Conservation of mass in tidal loading computations

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Summary. A re-examination of methods for including mass conservation in tidal loading shows that the spherical harmonic correction of Farrell is incorrect. The effect of unconserved mass for a nearly ocean-covered earth shows that the proper spherical harmonic expansion of the Newtonian Green function is the average of the internal and external expansions.

Introduction

Calculations of the effects of ocean tides upon Earth tides, commonly called tidal loading, are now commonplace. Such calculations often use numerical models of the ocean tide. For a variety of reasons (Cartwright 1977, section 6; Schwiderski 1980, section 4) these models may not conserve mass. Farrell (1972a) improved the fit of computed to observed gravity tides when he added mass conservation to a model that had lacked it. Since then loading calculations have customarily forced mass conservation of the ocean tide. My purpose here is to review the ways in which this may be done. One common method, suggested by Farrell (1973), turns out to be incorrect. Examining it throws light on what expression to use for the direct gravitational attraction of the ocean tide, an issue discussed by Groten & Brennecke (1973, 1974), Merriam (1974, 1980) and Pekeris (1978).

Any ocean tide model can easily be forced to conserve mass, though perhaps at the cost of violating other constraints. A numerical model will comprise cells within which the complex amplitude is constant. Letting this amplitude be \( H_i \) in the \( i \)th cell (with area \( A_i \)), we need merely change it to

\[
H_i - b(H_i) \left( \frac{\sum_i H_i A_i}{\sum_i b(H_i) A_i} \right)
\]

where \( b \) is any function; the sum may be over all ocean cells or done separately for different oceans (Warburton, Beaumont & Goodkind 1975). Melchior, Moens & Ducarme (1980) use \( b(x) = |x| \), making a larger correction where the tide is larger. More often, \( b(x) \) is taken to be 1, which corresponds to removing a uniform sheet of water. I shall take this simple method as the one to compare others with.
Mass conservation through spherical harmonics

We concern ourselves with ocean loading for gravity and vertical displacement. These loads are

\[ L = \int_0^{2\pi} \int_0^{\pi} G(\Delta)H(\theta, \lambda) \rho a^2 \sin \theta \, d\theta \, d\lambda \]

where \( H \) is the complex amplitude of the tidal constituent at colatitude \( \theta \) and longitude \( \lambda \), \( \rho \) is the density of seawater (nearly 1020 kg m\(^{-3}\)), \( a \) the radius of the Earth (6371 km) and \( G \) the Green function for mass loading, which depends only on \( \Delta \), the angular distance from the point of observation \((\theta', \lambda')\) to \((\theta, \lambda)\). We may write \( G(\Delta) \) as

\[ G(\Delta) = \sum_{n=0}^{\infty} c_n P_n(\cos \Delta) \]

and then use the addition rule for spherical harmonics to get

\[ L = 4\pi \rho a^2 \sum_{n=0}^{\infty} \frac{c_n}{2n + 1} \sum_{m=-n}^{n} Y_{nm}(\theta', \lambda') H_{nm} \]

where the \( Y_{nm} \)s are fully normalized spherical harmonics and are the spherical harmonics of the tide height. If the tide conserves mass, the integral of the tide height over the oceans (and therefore the Earth) will be zero. \( \rho_\infty \) is then zero, and the sum (2) starts at \( n = 1 \). Farrell (1973) suggested setting \( c_0 \) to 0 instead. This is not correct. Suppose the non-conserved mass to be distributed over the oceans as a sheet of thickness \( H' \); the load from this is

\[ 4\pi \rho a^2 \sum_{n=0}^{\infty} \frac{c_n}{2n + 1} \sum_{m=-n}^{n} Y_{nm}(\theta', \lambda') H' C_{nm} \]

where the \( C_{nm} \) are the spherical harmonic coefficients of the ocean function (Balmino, Lambeck & Kaula 1973). This sum is non-zero even if \( c_0 = 0 \). (Farrell 1973 pointed this out, but did not evaluate the degree of inaccuracy, and it seems to have been forgotten since.) Another way of seeing this is to note that the \( n = 0 \) part of the Green function (1) does not depend on distance: all loads, near or far, have the same effect. A uniform sheet of mass over the oceans would then produce the same effects at all places, whereas in reality these effects must depend upon location.

To see the size of the error, we first find the size of the correction made by putting \( c_0 \) to 0. Using (3), dividing out the layer thickness, and letting \( \gamma \) be the fraction of the surface that is ocean-covered, we get

\[ 4\pi \rho a^2 c_0 \gamma \]

For gravity, the coefficients in (1) are

\[ c_n = \frac{f}{a^2} \left[ -\frac{1}{2} - (n + 1) k_n + 2h_n \right] \]
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Table 1. Loads from a uniform ocean.

<table>
<thead>
<tr>
<th>Place</th>
<th>N latitude</th>
<th>E longitude</th>
<th>Gravity (µgal cm⁻¹)</th>
<th>Vertical displacement (mm cm⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice Springs</td>
<td>-23.72</td>
<td>133.83</td>
<td>-0.366</td>
<td>-0.134</td>
</tr>
<tr>
<td>Boulder</td>
<td>40.03</td>
<td>-105.15</td>
<td>-0.324</td>
<td>-0.084</td>
</tr>
<tr>
<td>Brussels</td>
<td>50.80</td>
<td>4.39</td>
<td>-0.331</td>
<td>-0.238</td>
</tr>
<tr>
<td>Nairobi</td>
<td>-2.60</td>
<td>36.85</td>
<td>-0.326</td>
<td>-0.191</td>
</tr>
<tr>
<td>Pinyon Flat</td>
<td>33.61</td>
<td>-116.46</td>
<td>-0.385</td>
<td>-0.264</td>
</tr>
<tr>
<td>La Jolla</td>
<td>32.80</td>
<td>-117.15</td>
<td>-0.412</td>
<td>-0.348</td>
</tr>
<tr>
<td>South Pole</td>
<td>-90</td>
<td>-</td>
<td>-0.347</td>
<td>-0.008</td>
</tr>
</tbody>
</table>

Positive values for decrease in g and upward motion. Zero harmonic correction for land–sea model used ($\gamma = 0.6865$) would be $-0.452 \mu$gal cm⁻¹, $-0.513$ mm cm⁻¹.

where $f$ is the Newtonian constant of gravitation, and the $h_1$ and $k_2$ are the load Love numbers. As $k_0 = 0$, the correction made by setting $c_0$ to 0 is

$$-4\pi pf\gamma (\frac{1}{2} - 2h_0)$$

which for $h_0 = -0.1346$ and $\gamma = 0.71$ is $-4.67 \times 10^{-7}$ s⁻², or $-0.467$ µgal cm⁻¹. While this may seem small, in the interiors of continents the total gravity load is only 1–2 µgal, so that a few centimetres of unconserved mass can have a large effect on the computed results. The actual tide is larger than a few centimetres, but its contributions nearly cancel, whereas all parts of the uniform sheet are in phase. For vertical displacement $c_n = f h_n / g a$, with $g = 9.82186$ m s⁻², giving a correction (per thickness of unconserved mass) of $-0.0530$.

Table 1 gives exact corrections at several locations, computed for a uniform sheet over the 1° ocean cells of Schwiderski (1980), for which $\gamma = 0.6865$. These corrections vary from place to place; for gravity, the simple correction (6) seems to overestimate by about 50 per cent.

Mass conservation and the gravity Green function

Followers of the literature on Earth tides will have noticed that the coefficients given by (5) for the gravity Green function are not those usually used. Farrell (1972b) wrote them as

$$c_n = \frac{f}{a^2} [n - (n + 1)k_n + 2h_n]$$

while Pekeris (1978) argued that they should be

$$c_n = \frac{f}{a^2} [-(n + 1) - (n + 1)k_n + 2h_n]$$

Merriam (1974) argued in favour of the form (5), and later pointed out (Merriam 1980) that this expansion converged more rapidly and was thus superior for computational purposes. My purpose here is to point out that (5) also gives the correct result for a uniform layer on a nearly ocean-covered earth, while (7) and (8) do not, and explain why this is so.

The expansions (5, 7, 8) differ only in the first term, which is the part of the Green function for the direct attraction of the water on a rigid earth, often called the Newtonian part. Most derivations of this (e.g. Pekeris 1978) expand the induced potential in spherical...
harmonics and then sum them to form the Green function. A much more straightforward route is to use the inverse square law, and compute the vertical part of the attraction. If we assume a spherical earth of radius \(a\), and let the elevation of our point of observation be \(e_a\), the Green function so derived is

\[
G(\Delta) = \frac{-f}{a^2} \left[ \frac{\epsilon + 2\sin^2 \Delta/2}{[4(1 + \epsilon)\sin^2 \Delta/2 + \epsilon^2]^{3/2}} \right]
\]

assuming \(\epsilon \ll 1\) (Trevor Baker 1977, unpublished computer program). Clearly, for \(|\epsilon|\) very small, \(G(\Delta)\) tends to the function

\[
-\frac{1}{a^2} \frac{1}{4\sin \Delta/2}
\]

given by Farrell (1972b, equation 48).

Now consider a rigid earth covered by a uniform layer of water of thickness \(H\), except for a cap of angular radius \(\Delta_0\). The attraction at the centre of this cap may be found by integrating (10) to get

\[
-2\pi f \rho H (1 - \sin \Delta_0/2).
\]

For \(\Delta_0 \ll 1\), this is \(-2\pi f \rho H\). In this case we have a nearly ocean-covered earth, so removing the \(c_0\) term in (3) should give nearly the same result. If we use (4) with \(\gamma = 1\), and (5) with \(k_0\) and \(h_0\) both 0, we do indeed get the same result as by direct integration. Were we to use (7) or (8), we would get very different results, in one case no correction and in the other one twice as large.

This apparent inconsistency is easily explained by the results of Pekeris (1978), who pointed out that

\[
\sum_{n=0}^{\infty} \frac{1}{2}P_n(\cos \Delta) = \frac{1}{4\sin \Delta/2}
\]

\[
\sum_{n=0}^{\infty} -nP_n(\cos \Delta) = \frac{1}{4\sin \Delta/2} - \delta(1 - \cos \Delta)
\]

\[
\sum_{n=0}^{\infty} (n + 1)P_n(\cos \Delta) = \frac{1}{4\sin \Delta/2} + \delta(1 - \cos \Delta).
\]

So long as \(\Delta > 0\), which is true for all observations not made on an ocean platform or on the sea bottom, all the expansions are equivalent. The delta function alters the integral over \([0, \pi]\) of the function, which is the same as the \(n = 0\) coefficient of the expansion in Legendre functions. It also causes the slow convergence of the sums pointed out by Merriam (1980). If we use the Green function in computing loads this issue is somewhat academic, since we can use (9) directly, without worrying about which spherical harmonic expansion is ‘correct’.

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References


