, the total array aperture equals the maximum shot distance ($x_{\text{max}}$), and very different phase velocities may be present in different parts of the aperture. In a representative single slant stack trace $Q(\tau, p)$ for apparent velocity $1/p_k$, an energy concentration will

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Figure 1. Schematic of a representative local slant stack. The (solid line) $T-x$ curve in (a) corresponds to the (solid line) $p-x$ curve in (b). Performing a local slant stack centred at $x_i$ in the $T-x$ plane produces a data slice in the $p-x$ plane at $x = x_i$. This slice is a vector whose amplitude ($\overline{Q}(\tau, p)$) at each $p_k$ position is the maximum amplitude in the slant stack for apparent velocity $1/p_k$. (c) is a section through (b) at $x = x_i$. The dotted line of slope $p_i$ and time intercept $\tau_i$ is tangent to the travel time curve at $(T_i, x_i)$.  

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be seen at $\tau_i$ (Fig. 1a) where coherent energy of slope $p_i$ is projected by the slant stack integral.

$$Q(\tau, p_i) = \int_0^{x_{\text{max}}} S(\tau + p_i x, x) \, dx$$  \hspace{1cm} (1)$$

where $S(T, x)$ is the observation $(T-x)$ wavefield. Since integral (1) is over all $x$, it is not known that the coherent energy contributing to the concentration at $\tau_i$ is centred at $x_i$ (Fig. 1). However, if the stack had been localized to the neighbourhood of $x_i$, say $x_i - \delta$ to $x_i + \delta$, then $x_i$ would be known to $\pm \delta$.

Consider now the representative local $x$ aperture $x_i - \delta$ to $x_i + \delta$ and the result of stacking over a broad range of $p$ values. For each $p_i$, a vector $Q(\tau, p_i)$ can be defined:

$$Q(\tau, p_i) = \int_{x_i-\delta}^{x_i+\delta} S(\tau + p_i x, x) \, W(x_i - x) \, dx$$  \hspace{1cm} (2)$$

where $W$ defines a weighting of each contribution to the integral that depends on its distance from $x_i$. Where an arrival branch exists, $Q(\tau, p_i)$ will be large; where no arrival branch exists, $Q(\tau, p_i)$ will be small. Also, at any $x_i$, there may be many arrival branches present, but, assuming the medium is laterally homogeneous, or nearly so, each of these will correspond to a different $p_i$. For each $p_i$ (at a fixed $x_i$) the maximum value ($\bar{Q}(\tau, p_i)$) in the $\tau$ vector $Q(\tau, p_i)$ can be found and plotted at the position ($p_i, x_i$) in the $p-x$ plane. Repeating this for a series of $x_i$ values allows a $Q$ to be assigned to every point in the $p-x$ plane. The resulting $p-x$ display should contain a high amplitude locus that corresponds to the $p-x$ curve. This is demonstrated in the examples presented in the following section.

**Implementation and synthetic examples**

Implementation of the localized slant stack algorithm is straightforward as the basic operation is a standard slant stack. The two modifications required are the addition of a loop to stack repetitively and the shifting of the local aperture for each successive stack.

The choice of aperture width requires some discussion. If the aperture is too wide, the desired $x$-localization is lost, and with it, resolution in the $x$-direction. If the aperture is too narrow, $p$ resolution is lost because discrimination between $p$ values relies on having an adequate baseline over which to measure slopes. (If one considers the limit of an aperture that is only one trace wide it is clear that all $p$ values will have the same $\bar{Q}$ and this will be equal to the largest amplitude in the trace; it takes a finite aperture to measure an apparent velocity.) The optimum width is that distance over which constructive interference is obtained for the $p$ under consideration. This has previously been referred to as a Fresnel zone (Schultz 1976; McMechan & Ottolini 1980). Unfortunately, the spatial wavenumber ($k_x$) is a function of $p$:

$$k_x = \omega p$$  \hspace{1cm} (3)$$

so the optimum aperture width changes with position and with the local temporal frequency ($\omega$) in the data. A specific example of two apertures is shown below (Fig. 3). The effect of aperture shape ($W$ in equation 2) is not investigated in any detail; a simple triangular window was used to decrease linearly the weighting of each trace lying within the aperture from 1.0 at the centre to 0.0 at the edges:

$$W(x_i - x) = 1.0 - \frac{|x_i - x|}{\delta}$$  \hspace{1cm} (4)$$
where $\delta$ is the half-width of the aperture expressed in the same units as $x$. This shape satisfies the requirements that the contributions in each localized stack come predominantly from traces near the centre of the aperture and that edge effects related to the finiteness of the aperture are reduced, but has no specific justification other than satisfactory performance.

For the purpose of illustration of localized slant stacking, three synthetic profiles were computed. These are shown in Fig. 2 and contain a single refracted branch (a), a single reflected branch (b), and a triplication (c). Amplitudes are scaled proportionately to shot distance. Each of these three examples will now be discussed in turn. For each example, the results are plotted with the 32 level grey scale of Henderson & Tanimoto (1974). Each grey level is obtained by overprinting a specific combination of characters; output is via a line printer. Perceived blackness is proportional to amplitude.
Fig. 3 contains the results of localized slant stacking of the refraction branch in Fig. 2(a). The trace at each $x$ in Fig. 3 is normalized to have the same maximum $Q$. Fig. 3(a) illustrates the effect of using an insufficiently wide aperture: $\delta$ is three traces (= 1.5 km) so at each $x_i$, the aperture encloses the $T-x$ trace at $x_i$ plus three traces on either side. In Fig. 3(a), there is a diffuse trend visible from high to low $p$ with increasing $x$, but no coherent image. For Fig. 3(b), $\delta$ is 10 traces (= 5 km); here, the image is adequate. The larger amplitudes at each $x$ in Fig. 3(b) are replotted in Fig. 3(c). The position of these points should give a good estimate of the $p-x$ curve. For comparison, the solid line superimposed on Fig. 3(c) is the analytically computed $p-x$ curve.

One aspect of the image in Fig. 3(b) requires further comment. Over most of the $p$ range, the $p-x$ image appears to be discrete in $p$ (i.e. large and small amplitudes alternate along the image). This behaviour is, in fact, expected because linear (constant $p$) interpolation was
Figure 4. The $p-x$ amplitude distribution produced by localized slant stacking of the synthetic reflection branch in Fig. 2(b) using a local aperture half-width of 10 $T-x$ traces. The larger amplitudes extracted from the total distribution (a) are shown in (b) along with the analytically computed $p-x$ curve (the solid line). In (b), the arrow indicates the position of the critical reflection.

used between ray emergent points in the $T-x$ plane in constructing the synthetic seismo-
gram profile. This means simply that the sampling rate (in $p$) used in slant stacking is greater
than that in the profile. It is of interest to note that localized stacking apparently has
sufficient resolving power to reveal this since it is not obvious in the $T-x$ profile.

Fig. 4 contains the results of localized slant stacking of the reflection branch in Fig. 2(b)
with $\delta = 10$ traces ($= 5$ km). The trace at each $x$ is normalized to have the same maximum $Q$
so that the small amplitude pre-critical reflections are seen as easily as the larger post-
critical reflections. Fig. 4(a) shows the complete $p-x$ $Q$ amplitude distribution; Fig. 4(b)
contains only the larger amplitudes at each $x$. The solid line superimposed on Fig. 4(b) is
the analytically computed $p-x$ curve.

The most striking feature of Fig. 4(b) is that the maximum amplitude locus in the $p-x$
image deviates in a systematic way from the computed $p-x$ curve. A similar deviation was
noted by Clayton & McMechan (1981) in the $p-r$ plane. The reason for this deviation is the
progressive phase shift that occurs along a post-critical reflection branch. In Fig. 4(b), the
position of the critical reflection is indicated by the arrow. At lesser distances, the image
and the computed $p-x$ curve correspond well. Immediately to the right of the arrow, the
deviation is largest. Then, as distance increases, the image and the computed curve converge.
The source of this behaviour may be seen in Fig. 5, which contains a blow-up of part of
Fig. 2(b). Basically, $p$ is usually assumed to be the reciprocal of the measured apparent phase
velocity across an array. This assumption is correct only if the arrival pulse shape is stable
with distance (or equivalently, that phase and group velocities are identical). In Fig. 5 it can
be clearly seen that the wavelet changes, with increasing distance, from having mixed delay toward having minimum delay. The phase energy moves forward within the wavelet itself to produce a high measured apparent phase velocity and a correspondingly reduced \( p \) (as seen in Fig. 4). This phenomenon has apparently not been noted before because the use of small array apertures (relative to the length of a whole branch) does not allow comparison of phase shifts as a function of distance. The main implication of this observation is that the validity of all previous interpretations involving \( p \) measurements from post-critical reflections may be questioned.

Fig. 6 contains the results of localized slant stacking of the triplication in Fig. 2(c) with \( \delta = 10 \) traces (= 5 km). This figure contains a number of parts because of difficulty in finding a single suitable amplitude scaling. The problem arises because the un-normalized \( \bar{Q} \) distribution in the \( p-x \) plane obtained by localized slant stacking contains amplitude information derived from the \( T-x \) plane. In this example, amplitudes on the reflection branch are much larger than the refractions, so scaling to produce clear visibility of one set of arrivals is unsatisfactory for the other. To resolve this, different parts can be normalized separately, and then recombined into a single image. Fig. 6(a) contains the \( p-x \) \( \bar{Q} \) distribution in which \( x \)-vector is scaled to have the same maximum amplitude. This produces a clear image of the high amplitude reflection branch (in the centre of the plot) and of those parts of the refraction branches in the upper left and lower right corners (at distances where the reflections are of low amplitude or do not exist). Fig. 6(b) shows the same \( \bar{Q} \) distribution in which each \( p \)-vector is normalized to the same maximum amplitude. This produces increased visibility of the refraction branches at the expense of enhancing low amplitude background values at the top and bottom \( p \) values.

Combining the results of \( x \) and \( p \) normalizations gives a good estimate of the location and shape of the complete \( p-x \) curve; this is shown in Fig. 6(c). For comparison, the analytically computed \( p-x \) curve is presented in Fig. 6(d). In Fig. 6(c), the \( T-x \) triplication (Fig. 2c) has been correctly unfolded by localized slant stacking, and the \( p-x \) estimates have a predictable relationship to the computed \( p-x \) curve. Much relative amplitude information is still present; significantly decreased amplitudes are observed as expected near A and B where the \( p-x \) curve is nearly horizontal (cf. Wiggins 1976). Also, to the right of the critical reflection (C), \( p \) values are systematically low as explained above in the analysis of Fig. 4.

Through these three examples, the feasibility of extracting a complete \( p-x \) image from \( T-x \) data is demonstrated.
Discussion and synopsis

This study demonstrates the possibility of constructing a $p-x$ image directly from $T-x$ data by localized slant stacking. Three synthetic examples are successfully processed to illustrate the method.

The approach taken is simply to reverse the process described by Wiggins (1976) who constructed synthetic seismograms from $p-x$ curves. The underlying basis is that those parts of $p-x$ curve that contribute most to a seismogram at any distance $x_i$ are those that are nearest to $x_i$. Conversely, the $p$ values corresponding to any $x_i$ can be extracted from a seismogram profile by slant stacking the energy in the neighbourhood of $x_i$. 

Figure 6. The $p-x$ amplitude distribution produced by localized slant stacking of the $T-x$ triplication in Fig. 2(c). In (a) each $x$-vector is normalized to be of the same maximum amplitude. In (b) each $p$-vector is normalized to be of the same maximum amplitude. (c) is a combination of the larger amplitudes in (a) and (b). (d) contains the analytically computed $p-x$ curve.
A number of questions raised remain open for further research. The shape and width of the local aperture used in this paper are adequate for the purpose of a feasibility study, but it is expected that considerable optimization can be done and would result in sharper $p-x$ images. It should also be possible to make specific use of the amplitude information in un-normalized $p-x$ images in interpretations and to find a method for correcting measured apparent phase velocities of post-critical reflections for their systematic phase shifts.

Two obvious extensions of this paper are to find a transformation from $p-x$ to velocity-depth to compare with the existing $p-\tau$ to velocity-depth transformation, and to find a $p-x$ to $p-\tau$ transformation.

**Acknowledgments**

The author gratefully acknowledges critical reviews of the paper by R. Clayton and D. Weichert. Contribution from the Earth Physics Branch No. 1004.

**References**


