Dynamics of Higher Dimensional Universe Models

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(Received March 3, 1984)

Dynamics of higher dimensional space defined by a product of constant curvature subspaces is studied using the classical Einstein equation. We deal with mainly vacuum case with cosmological constant and check the behaviour of timelike subspace as well as spacelike one. The Kasner-de Sitter solution which isotropizes the expansion is obtained. The bouncing of the contracting subspace may occur for spacelike-negative curvature or timelike-positive curvature cases.

§ 1. Introduction

Recently, the generalized Kaluza-Klein theory has received much attention. In this theory, higher dimensional space is supposed to have separated into two subspaces, the conventional spacetime and the compactified internal space. The gauge fields are considered to have the origin endowed by the symmetry of this internal subspace, whose size is related to the gauge coupling constant and will remain constant with time to ensure its constancy.5)

On the other hand, our universe is expanding and this cosmic expansion is believed to be traced back into the Planck era. In such era, the dynamics of these two subspaces might have coupled to each other and the origin of the internal space would be searched for through those dynamics. Thus, the problem of Kaluza-Klein cosmology has arisen, with which this paper is concerned.

One of the difficulties in this problem is the quantum effects such as gravitational Casimir effect, particle creation and other loop corrections. These effects will be crucial to this problem but the unambiguous estimation for them seems to be still under investigation. Therefore, in this paper, we intend to study the classical dynamics without the quantum effect. Our purpose is not to present a realistic model but to clarify a starting case, from which a realistic model should be derived adding many other corrections.

We study the dynamics of the space given by a product of constant curvature subspaces, both spacelike and timelike subspaces, using the Einstein equation with cosmological constant. For the vacuum and zero curvature case, the equation is simply integrated to give a solution which evolves from the Kasner solution to the de Sitter solution. This implies that the dimensional reduction does not occur for a positive cosmological constant. If the curvature term is included, its effect will dominate as the subspace contracts. A possibility of the bouncing will be discussed. The conditions for the constant size subspace will be also discussed.

In §2, the model space and the Einstein equation are presented. In §3, the dynamics is studied for the vacuum and zero curvature case and the Kasner-de Sitter solution is derived. In §4, the behaviour of the dynamics is clarified in detail and the difference from the mixmaster model is explained. In §5, some effects of the terms neglected in §3 are


\section{Model space and basic equations}

As the model space of \( N + 1 \) dimension, we take a product of the constant curvature subspaces with dimension \( n_a \) and \( \sum n_a = N \). The metric is written as

\[
-g_{AB} = \begin{pmatrix}
-1 & 0 & 0 \\
\varepsilon_1 a_1(t)^2 \tilde{g}_{\alpha \alpha}^{(1)} & \varepsilon_2 a_2(t)^2 \tilde{g}_{\beta \beta}^{(2)} & 0 \\
0 & 0 & \ddots
\end{pmatrix},
\]  

(2.1)

where each subspace is characterized by the signs of the signature and the curvature; spacelike \( \varepsilon_a = 1 \) and timelike \( \varepsilon_a = -1 \) and positive \( K_a = 1 \) and negative \( K_a = -1 \) curvatures. The Riemann tensor of a subspace is given as \( R^{(a)}_{ijkl} = \frac{K_a}{N}(\tilde{g}_{ij}\tilde{g}_{kl} - \tilde{g}_{ik}\tilde{g}_{jl})^{(a)} | a_a^2 \).

Here, we have not restricted the subspace to be spacelike only for generality. The positive definiteness of the Yang-Mills field term breaks down for the timelike subspace and it has been taken for granted that the subspace is spacelike. We shall put aside this problem for a while. We are rather interested in checking how differently the spacelike subspace and the timelike subspace behave in dynamics.

The Ricci tensors are computed as

\[
R_i^0 = \sum_n n_a \left( \frac{\dot{a}}{a} \right)_a,
\]

(2.2)

\[
R_{ij}^{(a)} = \left( \left( \frac{\dot{a}}{a} \right)_a + (n_a - 1) \frac{\dot{a}^2 + \varepsilon K}{a^2} \right) + \sum_{\beta \neq a} n_{\beta} \left( \frac{\dot{a}}{a} \right)_\beta \left( \frac{\dot{a}}{a} \right)_\rho \delta_{,j},
\]

(2.3)

The high dimensional Einstein equation

\[
R_{AB} - \frac{1}{2} g_{AB} R + \Lambda g_{AB} = -x T_{AB}
\]

(2.4)

becomes

\[
\sum_n n_a \left( \frac{\dot{a}}{a} \right)_a (\dot{a}^2 + \varepsilon K) + \sum_{\beta \neq a} n_{\beta} \left( \frac{\dot{a}}{a} \right)_\beta \left( \frac{\dot{a}}{a} \right)_\rho - \Lambda = x T^{(a)},
\]

(2.5)

\[
(n_a - 1) \left( \frac{\dot{a}}{a} \right)_a + \sum_{\beta \neq a} n_{\beta} \left( \frac{\dot{a}}{a} \right)_\beta + \frac{(n_a - 1)(n_a - 2)}{2} \frac{\dot{a}^2 + \varepsilon K}{a^2} a + \sum_{\beta \neq a} n_{\beta} \left( \frac{\dot{a}}{a} \right)_\beta \left( \frac{\dot{a}}{a} \right)_\rho
\]

\[
+ \sum_{\beta \neq a} \sum_{\gamma \neq a} n_{\beta} n_{\gamma} \left( \frac{\dot{a}}{a} \right)_\beta \left( \frac{\dot{a}}{a} \right)_\gamma - \Lambda = -\varepsilon x T^{(a)},
\]

(2.6)

where the stress tensor \( T_{AB} \) is assumed to be isotropic in each subspace as follows,
\[ T^{a}_{b} = \begin{pmatrix} T^{(0)}_{b} & -\varepsilon_{1} T^{(1)} b_{a}^{b} \\ -\varepsilon_{2} T^{(2)} b_{a}^{n} & \ldots \end{pmatrix} \] (2.7)

For the vacuum case \( T^{a}_{b} = 0 \), the equations are reduced to

\[ \sum_{a} n_{a} \left( \frac{\dot{a}}{a} \right)_{a} = \frac{2\Lambda}{N-1} \] (2.8)

and

\[ \left( \frac{\dot{a}}{a} \right)_{a} + \left( \sum_{a} n_{a} \left( \frac{\dot{a}}{a} \right)_{a} \right) = \frac{2\Lambda}{N-1} - (n_{a} - 1) \left( \frac{\varepsilon K}{a^{2}} \right)_{a}. \] (2.9)

Taking the summation of (2.9), we get

\[ \dot{X} + X^{2} = \frac{N}{N-1} \frac{2\Lambda}{\sum_{a} n_{a} (n_{a} - 1)} \left( \frac{\varepsilon K}{a^{2}} \right)_{a}, \] (2.10)

where \( X = \sum_{a} n_{a} \left( \frac{\dot{a}}{a} \right)_{a} \). Substituting (2.8) into (2.10), an integral equation is obtained as

\[ X^{2} = \sum_{a} n_{a} \left( \frac{\dot{a}}{a} \right)_{a}^{2} = 2\Lambda - \sum_{a} n_{a} (n_{a} - 1) \left( \frac{\varepsilon K}{a^{2}} \right)_{a}. \] (2.11)

If \( K_{a} = 0 \) for all \( a \), Eq. (2.10) is integrable and, by substituting the solution of \( X \) into (2.9), \( a_{a} \) is obtainable from

\[ \left( \frac{\dot{a}}{a} \right)_{a} + X \left( \frac{\dot{a}}{a} \right)_{a} = \frac{2\Lambda}{N-1}. \] (2.12)

§ 3. Kasner-de Sitter solution

We investigate the solution of (2.11) in this section.\(^{9)\) For \( K_{a} = 0 \), Eq. (2.10) is written as

\[ \dot{X} + X^{2} = \gamma^{2} \lambda, \] (3.1)

where \( \lambda = \lambda |\lambda| \) and \( \gamma = [2N|\lambda|/(N-1)]^{1/2} \). For \( K_{a} = 0 \), Eq. (2.11) is

\[ X^{2} - \sum_{a} n_{a} \left( \frac{\dot{a}}{a} \right)_{a}^{2} = 2\lambda |\lambda|. \] (3.2)

The discussion in this section is the same both for spacelike and timelike subspaces.

The solutions of (3.1) are listed up as follows:

(i) \( \lambda = 0 \)

(a) \( X = 0 \), \( (3.3) \)

(b) \( X = t^{-1} \), \( (3.4) \)

(ii) \( \lambda = +1 \)

(a) \( X = \pm \gamma \), \( (3.5) \)

\(^{9)\) The author is aware of the paper in Ref. 9), where a similar analysis was done in the case of the modified Einstein theory based on the Weitzenböck spacetime with absolute parallelism.
(b) \( X = \gamma \coth \gamma t \), \hfill (3.6)

(c) \( X = \gamma \tanh \gamma t \), \hfill (3.7)

(iii) \( \lambda = -1 \)

(a) \( X = \gamma \cot \gamma t \), \hfill (3.8)

where the origin of \( t \) has been chosen appropriately.

For \( X \) of (3.3), \( \dot{a}_a = 0 \) and the space is just \((1 + N)\) dimensional Minkowski space.

For \( X \) of (3.4), we get

\[
\left( \frac{\dot{a}}{a} \right)_a = \frac{P_a}{t} \tag{3.9}
\]

with constants \( P_a \) satisfying the conditions

\[
\Sigma_a n_a P_a = \Sigma a P_a^2 = 1 \tag{3.10}
\]

as seen from (3.2) and (3.4). The \( a_a \) is given as

\[
a_a = A_a t^P_a, \tag{3.11}
\]

\( A_a \) being integration constants. This is a generalization of the Kasner solution known in three space.

For \( X \) of (3.5), Eq. (2.12) derives

\[
\left( \frac{\dot{a}}{a} \right)_a = \pm \frac{\gamma}{N} - C_a e^{\mp \gamma t},
\]

where constants \( C_a \) are required to satisfy \( \Sigma_a C_a n_a = 0 \) from (3.5). But, another condition (3.2) requires \( \Sigma_a C_a^2 n_a = 0 \) and \( C_a = 0 \) for all \( a \). Thus, \( a_a \) is given as

\[
a_a = A_a \exp \left[ \pm \frac{2A}{\sqrt{N(N-1)}} t \right], \tag{3.12}
\]

which is the de Sitter solution.

For \( X \) of (3.6), we get

\[
\left( \frac{\dot{a}}{a} \right)_a = \frac{C_a}{\sinh \gamma t} + \frac{\gamma}{N} \coth \gamma t
\]

from (2.12) and \( a_a \) is

\[
a_a = A_a \left( \sinh \frac{\gamma t}{2} \right)^{P_a} \left( \cosh \frac{\gamma t}{2} \right)^{2(N-P_a)}
\]

\[
P_a = \frac{C_a}{\gamma} + \frac{1}{N}, \tag{3.13}
\]

which obeys the same condition of (3.10). This solution is reduced to the Kasner solution in the limit \( \gamma t \ll 1 \) and it does to the de
Sitter solution for $\gamma t \gg 1$. We call this solution as the Kasner-de Sitter solution.

For $X$ of (3·7), a real solution does not exist, which will be illustrated in §4.

For $X$ of (3·8), the solution is

$$a_\alpha = A_\alpha \left( \sin \frac{\gamma t}{2} \right)^{P_\alpha} \left( \cos \frac{\gamma t}{2} \right)^{2N-P_\alpha},$$

(3·14)

where $P_\alpha$ also obeys the condition (3·10). This solution is reduced to the Kasner solutions of $a_\alpha \sim t^{P_\alpha}$ for $\gamma t \ll 1$ and of $a_\alpha \sim |\pi - \gamma t|^{2N-P_\alpha}$ for $|\pi - \gamma t| \ll 1$.

§ 4. Property of the Kasner-de Sitter solution

Taking such a simple case as

$$ds^2 = dt^2 - \varepsilon_1 a(t)^{2n_1} \sum_{a=1}^{n_1} dx^a dx^a - \varepsilon_2 b(t)^{2n_2} \sum_{m=1}^{n_2} dx^m dx^m,$$

(4·1)

we illustrate a general feature of the solution obtained in §3. From (3·10), $P_\alpha$ is obtained as

$$P_\alpha = \frac{n_1 \pm \sqrt{n_1 n_2 (N-1)}}{n_1 N}$$

and

$$P_\alpha = \frac{n_2 \pm \sqrt{n_1 n_2 (N-1)}}{n_2 N}.$$  

(4·2)

Writing $x = \dot{a}/a$ and $y = \dot{b}/b$, Eq. (3·2) becomes

$$(n_1 x + n_2 y)^2 - (n_1 x^2 + n_2 y^2) = 2\lambda |\lambda|,$$

which represents two straight lines for $\lambda = 0$ and a hyperbola for $\lambda = \pm 1$. On the other hand, the solutions for $X$ describes a moving straight line of $n_1 x + n_2 y = X(t)$ and the cross points give the solution as shown in figure. For the Kasner solution, the cross points approaches the origin of this plane. For the cases of (3·6) and (3·8), the cross points approaches the vertex of the hyperbola. This figure will illustrate why the case of (3·7) does not have a solution, because $X$ is restricted in $-\gamma < X < \gamma$ and there is no cross point with the hyperbola. The behaviour of the time reversal solutions for the above solutions will be understood from this figure.

As first pointed out by Chodos and Detweiler, the Kasner solution of (3·11) might describe the dimensional reduction of the internal space, since some of $P_\alpha$ is negative. However, for $\Lambda > 0$, the Kasner type anisotropic behaviour ceases when the expansion rate has decreased. Then, all the subspaces turn to expand in isotropic way as the de Sitter solution. Maeda has pointed out that the isotropization of the Kasner solution may occur through the particle creation. The above solution teaches us another such isotropization mechanism. Therefore, the dimensional reduction should be considered avoiding or overcoming these effects.
§ 5. Discussion on other effects

(i) Curvature effect

If we include the curvature term, the dynamics is no longer described by the solutions given in §3. The integral equation (2·5) is written for \(a=a_1\) and \(b=a_2\) as

\[
\frac{n_1(n_1-1)}{2} \frac{\dot{a}^2 + \varepsilon_1 K_1}{a^2} + \frac{n_2(n_2-1)}{2} \frac{\dot{b}^2 + \varepsilon_2 K_2}{b^2} + n_1 n_2 \left( \frac{\dot{a}}{a} \right) \left( \frac{\dot{b}}{b} \right) = 1. \tag{5·1}
\]

Writing \(a= Ae^\alpha\) and \(b= Be^\beta\) and taking the principal axis transformation from \((\alpha, \beta)\) to \((\bar{\alpha}, \bar{\beta})\), the relation (5·1) is rewritten as

\[
\lambda_1 \left( \frac{\ddot{\bar{\alpha}}}{dt} \right)^2 - \lambda_2 \left( \frac{\ddot{\bar{\beta}}}{dt} \right)^2 + U = \Lambda, \tag{5·2}
\]

where

\[
\lambda_{1,2} = \sqrt{(n_1^2 + n_2^2 - N)^2 + 4n_1 n_2 (N+1) \pm (n_1^2 + n_2^2 - N)} \tag{5·3}
\]

and

\[
U = \frac{\varepsilon_1 K_1 n_1(n_1-1)}{a^2} + \frac{\varepsilon_2 K_2 n_2(n_2-1)}{b^2}.
\]

As \(\lambda_{1,2}\) are positive, the “kinetic” term is Lorenzian. Therefore, in the limit \(\gamma t \ll 1\), the dynamics of \(\bar{\alpha}\) and \(\bar{\beta}\) tend to a “free” motion such as

\[
\lambda_1 \left( \frac{\ddot{\bar{\alpha}}}{dt} \right)^2 \approx \lambda_2 \left( \frac{\ddot{\bar{\beta}}}{dt} \right)^2 \gg |U|, |\Lambda| \tag{5·4}
\]

without colliding the potential barrier: \(|Pa|<1\) from (3·10) and \(|U| \sim t^{-2|\varepsilon|}\) for \(P_a<0\) while the kinetic terms change like \(t^{-2}\). This behaviour is essentially different from the chaotic oscillation seen in the mixmaster model.¹⁰

The other equations (2·6) for \(a\) and \(b\) are written as

\[
\frac{d^2 a}{d\tau^2} = -\varepsilon_1 K_1 n_1(n_1-1)(a^n b^m)^2 \frac{a^n b^m}{a^2}
\]

and

\[
\frac{d^2 b}{d\tau^2} = -\varepsilon_2 K_2 n_2(n_2-1)(a^n b^m)^2 \frac{a^n b^m}{b^2}, \tag{5·5}
\]

where \(\tau\) is a new time coordinate defined as \(a^n b^m d\tau = dt\) and \(\Lambda\) is taken zero. In the “free” motion stage described by the Kasner solution, \((a^n b^m)^2 = t\). Then, the curvature term on the right-hand side increases as \(t^{(1+|\varepsilon|)}\) with \(t\) and it will dominate for \(t>a, b\) either contracting or expanding subspaces. Consider the case where \(a\) is decreasing with \(t\) and is smaller than \(b\). If \(\varepsilon_1 K_1<0\), the contraction will be accelerated much more by the curvature. A necessary condition of the bouncing is \(\varepsilon_1 K_1<0\), that is, spacelike-negative or timelike-positive. If \(a\) is increasing and much smaller than \(b\), a necessary condition for the turn to contraction is \(\varepsilon_1 K_1>0\). Further details of the dynamics will be published.
elsewhere.

(ii) Static subspaces

The stationary conditions of the two subspaces, i.e., \( \dot{a}_a = \ddot{a}_a = 0 \), are written as

\[
(n_a - 1) \left( \frac{\varepsilon K_a}{a^2} \right)_a = \chi \{ T^{(a)} + \sum_a \varepsilon_a T^{(a)} \}
\]

and

\[
\Lambda = \frac{\chi}{2} \left( (N - 2) T^{(0)} + \sum_a n_a \varepsilon_a T^{(a)} \right),
\]

from (2.5) and (2.6). These conditions are the generalization of that for the Einstein static universe. For the vacuum case, both the curvature and \( \Lambda \) must be zero.

Next, we consider the stationary condition of the extra dimensional subspace, e.g., \( \dot{b} = \ddot{b} = 0 \) in the case of two subspaces. This condition is written as

\[
n_1 (n_1 - 1) \frac{\ddot{a}}{a} = 2 \Lambda - \chi \left( (N - 2) T^{(0)} + n_1 \varepsilon_a T^{(1)} + n_2 \varepsilon_2 T^{(2)} \right),
\]

\[
\frac{n_1 (n_1 - 1) \ddot{a}^2 + \varepsilon_1 K_1}{a^2} = \frac{n_1 (n_1 - 1) \ddot{a}^2}{a^2} + \frac{\chi}{2 (N - 1)} \left( (2 n_1 + 3 n_2 - 2) T^{(0)} \right)

+ n_1 n_2 \varepsilon_1 T^{(1)} - n_2 (n_1 - 1) \varepsilon_2 T^{(2)}
\]

and

\[
(N - 1) (n_2 - 1) \frac{\varepsilon_2 K_2}{b^2} = 2 \Lambda - \left( T^{(0)} + n_1 \varepsilon_1 T^{(1)} - (n_1 - 1) \varepsilon_2 T^{(2)} \right).
\]

For the vacuum, the static space has been completely decoupled from the dynamics of another subspace as follows,

\[
\frac{\ddot{a}^2 + \varepsilon_1 K_1}{a^2} = \frac{2 \Lambda}{n_1 (N - 1)}
\]

and

\[
(n_2 - 1) \frac{\varepsilon_2 K_2}{b^2} = \frac{2 \Lambda}{N - 1}.
\]

If not vacuum, the second term on the right-hand side of (5.8) will be changing with time since

\[
\dot{T}^{(0)} + n_1 \frac{\ddot{a}}{a} (T^{(0)} + \varepsilon_1 T^{(1)}) = 0
\]

from the conservation law \( T^{AB} \); \( \varepsilon = 0 \). Therefore, a stationary solution would not exist in this case.

(iii) Negative curvature and timelike subspace

In the Kaluza-Klein theory, the invisibility of the extra subspace is assured by its compactness with the Planck length and the low energy theorem. The compactness does not necessarily require the positive curvature. Even flat or negative curvature space can
be compactified, though their topology may not be simple like $S^n$.\textsuperscript{11} Then, the sign of $K_a$ would be meaningful for all the cases $\pm 1$ and 0.

On the other hand, the signature $\varepsilon_a$ is considered to be definitely spacelike, $\varepsilon_a=1$, in order to have a right sign of the Yang-Mills Lagrangian. However, the invisibility at the low energy would be assured even for timelike subspace if it is compactified. Therefore, we should investigate such problem why the extra subspaces are all spacelike or what happens if there exist both spacelike and timelike extra subspaces. For that purpose, the study of the timelike subspace would be necessary.

References

5) S. Randjbar-Daemi, A. Salam and J. Strathdee, ICTP Preprint IC/83/208.