Applications of the Kirchhoff-Helmholtz integral to problems in seismology

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Summary. A numerical method for evaluating the Kirchhoff-Helmholtz integral is described. The Kirchhoff response is calculated by discretizing the surface, specifying simple point sources on each element of the surface, and summing the contribution from the elements. The results of the method are compared to those of an asymptotic, first motion approximation of the analytical solution of SH-waves impinging on a rigid sphere. The agreement between the results of the two methods is excellent for source and receiver distances which are large compared to the radius of the sphere. The method is applied to the calculations of reflections from mountain topography and a planar surface with an aperture. The phase shifts of pulses are consistent with optics; the amplitudes are not. The method does predict frequency dependence of the scattered amplitudes. Calculations are presented to model spall which produce travel-time and amplitude anomalies consistent with observations from nuclear blasts.

Introduction

Many wave propagation phenomena cannot be adequately modelled by existing solutions to plane-layered media. Yet the increasing use of broadband seismic data to determine source dislocations, Q, and velocity structure requires a knowledge of effects of material irregularities in the medium on seismic wave propagation. Certainly, documented amplitude and \(dT/d\Delta\) anomalies of teleseismic arrivals at large arrays (Glover & Alexander 1969; Walck & Minster 1980) which vary as a function of azimuth suggest the existence of non-flat boundaries at depth.

Numerical schemes which handle material irregularities are in abundance. Finite difference and finite element codes have been used successfully (Boore, Larner & Aki 1971; Smith 1975) and can be applied to a variety of materials; however, the expense of calculating the response at distances which are large compared to the wavelength of interest is prohibitive. Rayleigh-FFT techniques have been exploited for these problems (Aki & Larner 1970). Similarly, implementation of these methods for analysis of three-dimensional scattering is also costly. Geometric ray methods are useful for predicting scattering of signals which have wavelengths that are short compared to the size of the heterogeneity (Hong &
But existing ray methods do not predict frequency-dependent amplitudes of scattered pulses and do not handle diffracted arrivals.

Discussion of techniques

The method presented in this paper is an integral equation approach and is based on the evaluation of the scalar integral equation variously called the Kirchhoff, Helmholtz or Huygen’s integral. This equation is a formulation of the wave equation in terms of a linear surface integral over the boundary of a continuous volume. That is, the scalar wave equation is

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} (r, t) - \nabla^2 u(r, t) = \Phi(r_0, t). \quad (1)$$

Here $u(r, t)$ is the field at a point $r$ resulting from a source potential $\Phi(r_0, t)$ and $c$ is the wave speed. Following the formalism discussed by Mow & Pao (1971), we consider the motion of a homogeneous body $V$ with a smooth boundary $\partial V$ with outward pointing normal $n$. Then if $r \in V$ and $t \in (0, \infty)$

$$u(r, t) = \int_V \int_0^\infty G(r, r_0, t-t_0) \Phi(r_0, t) \, dt_0 \, dV_0 \quad + \frac{1}{c^2} \int_V \left\{ G(r, r_0, t) \frac{\partial u}{\partial t}(r_0, 0) - u(r_0, 0) \frac{\partial G}{\partial t}(r, r_0, t) \right\} \, dV_0 \quad + \int_{\partial V} \int_0^\infty \left\{ G(r, r_0, t-t_0) \nabla_0 u(r_0, t) - u(r_0, t) \nabla_0 G(r, r_0, t-t_0) \right\} \cdot n(r_0) \, dt_0 \, dS_0. \quad (2)$$

Here $G(r, r_0, t-t_0)$ is the fundamental singular solution of the scalar wave equation

$$\frac{1}{c^2} \frac{\partial^2 G}{\partial t^2} - \nabla_0^2 G(r, r_0, t-t_0) = \delta(t-t_0) \delta(r-r_0). \quad (3)$$

Let us define $f(r, t)$ as the sum of the first two integrals in equation (2). Then $f(r, t)$ can be interpreted as the whole-space solution of the problem with sources $\Phi(r_0, t_0)$ and initial values $\partial u/\partial t (r_0, 0)$ and $u(r_0, 0)$. Hence $u(r, t)$ is a sum of the direct pulse and a reflected pulse from the surface which is described by the third integral in equation (2).

If $r \notin V$, then the left-hand side of equation (2) is zero. If $r \in \partial V$ then

$$\frac{1}{2} u(r, t) = \int_0^\infty P \int_{\partial V} \left\{ G(r, r_0, t-t_0) \nabla_0 u(r_0, t) - u(r_0, t) \nabla_0 G(r, r_0, t-t_0) \right\} \cdot n(r_0) \, dt_0 \, dS_0 + f(r, t). \quad (4)$$

Here $P$ denotes the principal value of the integral. A detailed derivation of equation (4) can be found in Cole (1980). This result requires that $G$ has a specific asymptotic behaviour at its singularity. The function $G$ used in the Kirchhoff formulation here meets this requirement; specifically

$$G(r, r_0, t-t_0) = \frac{\delta(t-\tau)}{4\pi |r-r_0|}; \quad \tau = \frac{|r-r_0|}{c}. \quad (5)$$
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Substitution of this function into equation (2) gives a familiar optics formula ([Born & Wolf 1964])

\[ u(r, t) = \frac{1}{4\pi} \int \frac{1}{r} \left[ \frac{\partial u}{\partial n} \right] - \left[ u \right] \frac{\partial r}{\partial n} \frac{\partial r}{\partial r} \left( \frac{1}{r} \right) + \frac{1}{rc} \frac{\partial r}{\partial n} \left( \frac{\partial u}{\partial t} \right) \right] dS_0 + f(r, t) \]  

(6)

where \( r = |r - r_0| \), the distance from receiver to surface, \( \partial u/\partial n = Vu \cdot n \) and \( \partial r/\partial n = V(r - r_0) \cdot n \). The square brackets denote the values of the functions on \( \partial V \) at the time \( t - r - r_0/|c| \).

The integrals (2), (4) and (6) are formally exact and are a mathematical representation of Huygen’s principle; that is, a disturbance at a receiver point is a superposition of secondary waves proceeding from a surface existing between that point and the source. Diffraction phenomena arise from the mutual interference of these secondary disturbances. However, one needs the value of the potential and its normal derivative on the surface to calculate \( u(r, t) \). Equation (4) may be solved for \( u(r, t) \) or \( Vu \cdot n \) on the surface subject to some constraints imposed by boundary conditions (e.g. continuity of \( u(r, t) \) or \( Vu \cdot n \) across the boundary). This approach is taken by Mitzner (1967) who sets \( Vu \cdot n \) equal to zero and solves for \( u(r, t) \). However, this approach may be costly for high-frequency scattering.

Alternatively one may estimate the values on the surface by invoking an approximation. This approach is used in this paper. Assuming a point source, the boundary values on the surface are taken to be

\[ u(r, i) = \left[ f \left( t - \frac{r_0}{c} \right) / r_0 \right] \left( 1 + R \right) \]  

(7)

\[ \frac{\partial u}{\partial n} = \frac{\partial r_0}{\partial n} \left( 1 - R \right) \left[ -f \left( t - \frac{r_0}{c} \right) / r_0^2 - \frac{1}{c} f' \left( t - \frac{r_0}{c} \right) \right] . \]  

(8)

Here \( r_0 \) is the distance from the source to the surface, \( R \) is the approximate plane-wave flat interface reflection coefficient, and \( f(t) \) and \( f'(t) \) are the source time function and its derivative, respectively. The reflection coefficient will depend on incidence angle. This approximation is variously called the Kirchhoff, physical optics, or the tangent plane hypothesis and is widely used by workers in electromagnetic scattering investigations ([Davies 1954]). It assumes that the incident pulse is of a sufficiently high frequency so that locally the amplitude decay is described by both geometric ray theory and plane-wave reflection coefficients. Therefore every point on the surface reflects the incident pulse as though there were an infinite plane tangent to the surface at that point. The values of the potential and its normal derivative at a point are independent of the boundary values at other points. Hence the effects of multiple scattering and diffractions along the surface are neglected.

Upon substituting the values (7) and (8) for \( u \) and \( \partial u/\partial n \) one obtains for the reflected potential

\[ u_i(r, t) = \frac{1}{4\pi} \int_{\partial V} R \left( \frac{1}{rr_0} \frac{\partial r_0}{\partial n} + \frac{1}{r^2} \frac{\partial r}{\partial n} \right) f \left( t - \frac{r - r_0}{c} \right) dS_0 \]  

\[ + \frac{1}{4\pi} \int_{\partial V} R \left( \frac{1}{rr_0 c} \frac{\partial r_0}{\partial n} + \frac{1}{r_0 c} \frac{\partial r}{\partial n} \right) f' \left( t - \frac{r - r_0}{c} \right) dS_0. \]  

(9)

This equation is similar to those derived by Trorey (1970, 1977), Hilterman (1970, 1975), and Berryhill (1977). These authors have derived convolutional forms of the Kirchhoff integral which make computation rapid. Hilterman has verified his results with small-scale experimental modelling of a point source in air impinging on rigid anticlines,
synclines, and normal faults. The agreement between his numerical calculations and experiments is, in general, excellent. However, these analytical forms of the solution place severe restrictions on either the source-receiver geometry or the surface geometry.

The method presented in this paper differs from Trorey and Hilterman in that the source and receiver are allowed to be at separate locations, the surface geometry is arbitrary, and the integral is approximated by the following expression

$$u_i(r, t) = \frac{1}{4\pi} \sum_{k=1}^{N} R \left\{ f \left( t - \frac{r_0}{c} - \frac{r}{c} \right) Q_k^{(1)} + f' \left( t - \frac{r_0}{c} - \frac{r}{c} \right) Q_k^{(2)} \right\} \Delta S_k$$

(10)

where

$$Q_k^{(1)} = \frac{1}{rr_0^2} \frac{\partial r_0}{\partial n} + \frac{1}{r_0 r} \frac{\partial r}{\partial n}$$

and

$$Q_k^{(2)} = \frac{1}{rr_0 c} \frac{\partial r_0}{\partial n} + \frac{1}{rr_0 c} \frac{\partial r}{\partial n}.$$

An important part of the procedure is the discretization of the surface. The rough surface is specified by a function $z(x_i, y_i)$ where $(x_i, y_i)$ is a location on a horizontal grid of regularly spaced points separated by a distance $\Delta x$ and $\Delta y$ (see Fig. 1). From this information, one can readily calculate $\partial z/\partial x$ and $\partial z/\partial y$. Then the following formulae are used to calculate $\Delta S_k$, $\partial r_0/\partial n$, and $\partial r/\partial n$:

$$\Delta S = \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \Delta x \Delta y$$

(11)

$$\frac{\partial r}{\partial n} = \frac{[(x_s - x_i) \partial z/\partial x + (y_s - y_i) \partial z/\partial y - (z_s - z_i)]}{\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}}$$

(12)

$$\frac{\partial r_0}{\partial n} = \frac{[(x_0 - x_i) \partial z/\partial x + (y_0 - y_i) \partial z/\partial y - (z_0 - z_i)]}{\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}}.$$

(13)

Here $x_s$, $y_s$, and $z_s$ are the station coordinates and $x_0$, $y_0$, and $z_0$ are the source coordinates. These values are calculated for a finite surface. However, the Kirchhoff integral is formally stated for a closed surface. This problem is circumvented by integrating over a closed surface consisting of the part of the surface one is interested in $(S_o)$ and a portion of a sphere of large radius $(S_B)$ (see Fig. 2). One then argues that the contributions from the sphere arrive at the receiver at times later than those of interest and neglects them.

For the time function $f(t)$ we have chosen a ramp function. This choice circumvents the problems of numerically simulating a delta function. So, on each element we add the sum of a ramp function multiplied by $Q_k^{(1)}$ and a step function multiplied by $Q_k^{(2)}$ appropriately in time. That is, each element is illuminated and contributes to the total response at a time $\tau = (r + r_0)/c$. This two-way travel time is calculated and the responses from all the elements are summed cumulatively in order of increasing $\tau$ as displayed in Fig. 3.

For problems presented in this paper, the numerical ramp response is convolved with the analytical third derivative of a modified Haskell explosion source function, specifically

$$\Psi(t) = \Psi_0 [1 - \exp(-kt) \{1 + kt + (kt)^2/2 - B(kt)^3\}].$$

(14)
Here $\Psi_0$ is the source strength, $k$ scales inversely as the source strength and $B$ is the overshoot constant. This convolution is mathematically equivalent to the first derivative of the reflected potential caused by a source described by (14). From trial and error, we have determined that such a convolution eliminates the spikiness introduced by simple differencing and, therefore, is the preferred method for differentiation in the calculations presented in this paper.
We have carried out experiments to determine the grid size required to produce a smooth seismogram. As an example we show, in Fig. 4, the variation of the waveform and maximum amplitude of a seismogram as a function of grid size for a sample reflection problem. The reflecting flat surface is specified for seismograms A, B and C by grid areas of a wavelength of 4 km. Hence for seismograms A, B and C the number of grids per wavelength is 11.4, 8 and 4 respectively. Seismogram C shows that the coarse discrimination of the surface has introduced high-frequency noise which degenerates the waveform and the maximum amplitude. For each problem we calculate, the grid size is selected by trial and error. The grid is made progressively finer until the solution is unvarying as shown in Fig. 4.

\[ x \leq R: \text{Refl. Coeff.} = \cos \left( \frac{\pi x}{2R} \right) - 1 \]
\[ x > R: \text{Refl. Coeff.} = -1 \]

Figure 4. This figure shows the change of response as the element size increases. The model used here is a free surface with a hole. The source and receiver are directly below this hole. The response is convolved with a modified Haskell source and a short-period WWSSN instrument to produce the above seismograms.
Applications

The development of any numerical procedure necessitates a comparison with known exact solutions. To see that the integral is being calculated correctly, the results of this code are compared with an analytical formula developed by Hilterman (1975). For the source and the receiver together above an arbitrary rigid surface, Hilterman reduces the Kirchhoff integral to the following form:

\[ u_t(r, t) = \frac{1}{2\pi c} \left[ \left( f(t) \ast \frac{2}{t} \frac{d\Omega}{dt} \right) + \left( \frac{df}{dt} \ast \frac{d\Omega}{dt} \right) \right] . \] (15)

Here * denotes convolution. \( d\Omega/dt \) is the increase of solid angle with a vertex at the source-receiver point subtended by the surface as a function of time (Hilterman 1975). We choose to calculate the response to a point source in a fluid overlying a rigid hemisphere imbedded in an infinite rigid plane (see Fig. 5 for the geometry). For such a geometry we have determined \( d\Omega/dt \) to be

\[ \frac{d\Omega}{dt} = \frac{\pi}{| R |} \left\{ \frac{2(| R |^2 - a^2)}{ct^2} - \frac{c}{2} \right\} H(t - \tau_0) \quad \text{for} \quad \tau_0 < t < \tau_1 \] (16)

where \( \tau_0 = 2(| R | - a)/c, \) the minimum two-way travel time, and \( \tau_1 = 2(| R |^2 + a^2)^{1/2}/c. \) \( H(t) \) is the assumed source time function and is the unit step function. Using these results, we calculate the following two terms \( \phi_A \) and \( \phi_B \) analytically and numerically to check on the accuracy of the integral approximation.

\[ \phi_A = \frac{1}{2\pi c} \int \frac{2}{t} \frac{d\Omega}{dt} dt \] (18)

\[ \phi_B = \frac{1}{2\pi c} \int_{\tau_0}^t \frac{dH}{d\tau} \frac{d\Omega}{d\tau} \ d\tau. \] (19)

The comparison is shown in Fig. 5. In this calculation, the source-receiver point is 20 km above the centre of the hemisphere with a radius of 5 km. The velocity of the medium is 6 km s\(^{-1}\). The agreement is good for \( \tau_0 + 15 \) s. The results differ because the integral is calculated numerically over a finite surface. The conclusion from this experiment is that the numerical evaluation of the integral is adequate.

We further test the code by comparing the Kirchhoff solutions with analytical high-frequency solutions which satisfy the given boundary conditions. Again we choose to calculate the potentials caused by a point source impinging on a rigid sphere; however we allow the source and receiver to separate. The Kirchhoff results are compared to those from a first motion approximation of an asymptotic form of a solution obtained by Gilbert & Helmberger (1972). They solve the problem of the reflection of an \( SH \) pulse from a fixed and rigid sphere. Fig. 6 illustrates the geometry and parameters used in this problem. The displacement as a function of spherical polar coordinates \( (r, \theta) \) obeys the following equation.

\[ \mu \nabla^2 u(r, \theta, t) - \rho \frac{\partial^2 u}{\partial t^2} (r, \theta, t) = \frac{-f(t) \delta(r-r_0) \delta(\theta - \theta^*)}{2\pi r^2 \sin \theta} . \] (20)
Figure 5. The geometry of the point source problem is shown at the top of figure. Here $a$ is the radius of the hemisphere. $r$ is the vector from the source-receiver point to an arbitrary position on the surface. $R$ is the vector extending from the centre of the hemisphere to the source-receiver point. Below are the two parts of the solution $\phi_A$ and $\phi_B$. The dotted lines are the values computed by the numerical integration. The solid lines are the values computed by the analytical Hilterman solution.

Figure 6. Geometry of the spherical problem.
Here $\mu$ is the rigidity of the medium, $\rho$ is the density, and $c$ is the wavespeed ($c = \sqrt{\mu/\rho}$). The Gilbert & Helmberger asymptotic solution for the reflected pulse is

$$\bar{u}_r = \frac{-f(s)s^{1/2}}{2\pi \mu (2\pi r_0 \sin \theta)^{1/2}} \text{Im} \int_C \gamma^{1/2} \exp \left[-s(\gamma \theta + \Psi)\right] \gamma^{1/4} \frac{(\gamma^2 - c^2)^{1/4} (\gamma^2 - c^2 - \gamma^2)^{1/4}}{(r^2/c^2 - \gamma^2)^{1/4}} \, d\gamma$$

(21)

$\bar{u}_r, f(s)$ are the Laplace transform of $u(r, \theta, t)$ and $f(t)$. The variables $\gamma$ and $\Psi$ and the path of integration are defined in Gilbert & Helmberger (1972). After performing the first motion approximation we obtain for the reflected pulse

$$u_r = \frac{-1}{4\pi \mu} \left[ \frac{a(\cos i)(\sin \rho)}{\sin \theta r_0 (R_0 r \cos \rho + R_0 \cos \rho_0)} \right]^{1/2} \bar{F} \left[ t - \left( \frac{R_0 + R}{c} \right) \right]$$

(22)

and for the direct pulse

$$u_d = \frac{1}{4\pi \mu} f(t - R'/c)/R'$$

(23)

where $R'$ is the distance between the source and the receiver.

If the source and receiver are together, it can be shown that solution (22) and a far-field first motion approximation to the Kirchhoff solution are equivalent. First let us define a geometric spreading factor $S$ such that equation (22) may be rewritten as

$$4\pi \mu u_r = -Sf[t-(R_0 + R)/c].$$

(24)

Now we examine the Kirchhoff solution when the source and receiver are together. Following an approach developed by Hilterman the first term in equation (15) is discarded as a near-field term.

$$4\pi \mu u_r(r, t) = -\frac{1}{2c} \left\{ \frac{df}{dt} \ast \frac{d\Omega}{dt} \right\}.$$  

(25)

Then a first motion approximation is made.

$$\frac{d\Omega}{dt} \bigg|_{t=\tau_0} H(t-\tau_0).$$

(26)

Then

$$4\pi \mu u_r(r, t) = -\frac{d\Omega}{dt} \bigg|_{t=\tau_0} f(t-\tau_0).$$

(27)

From equation (16) we find that

$$4\pi \mu u_r(r, t) = -\frac{a}{2R_0 + a} f(t-\tau_0).$$

(28)

If we now take the limiting value of $S$ as $R \to R_0$ and $\theta \to 0$ in equation (22), the result is

$$S = \frac{a}{2(R_0 + a)R_0}.$$  

(29)

Hence, the two far-field high-frequency solutions are formally equivalent when the source and receiver are together. A similar result is obtained by Hilterman (1975) for a rigid planar surface.
The Kirchhoff solutions are also calculated when the source and receiver are separated and the maximum amplitudes of the synthetics are compared with those amplitudes predicted by equation (22). The time history of the input source for this problem is described by the third derivative of equation (14) with the overshoot constant \( B = 2 \) and \( k = 10 \). The medium has a shear wave velocity of 5 km s\(^{-1}\); thus the wavelength of the input source is approximately 4 km. The gridlength used in these calculations is 0.1 km, making the number of grids per wavelength equal to 40. The total grid area needed to describe the surface of the sphere is 400 km\(^2\). The gridlength was selected to give an extremely fine sampling of the surface so that we may investigate the effects of a wide range of pulse widths as input time histories. Six ramp responses for this problem required 595.8 s of CPU time on a PRIME750.

Table 1 shows the parameters used in these numerical experiments and the numerical and theoretical amplitudes. The results compare favourably; the accuracy of the Kirchhoff solutions is better than or equal to 1 per cent.

The above two experiments indicate that the Kirchhoff code correctly predicts reflections from curved surfaces with large reflection coefficients and far-field receivers. Similar efforts have been carried out by workers in the field of electromagnetic scattering. Jiracek (1972) computes the amplitudes of electromagnetic waves caused by an incident transverse electric plane wave impinging on a perfectly conducting two-dimensional sinusoidal surface. He compares results obtained from a Rayleigh-FFT method, an integral equation solver, and the Kirchhoff method. The most obvious failure of the Kirchhoff technique to predict correct amplitudes occurs when the incident angle is past critical angle. This result is not surprising in light of assumptions made in estimating the boundary values on the surface. However, for angles less than critical, the Kirchhoff code is adequate and inexpensive for problems involving three-dimensional rough surfaces.

We can gain further insight into the usefulness of this formalism by comparison of the Kirchhoff solutions with optical solutions for problems of geophysical interest. First, the technique is applied to the calculation of reflections from a mountain with a buried source. In the second application, reflections from a plane where the reflection coefficient varies as a function of position on the surface are computed. In both calculations, particular attention will be paid to those propagation paths where classical ray theory fails.

The first application of the code is the calculation of the reflected potentials from an isotropic source underneath an idealized mountain (see Fig. 7). The topography of the mountain is calculated from this formula where \( z \) is the height of the surface.

\[
z = \frac{C}{2} \left[ 1 - \cos \left( \frac{2\pi}{W} \left( \left( x^2 + y^2 \right)^{1/2} - \frac{w}{2} \right) \right) \right].
\]

Table 1. Parameters used in calculation of response from a sphere with radius \( a = 10 \) km.

<table>
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<th>( R ) (km)</th>
<th>( R_a ) (km)</th>
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<th>Reflected amplitude (theoretical)</th>
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Here $C$, the maximum height, is 5 km, and $W$, the width is 33.33 km. The acoustic reflection coefficient is $-1$ everywhere on the surface. The topography is specified on a $150 \times 150$ km grid with each element of the grid being 0.5 km long.

Since the angle between the normal of the surface and the incident source ray is calculated by the code, it is simple to plot the path of the reflected rays. These rays are traced for two depths below the baseline of the free surface. In the first plot, Fig. 7, we can see the rays from a source at 10 km which reflect off the free surface and travel to a depth of 50 km. This figure shows the position of the ray caustic, foci, and the shadow zone caused by the convex shape of the mountain. These features will influence the waveforms considerably.

In Fig. 8 the rays are traced to a depth of 1000 km. The Kirchhoff responses are calculated at this depth at the marked positions which vary from 0 to 750 km horizontally. Upon closer inspection of Fig. 8, one can see slight asymmetries in the location of the rays with respect to the position of 0 km. These asymmetries are caused by the discretization of the surface of the mountain. The error in the value of the computed normal derivatives introduces 10 km of uncertainty into the location of the rays at this depth.

The calculated reflected responses are shown in Fig. 9. These pulses are convolutions of the ramp response with the Haskell function with the parameter $B = 0$ and $k = 25$. Hence the number of grids per wavelength is 10. As the horizontal distance of the receiver changes, we see systematic waveform variations which can be interpreted in terms of rays interacting with caustics. In the ranges of 0, 50, 100 and 150 km the synthetics have complicated pulse shapes caused by the interference of three families of rays. The first arrival is a simple pulse with a $\pi$ phase-shift which is a consequence of the reflection off the free surface. The

![Figure 7](https://academic.oup.com/gji/article-abstract/72/1/237/666446)

**Figure 7.** The rays which reflect off a mountain described by equation (30). The source is 10 km below the baseline indicated by the dashed line. Also shown are the caustics formed by such a mountain and the geometric focus.
second pulse is a reflected ray with a path which is tangent to the caustic formed by the mountain. This path results in a \( \pi + \pi/2 \) phase-shift of the pulse. The third arrival reflects off the mountain and travels through the geometric focus caused by the mountain; thus the phase shift of this arrival is \( \pi + \pi \). The maximum amplitude of these four distances is controlled by the interference of these rays. Clearly the high amplitude and the simple pulse of the first synthetic at 0 km is a result of the constructive interference of the first two rays. Past 150 km, the latter two arrivals arrive closely in time and their interference controls the amplitude and frequency content of the second pulse on the record. From Fig. 8, it is clear that a ray interpretation of pulses on records past 400 km is no longer valid. Ray theory predicts only one reflected pulse because the mountain creates a shadow zone; yet one sees two distinct pulses predicted by the Kirchhoff method. The second phase-shifted pulse

Figure 8. The rays which reflect off the mountain are traced to a depth of 1000 km.

Figure 9. The responses convolved with a modified Haskell source for receivers all at a depth of 1000 km and at horizontal distances which vary from 0 to 750 km away from the centre position.
decreases in amplitude and frequency content. As the horizontal distance of the receivers increases, the amplitude of the first reflection becomes the maximum amplitude of the record. If one calculates the maximum amplitudes of reflections off a plane for the same source-receiver geometry, one can see that the two sets of amplitudes merge. This behaviour is shown in Fig. 10 where the amplitudes as a function of horizontal distance for the two geometries have been calculated. The solid line shows the decay of amplitudes calculated for a planar surface. The triangles are amplitudes calculated for a mountain with a height of 2 km and a width of 10 km. The two sets of values coincide past 800 km.

The Kirchhoff results for this experiment are gratifying because one does not expect infinite amplitudes or abrupt shadow zones predicted by optics in real physical systems. This experiment also demonstrates that this technique produces the requisite phase shifts in an extremely simple manner unlike existing ray tracing techniques which must track the behaviour of a ray tube along the propagation path.

The second application of the code is the calculation of reflections off an acoustic planar free surface where the reflection coefficients are allowed to vary as a function of position on the surface. These calculations demonstrate the flexibility of the code and again emphasize the differences between the Kirchhoff solution and optics. (The wavespeed of the medium is 6 km s⁻¹ for all the following calculations.)

Initially one assumes that the reflection coefficient is zero for elements of the plane within a circular aperture of radius \( R \) and is \(-1\) for elements outside this aperture. The source is directly underneath the centre of the hole. From ray theory one expects that no reflected energy will arrive at a receiver directly underneath the source. Yet one calculates non-zero amplitudes for both long- and short-period WWSSN seismograms from the Kirchhoff code. These seismograms are displayed in Fig. 11 as a function of the radius of the aperture for a receiver 1000 km below the surface. Only the reflected \( pP \) phase has been calculated and convolved with WWSSN instruments.

This pulse is systematically delayed as the radius of the hole increases from 1 to 5 km. There is no change in the waveforms. Only the amplitudes of both sets of seismograms decrease. However, the amplitude of the seismograms for an aperture with a 5 km radius is more than half the magnitude of the amplitude of a \( pP \) phase reflected from a free surface without a hole. Clearly, then, ray theory is not a good approximation to the solution of this problem.
In addition, ray theory fails to predict any dependence of the reflected amplitudes on frequency. Intuitively one expects, for an aperture problem that the higher frequencies of a broadband signal will be reduced relative to the lower frequencies after reflection. This hypothesis is tested by calculating the reflected responses from sources of differing frequency content. In the following calculations the parameter $B$ of the modified Haskell source representation equals zero; however $k$ varies from 5 to 25. An increase in $k$ broadens the bandwidth of the incident signal (von Seggern & Blandford 1972). One computes two responses for a given source pulse. The first response is a reflection off the plane with a hole and the second is a reflection off the plane without a hole. The amplitude of the latter response has no frequency dependence; hence if the reflection from a hole has no frequency dependence, one predicts that the ratio of the amplitudes of the two reflections will be constant as a function of the parameter $K$. However, if there is a frequency dependence, the ratios should vary systematically.

From numerical experiments one confirms the frequency dependence of the reflected amplitudes. This result is shown in Fig. 12. Specifically, the amplitudes of the reflections from the aperture are always smaller than the amplitudes of planar reflections. Also the ratio of the two responses decreases when $B$ decreases if the receiver is located at position 2. This behaviour is displayed for apertures with three radii, 2, 3 and 4 km.

However, this behaviour does not occur if the receiver is located at position 1, 1000 km directly below the source. The ratios are approximately constant as $B$ decreases. This
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Figure 12. Ratio of maximum amplitude of reflection from a surface with and without a hole. The open circles are ratios measured at a receiver directly underneath the source and the centre of the aperture. The triangles are ratios measured at a receiver 68 km away from the centre position. Both receivers are 1000 km below the free surface. The radius of the hole $R$ is 2, 3 and 4 km. The ratios are plotted for these radii as a function of $k$ (a measure of the bandwidth).

Observation suggests that, here, the reflected amplitudes from the aperture do not depend on the bandwidth of the signal. Although this result is not intuitive, it is typical of analytical solutions to Fraunhofer diffraction from apertures in an opaque screen (Born & Wolf 1964). For example, the solutions of the intensity of light transmitted through a rectangular aperture has the function form of

$$\frac{\sin(A\omega x)}{A\omega x} \cdot \frac{\sin(B\omega y)}{B\omega y}$$

where $A$ and $B$ are geometric constants and $x$ and $y$ are the rectangular coordinates of the position of the receiver. The limit of the above function as $x$ and $y$ approach zero is 1 and is independent of the value of $\omega$.

Such experiments, which vary the reflection coefficients on the free surface, may be applicable to the analysis of the effects of spallation generated by nuclear blasts on teleseismic $P$-wave reflections. Spall is the physical separation of near surface layers during the explosion. Material above the bomb is either ejected or returns to produce an impact signal on near-field instruments. This non-linear and non-elastic behaviour of the material surrounding the source may result in amplitude and travel-time anomalies of reflected $sP$ and $pP$ phases.

The model used to simulate spall is one where the reflection coefficient is cosine taper; that is

$$R = \begin{cases} \cos \left( \frac{\pi x}{2R} \right) - 1 & \text{if } x < R \\ -1 & \text{if } x > R. \end{cases}$$
Here $x$ is the distance from the source epicentre on the free surface. One chooses this function for the reflection coefficient to simulate material reflecting more energy as the distance from the source increases. The model introduces complications into the short-period waveforms but only broadens the long-period waveforms. This effect and the source-receiver geometry is illustrated in Fig. 13. The geometry is the same as used in the aperture calculations. Unlike the first model, this model causes the amplitudes of both the long- and short-period reflections to decay quite rapidly. The amplitude decay is greater for the short- than for the long-period reflections. Hence the long-period energy is insensitive to the perturbation of the reflection coefficients relative to the short-period energy.

The source-receiver geometry is changed for this particular model to test the hypothesis that asymmetries of spalling with respect to the source location can introduce observable azimuthal variations of amplitudes and waveforms of teleseismic records of nuclear blasts. Such variations have been documented for teleseismic recordings of Nevada Test Site blasts (Helmberger & Hadley 1981). In addition, photographs of collapse craters from NTS blasts suggest that processes like spalling and subsidence occur along pre-existing planes of weakness which are not symmetrical with respect to the emplacement hole (Springer & Kinnaman 1971). Fig. 14 shows the results for stations at three azimuths. The source is placed 2 km to the right of the centre of the spall aperture and 1 km below the free surface. The receivers are all at horizontal distance which corresponds to a take-off angle of $20^\circ$ for the direct $P$-wave. One sees azimuthal variations of waveform and amplitudes for both long- and short-period reflections. The amplitude variations are not large but the waveform changes are dramatic for short-period records.
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This model is crude and consequently does not demonstrate that spalling affects teleseismic reflections. However, Shumway & Blandford (1980) report observing a systematic delay in arrival times of $pP$ phases from explosions. The simple aperture experiment provides an explanation for that delay. In addition the Kirchhoff technique allows one to specify more realistic dynamical information on the free surface and calculate more realistic models in a straightforward manner.

Conclusions

A numerical procedure has been presented for the evaluation of the Kirchhoff–Helmholtz integral assuming the tangent plane hypothesis. The method is a high-frequency one and produces results which compare well with existing asymptotic first motion solutions. The technique has been applied to two problems and compared to classical ray theory results. First, the reflections off an idealized mountain are calculated and have phase-shifts consistent with those predicted by optics; however, the amplitudes at triplications are finite unlike the classical ray result. In addition diffracted pulses are produced in the shadow zones. The second application is the calculation of reflections where the reflection coefficients vary as a function of position. For a hole in the free surface, the Kirchhoff method produces reflections where ray theory predicts no reflections. The method also produces amplitudes which are frequency-dependent. The results are applied and extended to model the effects of spallation on teleseismic reflections. Travel-time delays and amplitude anomalies are predicted. These anomalies are consistent with observations although the observations are not modelled.

In conclusion, the method has a broad range of applications. The method is inexpensive to run for modelling two- and three-dimensional rough surfaces. Although the method is appropriate for narrow angles of reflections and acoustic reflections, its range of applicability can be extended by introducing new time functions on the boundary. The code can
also be coupled with existing propagational techniques such as ray-tracing, Cagniard-de Hoop methods, or full wave theory. This coupling will enable one to handle more complicated and relevant seismological problems.

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References


